Size Theory and Shape Matching: A Survey

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Abstract

This paper investigates current research on size theory. The research reviewed in this paper includes studies on 1-D size theory, multidimensional size theory and their applications to shape matching. This survey will also address the connections between the size theoretical approach and other approaches to shape matching.

1 Introduction

Originally proposed as a method for comparison between manifolds, size theory(cf. [1] [5] [6] [7] [8] [9] [10] [11]) is a new approach towards geometrical/topological shape matching. In a size theoretical framework a shape is described by a size pair $(\mathcal{M}, \phi)$, where $\phi : \mathcal{M} \to \mathbb{R}^k$ is a k-dimensional measuring function. Natural pseudo-distance and matching distances are two important and related dissimilarity measures between size pairs.

The natural pseudo-distance between size pairs $(\mathcal{M}, \phi)$ and $(\mathcal{N}, \psi)$ is defined as $\inf_{f \in Hom(\mathcal{M},\mathcal{N})} \max_{P \in \mathcal{M}} |\phi(P) - \psi(f(P))|$. Despite its beautiful geometric formulation, the natural pseudo-distance is very difficult to compute since the infimum is taken over all the homeomorphisms between $\mathcal{M}$ and $\mathcal{N}$. The matching distance d_{match} between reduced size functions, being an easily computable lower bound, is proposed in [6] as an alternative matching function.

The 1-D size function does not generalize readily to the multidimensional case, some care must be taken in choosing a suitable foliation. The reduced size function basically studies the connectedness of sub-level sets. Other types of lower bounds also exists, for example, Frosini *et.al* [11] proposed a lower bound based on size homotopy groups. The Persistent Homology approach (cf. [12] [2] [3] [4]) uses higher dimensional topological information and it is noted that the zero-dimensional vineyard is closely related to multidimensional size functions.

The following sections are organized as follows: In section 2 we discuss 1-D size matching. In section 3 we discuss multidimensional size functions. In section 4 we relate the size-theoretic approach and other topological/geometrical approaches. In section 5 we conclude this survey and discuss possible future directions.

2 1-D size functions

Let $\Delta^+ = \{(x,y) \in \mathbb{R}^2 : x < y\}$. The 1-D reduced size function $l^*_{(\mathcal{M},\phi)} : \Delta^+ \to \mathbb{N}$ assigns to each point $(x,y) \in \Delta^+$ the number of connected components of the sub level-set $\{P \in \mathcal{M} : \phi(P) \leq x\}$ that is divided by the equivalence classes of $\langle \phi \leq y \rangle$-connectedness. It is easily seen that the size function is non-decreasing in the first argument and non-increasing in the second argument (cf. [10]).

Definition 1 *For every point $p = (x,y) \in \Delta^+$, the multiplicity $\mu(p)$ is defined as the minimum over all $\epsilon > 0$ with $x + \epsilon < y - epsilon$ of*

$$l^*_{(\mathcal{M},\phi)}(x+\epsilon, y-\epsilon) - l^*_{(\mathcal{M},\phi)}(x-\epsilon, y-\epsilon) - l^*_{(\mathcal{M},\phi)}(x+\epsilon, y+\epsilon) + l^*_{(\mathcal{M},\phi)}(x-\epsilon, y+\epsilon)$$

Definition 2 *For every vertical line r with equation $x = k$, the multiplicity $\mu(r)$ is defined to be the minimum over all $\epsilon > 0$ with $k + \epsilon < \frac{1}{\epsilon}$ of*

$$l^*_{(\mathcal{M},\phi)}(k+\epsilon, \frac{1}{\epsilon}) - l^*_{(\mathcal{M},\phi)}(k-\epsilon, \frac{1}{\epsilon})$$

Corner points are those points in the half-plane with non-zero multiplicities. Corner lines (corner points at infinity) are vertical lines with non-zero multiplicities. The corner points are relatively easy to verify since by definition one only has to evaluate the reduced size function values at the four corners of a moving window/interval of variable size. The following observation about corner points/lines is mentioned in [10] [1],

Theorem 1 *For every* $(\bar{x}, \bar{y}) \in \Delta^+$*, we have*

$$l^*_{(\mathcal{M},\phi)}(\bar{x}, \bar{y}) = \sum_{(x,y)\in\Delta^*, x\leq\bar{x}, y\leq\bar{y}} \mu((x,y))$$

where Δ^* *is* Δ^+ *extended by the points at infinity.*

This theorem leads to the key theorem in [10] that establishes the bijective correspondence of set of reduced size functions and formal series.

