Size Theory and Shape Matching

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Outline

What is Size Theory?

- Introduced by P. Frosini *et.al.*
- A geometrical/topological approach towards shape comparison.
- *Natural pseudo-distance* and *reduced size function*.

Model Setup

- Size pair: $(\mathcal{M}, \vec{\phi})$ where $\vec{\phi} : \mathcal{M} \rightarrow \mathbb{R}^k$
- To compare shapes $(\mathcal{M}, \vec{\phi})$ and $(\mathcal{N}, \vec{\psi})$, need to compute:

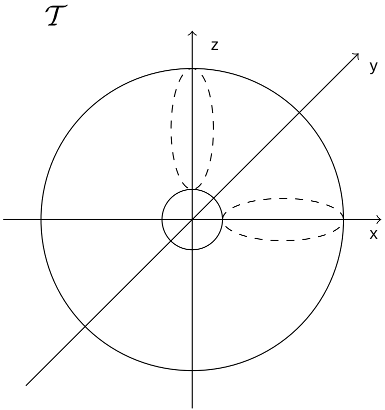
Definition

Natural pseudo-distance

$$inf_{h \in Hom(\mathcal{M},\mathcal{N})} max_{P \in \mathcal{M}} |\phi(P) - \psi(h(P))|$$

where $Hom(\mathcal{M}, \mathcal{N})$ is the set of all homeomorphisms between $\mathcal{M}$ and $\mathcal{N}$, the natural pseudo-distance equals ∞ if $Hom(\mathcal{M}, \mathcal{N}) = \emptyset$

A computable Natural pseudo-distance



Natural pseudo-dist is 2, ϕ and ϕ' are restrictions of $\zeta : \zeta(x, y, z) = z$

Lower Bounds for Natural pseudo-distance

1. Lower bounds via reduced size function
   - Donati and Frosini (2004)
   - d_{match} Frosini *et.al* (2008).
2. Lower bounds via size homotopy group
   - Frosini and Mulazzani (1999).

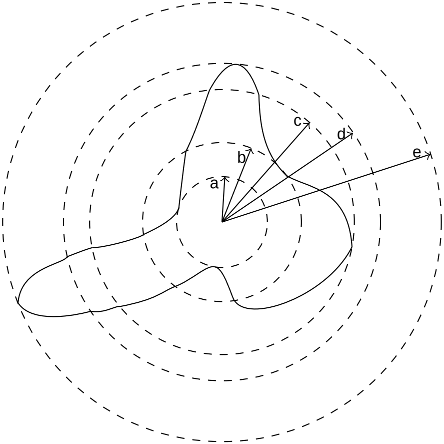
Reduced size function

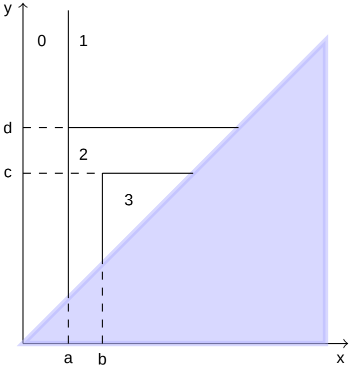
- Define the diagonal as $\Delta = \{(x, y) \in \mathbb{R}^2 : x = y\}$
- Define the open half-plane above diagonal as $\Delta^+ = \{(x, y) \in \mathbb{R}^2 : x < y\}$
- Homotopy: P and Q are $\langle \phi \leq y \rangle$-homotopic if they are connected by a path in the sub level-set $\mathcal{M}\langle \phi \leq y \rangle$, written as $P \cong_{\phi \leq y} Q$.

Definition

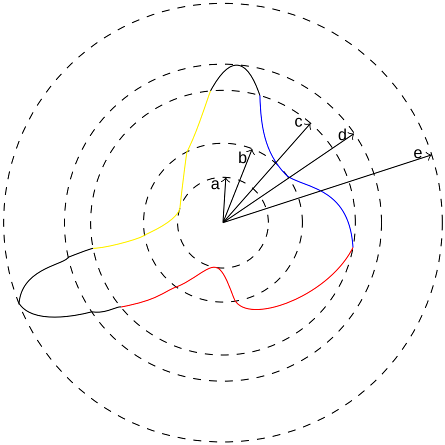
The reduced size function $l^*_{(\mathcal{M},\phi)} : \Delta^+ \to \mathbb{N}$ *is defined as* $l^*_{(\mathcal{M},\phi)}(x, y) = card(\mathcal{M}\langle \phi \leq x \rangle / \cong_{\phi \leq y})$.