Definition 3 *Any function* $m : \Delta^* \to \mathcal{N}$ *is said to be a formal series in* Δ^**. The set* $supp(p) = \{p \in \mathcal{R} : m(r) \neq 0\}$ *is called the support of* m*.*

Clearly the corner points/lines are the support of formal series.

Theorem 2 *Let* S_ρ *denote the set* $(x, y) \in \mathbb{R}^2 : x < y - \rho$*, We say* $l^*_{(\mathcal{M},\phi)} \cong_\rho l^*_{(\mathcal{N},\psi)}$ *if the the two reduced size functions differ only in a vanishing subset of* S_ρ*. Let* $\mathcal{L}_\rho$ *denote the quotient of all size functions under* $\cong_\rho$ *and let* Ω_ρ *denote the set of formal series. There exist a bijection* $\tilde{\alpha}_\rho : \mathcal{L}_\rho \to \Omega_\rho$ *for every* $\rho \geq 0$*.*

This theorem justifies the reduction to corner-point presentation since they completely characterize reduced size functions. Since reduced size functions have at most a finite number of corner points/lines in the upper half plane (cf [10]), the comparison between two size functions can be achieved through the comparison of the sequence of multiplicities of the respective corner points (lines). Indeed, the matching distance between two reduced size functions is defined as,

Definition 4 *If* (a_i) *and* (b_i) *are two representative sequence for* l_1^* *and* l_2^**, then the matching distance is defined by*

$$d_{match}(l_1^*, l_2^*) := inf_\sigma sup_i d(a_i, b_{\sigma(i)})$$

where $i \in \mathbb{N}$ *and* σ *varies among all bijections from* $\mathbb{N}$ *to* $\mathbb{N}$*. The pseudo-distance* d *measures the cost of moving one corner point to another and is defined as,*

$$d((x, y), (x', y')) = min\{max\{|x - x'|, |y - y'|\}, max\frac{y - x}{2}, \frac{y' - x'}{2}\}$$

d_{match} is stable with regard to the measuring function, it is noted in [7] that the matching distance between $l^*_{(\mathcal{M},\phi)}$ and $l^*_{(\mathcal{M},\psi)}$ is upper bounded by the infinity norm of $\phi - \psi$. As proved by in [6], d_{match} is a lower bound for the natural pseudo-distance. It improves an earlier result in [9], indeed it is the tightest lower bound using size functions [8]. It can be computed in $O(n^{2.5})$ time where n is the number of corner points [7]. Other lower bounds exist, for example in [11] a bound is obtained via size homotopy group.

3 Multidimensional size functions

In the multidimensional setting, Δ^+ denote the set $\{(\vec{x}, \vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\}$. The multidimensional size function can be reduced to the 1-D case using a foliation. For size pairs $(\mathcal{M}, \vec{\phi})$, a parametrized families of half-planes in $\mathbb{R}^k \times \mathbb{R}^k$ is considered instead of one.

Definition 5 *For every unit vector $\vec{l}$ s.t. $l_i > 0$ for every $i = 1, \ldots, k$ and for every $\vec{b} \in \mathbb{R}^k$ such that $\sum_{i=1}^k b_i = 0$, we shall say $(\vec{l}, \vec{b})$ is admissible. We denote the set of all admissible pairs by Adm_k. Given an admissible pair $(\vec{l}, \vec{b})$, we parametrize the half plane $\pi_{(\vec{l},\vec{b})}$ by*

$$\vec{x} = s\vec{l} + \vec{b} \qquad \vec{y} = t\vec{l} + \vec{b}$$

where $s, t \in \mathbb{R}$ and $s < t$.