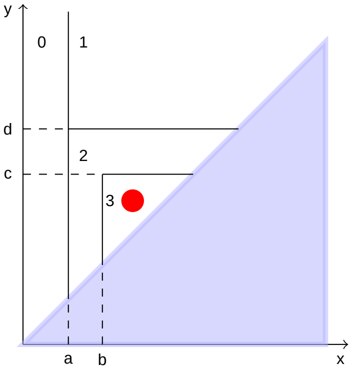
Reduced size function example





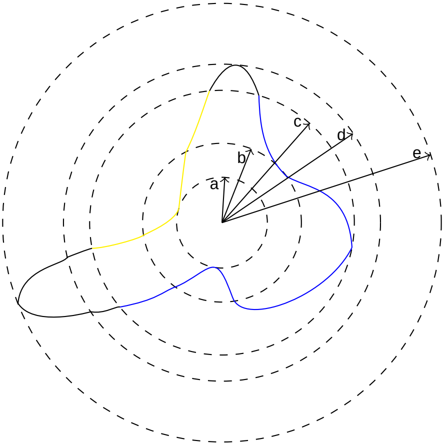
Reduced size function example

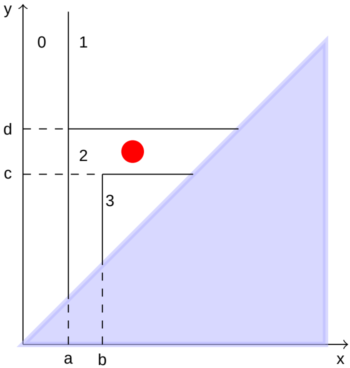




$$I^*_{(\mathcal{M},\phi)}(x, y) = 3$$

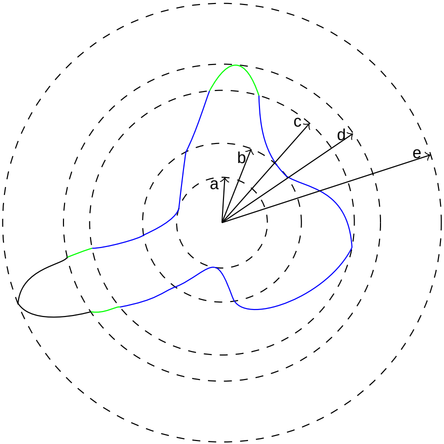
Reduced size function example

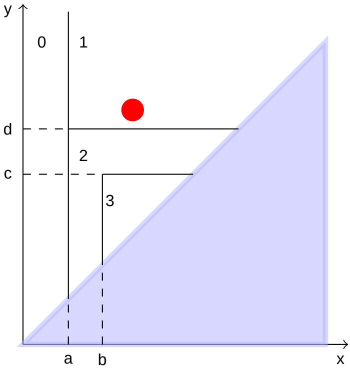




$$\ell^*_{(\mathcal{M},\phi)}(x, y) = 2$$

Reduced size function example





$$\ell^*_{(\mathcal{M},\phi)}(x,y) = 1$$

Properties of reduced size functions

- $l^*_{(\mathcal{M},\phi)}(x,y)$ is non-decreasing in x and non-increasing in y.
- $l^*_{(\mathcal{M},\phi)}(x,y) < +\infty$ for $x < y$
- $l^*_{(\mathcal{M},\phi)}(x,y) = 0$ for $x < min_{P\in\mathcal{M}}\phi(P)$.
- $l^*_{(\mathcal{M},\phi)}(x,y) = +\infty$ for any x and y $s.t.$ $\exists$ a non-isolated point $Q \in \mathcal{M}$ with $y < \phi(Q) < x$.
- For every $y \geq max_{P\in\mathcal{M}}\phi(P)$, $l^*_{(\mathcal{M},\phi)}(x,y)$ is equal to the number of arcwise connected components $\mathcal{N}$ of $\mathcal{M}$ such that $x \geq min_{P\in\mathcal{N}}\phi(P)$.

Formal Series

- Denote the set of vertical lines of $\mathbb{R}^2$ by $\mathcal{R}$, let $\Delta^* = \Delta^+ \cup \mathcal{R}$.

Definition

Any function $m : \Delta^* \to \mathcal{N}$ is said to be a formal series in Δ^*. The set $supp(p) = \{p \in \mathcal{R} : m(r) \neq 0\}$ is called the support of m.