Theorem 3 *Let $(\vec{l}, \vec{b})$ be an admissible pair, and $F^{\vec{\phi}}_{(\vec{l},\vec{b})} : \mathcal{M} \to \mathbb{R}$ be defined by setting:*

$$F^{\vec{\phi}}_{(\vec{l},\vec{b})} = max_{i=1,\ldots,k}\{\frac{\phi_i(P) - b_i}{l_i}\}$$

Then, for every $(\vec{x}, \vec{y}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ the following equality holds:

$$l_{(\mathcal{M},\vec{\phi})}(\vec{x}, \vec{y}) = l_{(\mathcal{M},F^{\vec{\phi}}_{(\vec{l},\vec{b})})}(s, t)$$

From above we can see that for each $(\vec{l}, \vec{b}) \in Adm_k$, we have an associated 1-D size function. Let $\sigma_{(\vec{l},\vec{b})}$ be the associated formal series, the family $\sigma_{(\vec{l},\vec{b})}$: $(\vec{l}, \vec{b}) \in Adm_k$ is the complete descriptor for the k-dimensional size function

$l_{(\mathcal{M},\vec{\phi})}$. Similar to the 1-D case, we have that $l_{(\mathcal{M},\vec{\phi})} \equiv l_{(\mathcal{N},\vec{\psi})}$ if and only if $d_{match}(l_{(\mathcal{M},F^{\vec{\phi}}_{(\vec{l},\vec{b})})}, l_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}) = 0$ for all admissible pairs. Like in the 1-D case, a stability result on the measuring functions could be obtained using the foliation defined in **Theorem 3**. As a consequence, the stability result leads to a lower bound of the natural pseudo-distance.

Theorem 4 *Let $(\mathcal{M},\vec{\phi})$ and $(\mathcal{M},\vec{\psi})$ be two size pairs with $\mathcal{M}$ and $\mathcal{N}$ homeomorphic. Setting $d((\mathcal{M},\vec{\phi}),(\mathcal{M},\vec{\psi})) = \inf_f max_P \|\vec{\phi}(P) - \vec{\psi}(f(P))\|$, then*

$$\sup_{(\vec{l},\vec{b})\in Adm_k} \min_{i=1,\ldots,k} l_i \cdot d_{match}(l^*_{(\mathcal{M},F^{\vec{\phi}}_{(\vec{l},\vec{b})})}, l^*_{(\mathcal{M},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}) \leq d((\mathcal{M},\vec{\phi}),(\mathcal{M},\vec{\psi}))$$

Denote the LHS by D_{match}, then D_{match} gives a computable lower bound for the natural pseudo-distance. The computational cost of D_{match} is roughly that of d_{match} multiplied by $|Adm_k|$.

As shown by [5] [1], the multidimensional size function could yield discriminatory power. The shapes to compare are $\mathcal{M}$ and $\mathcal{N}$ where $\mathcal{M}$ is the boundary of $[-1,1]^3$ and $\mathcal{N} = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$, The measuring functions are the restrictions of $\vec{\Phi}(x,y,z) = (|x|,|z|)$ to respective manifolds. And the admissible pair is given by $\vec{l} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $\vec{b} = (0,0)$. With the choice above, we have $F^{\vec{\phi}}_{(\vec{l},\vec{b})} = \sqrt{2}\max\{|x|,|z|\}$. Using **Theorem 3**, the 2-D size function reduces to 1-D.

$$l_{(\mathcal{M},\vec{\phi})}(x_1,x_2,y_1,y_2) = l_{(\mathcal{M},\vec{\phi})}(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}) = l_{(\mathcal{M},F^{\vec{\phi}}_{(\vec{l},\vec{b})})}(s,t)$$

Similar reduction holds for $\vec{\psi}$, thus the comparison reduces to the computation

$$D_{match} = \frac{\sqrt{2}}{2} d_{match}(l_{(\mathcal{M},F^{\vec{\phi}}_{(\vec{l},\vec{b})})}, l_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}) = \frac{\sqrt{2}}{2}(\sqrt{2}-1) = 1 - \frac{\sqrt{2}}{2}$$

See figure 1 for details.