Corner Points and Lines

Lemma

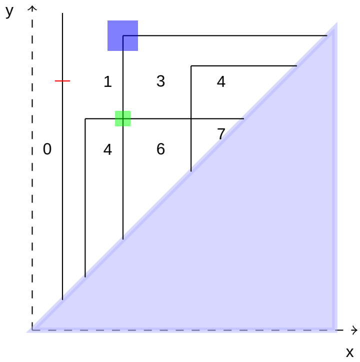
Let $x_1 \leq x_2 < y_1 \leq y_2$*, then*
$l^*_{(\mathcal{M},\phi)}(x_2, y_1) - l^*_{(\mathcal{M},\phi)}(x_1, y_1) \geq l^*_{(\mathcal{M},\phi)}(x_2, y_2) - l^*_{(\mathcal{M},\phi)}(x_1, y_2)$

Definition

For $p = (x, y) \in \Delta^+$*, define* $\mu(p)$ *as the minimum over all positive* ϵ *with* $x + \epsilon < y - \epsilon$ *of* $l^*_{(\mathcal{M},\phi)}(x + \epsilon, y - \epsilon) - l^*_{(\mathcal{M},\phi)}(x - \epsilon, y - \epsilon) - l^*_{(\mathcal{M},\phi)}(x + \epsilon, y + \epsilon) + l^*_{(\mathcal{M},\phi)}(x - \epsilon, y + \epsilon)$*.*
For every vertical line r*, with equation* $x = k$*, define* $\mu(r)$ *as the minimum over all positive* ϵ *with* $k + \epsilon < 1/\epsilon$ *of*
$l^*_{(\mathcal{M},\phi)}(k + \epsilon, 1/\epsilon) - l^*_{(\mathcal{M},\phi)}(k - \epsilon, 1/\epsilon)$

Corner Points and Lines

- A **corner point** is a point $p \in \Delta^+$ such that $\mu(p) > 0$
- A **corner line** is a vertical line $r \in \mathcal{R}$ such that $\mu(r) > 0$. *a.k.a* corner point at ∞.

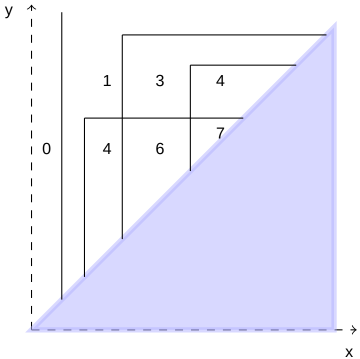


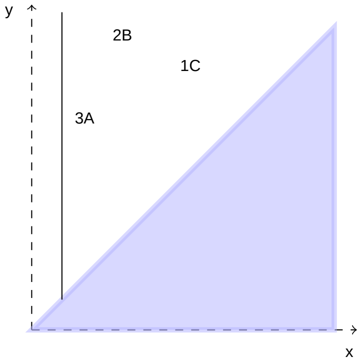
Corner Points and Lines

Theorem

For $(\bar{x}, \bar{y}) \in \Delta^+$, *we have*

$$\ell^*_{(\mathcal{M},\phi)}(\bar{x}, \bar{y}) = \sum_{(x,y)\in\Delta^*, x\leq\bar{x}, y\leq\bar{y}} \mu((x,y))$$





Equivalence of Corner Point representation

- Let S_ρ denote the set $(x, y) \in \mathbb{R}^2 : x < y - \rho$, We say $l^*_{(\mathcal{M},\phi)} \cong_\rho l^*_{(\mathcal{N},\psi)}$ if the the two reduced size functions differ only in a vanising subset of S_ρ
- Let $\mathcal{L}_\rho$ denote the quotient of all size functions under $\cong_\rho$ and let Ω_ρ denote the set of formal series. There exist a bijection $\tilde{\alpha}_\rho : \mathcal{L}_\rho \to \Omega_\rho$ for every $\rho \geq 0$.

Comparing reduced size functions

Definition

For any (x,y), (x',y') *in* $\bar{\Delta}^*$, *define the distance*

$$d((x,y),(x',y')) = min\{max\{|x-x'|,|y-y'|\}, max\frac{y-x}{2}, \frac{y'-x'}{2}\}$$

Definition

If a_i *and* b_i *are two representative sequence for* l_1^* *and* l_2^* *respectively, then the matching distance between* l_1^* *and* l_2^* *is given by:*

$$d_{match}(l_1^*, l_2^*) = inf_\sigma sup_i d(a_i, b_i)$$

where i *varies in* $\mathcal{N}$ *and* σ *varies among all the bijections from* $\mathcal{N}$ *to* $\mathcal{N}$.

Stability and Lower bounds

Theorem

Let $(\mathcal{M}, \phi)$ be a size pair. For every real number $\epsilon \geq 0$ and for every measuring function $\psi : \mathcal{M} \to \mathbb{R}$ such that $max_P |\phi(P) - \psi(P)| \leq \epsilon$, we have

$$d(l^*_{(\mathcal{M},\phi)}, l^*_{(\mathcal{N},\psi)}) \leq \epsilon$$

Theorem

Let $(\mathcal{M}, \phi)$ and $(\mathcal{N}, \psi)$ be two size pairs, with $\mathcal{M}$ and $\mathcal{N}$ homeomorphic, then

$$\inf_f \max_P |\phi(P) - \psi(P)| \geq d_{match}(l^*_{(\mathcal{M},\phi)}, l^*_{(\mathcal{N},\psi)})$$

where $f \in Hom(\mathcal{M}, \mathcal{N})$

Admissible Pair

Definition

For every unit vector $\vec{l} \in \mathbb{R}^k$ *such that* $l_i > 0$ *for every* $i = 1, \ldots, k$ *and for every vector* $\vec{b} \in \mathbb{R}^k$ *such that* $\sum_{i=1}^{k} b_i = 0$, *we say* $(\vec{l}, \vec{b})$ *is admissible, denote by* Adm_k. *For every* $(\vec{l}, \vec{b}) \in Adm_k$, *we define the half-plane* $\pi_{(\vec{l}, \vec{b})}$ *to be*

$$\vec{x} = s\vec{l} + \vec{b}$$
$$\vec{y} = t\vec{l} + \vec{b}$$

where $s < t \in \mathbb{R}$

Foliation

- For $(\vec{l}, \vec{b}) \in Adm_k$, a foliation could be defined as $F^{\vec{\phi}}_{(\vec{l},\vec{b})}(P) = max_{i=1,\dots,k}\{\frac{\phi_i(P)-b_i}{l_i}\}$

Theorem

For every $(\vec{x}, \vec{y}) \in \pi_{(\vec{l},\vec{b})}$, the following equality holds,

$$l^*_{(\mathcal{M},\phi)}(\vec{x}, \vec{y}) = l^*_{(\mathcal{M},F^{\vec{\phi}}_{(\vec{l},\vec{b})})}(s, t)$$

- The 1-D reduced size functions could be used for shape comparison

Stablity of matching

Lemma

If $(\mathcal{M}, \vec{\phi})$ *and* $(\mathcal{M}, \vec{\psi})$ *are size pairs with* $max_{P \in \mathcal{M}} \|\vec{\phi}(P) - \vec{\psi}(P)\| \leq \epsilon$, *then for each* $(\vec{l}, \vec{b}) \in Adm_k$, *it holds that*

$$d_{match}(l^*_{(\mathcal{M}, F^{\vec{\phi}}_{(\vec{l},\vec{b})})}, l^*_{(\mathcal{M}, F^{\vec{\psi}}_{(\vec{l},\vec{b})})}) \leq \frac{\epsilon}{\min_{i=1,\dots,k} l_i}$$

Lower bound

Theorem

Let $(\mathcal{M},\vec{\phi})$ *and* $(\mathcal{M},\vec{\psi})$ *be two size pairs with* $\mathcal{M}$ *and* $\mathcal{N}$ *homeomorphic. Setting* $d((\mathcal{M},\vec{\phi}),(\mathcal{M},\vec{\psi})) = \inf_f max_P \|\vec{\phi}(P) - \vec{\psi}(f(P))\|$*, then*

$$\sup_{(\vec{l},\vec{b})\in Adm_k} \min_{i=1,\dots,k} l_i \cdot d_{match}\left(l^*_{(\mathcal{M},F^{\vec{\phi}}_{(\vec{l},\vec{b})})}, l^*_{(\mathcal{M},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}\right) \leq d((\mathcal{M},\vec{\phi}),(\mathcal{M},\vec{\psi}))$$

- The L-H-S is the extended distance D_{match}.

Example

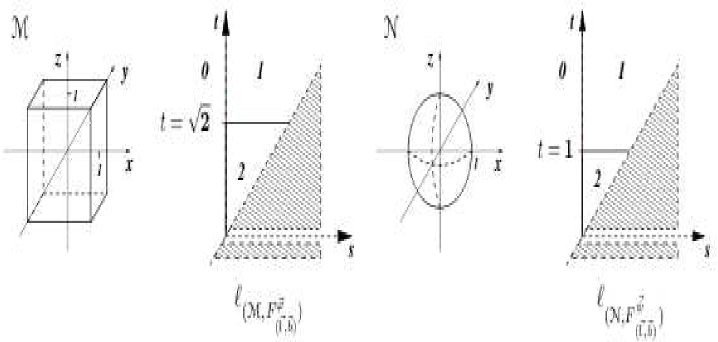


Figure: The topological spaces and reduced size functions

Connection to Persistent Homology

- $l_{(\mathcal{M},\phi)}(x,y) = \beta_0^{x,y}$ in 1-D.
- Reduced size functions plays the role of filtration functions.
- Corner points correspond to persistence intervals.
- Stabilty can be proved in both cases.