Clearly, the 2-D size function with a suitably chosen admissible pair can discriminate the surface of a cube from a 2-sphere while the 1-D size functions can't. However we should note that in practice such a desired admissible pair could be very tricky to find and may require a very fine discretization. Since the computational complexity depends linearly on the size of the admissible set, this will of course adds to the computational cost.

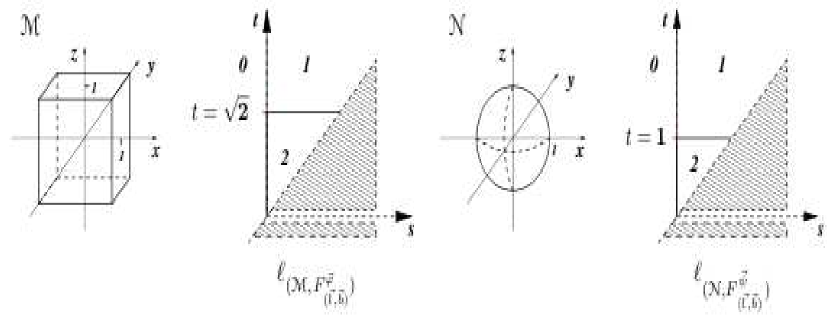

Figure 1: The Topological spaces $\mathcal{M}$ and $\mathcal{N}$ and the size functions (adapted from [5]). Note in particular $(0, \sqrt{2})$ and $(0, 1)$ are corner points of $l_{(\mathcal{M}, F^{\vec{\phi}}_{(\vec{l},\vec{b})})}$ and $l_{(\mathcal{N}, F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$ respectively.

4 Relation to Persistent Homology

In [5], a link between size functions and persistent homology (cf. [12] [2] [3] [4]) is discussed in detail. As the authors noted, the 1-D reduced size function $l_{(\mathcal{M},\phi_t)}(x, y)$ is equivalent to the persistent betti number $\beta_0^{x,y}$. The measuring function ϕ plays the role of a filtration function in persistence homology. And $l_{(\mathcal{M},\phi_t)}(x, y)$ is the number of 0-dimensional features in the set $\mathcal{M}\langle\phi < x\rangle$ that persist in $\mathcal{M}\langle\phi < y\rangle$. Corner points and corner lines correspond to finite and infinite persistence intervals respectively. And generally speaking, corner points farther away from the diagonal correspond to coarser (0-dimensional) homological features. In fact the representation of the 1-D reduced size function is equivalent to the persistence diagram in [4]. The multidimensional size function is a generalization in sense that it studies the connectedness of the set $\mathcal{M}\langle\vec{\phi} \preceq \vec{y}\rangle$ instead of the connectedness of the sub-level set in the 1-D case.

And assuming tameness of the measuring function, a stability result on multidimensional measuring functions is obtained in [5] similar to the one in [3].

5 Conclusion and Discussion

Size theory approach extracts topological information from geometrical shapes. By representing the connectivity of sub-level sets succinctly as finite sequence of corner points, matching distances of shapes can be computed efficiently. An noticable feature of size theory is that measuring functions could be arbitrary continuous functions. The size theory could easily be combined with a measuring function that encodes local geometry such as eccentricity.

From the definition of natural pseudo-distance, we can see that it makes sense only to consider comparison of homeomorphic manifolds. The seems rather restrictive especially in the discrete case. The formulation could be generalized using correspondences into the following form,

$$\inf_{R(\mathcal{M},\mathcal{N})} \max_{(a,b)\in R} |\phi(a) - \psi(b)|$$

where R denotes correspondence. It should be noted here that the concern is purely notational. In practice, non-hemeomorphic discretization does not raise any issue in computing the d_{match} for reduced size functions (Indeed, even homeomorphic manifolds leads to formal series with different cardinality). However it would be interesting to see if the generalization gives a similar lower bound result.

The computation of multidimensional size functions depends on the choice of foliations. In [1] and [5], the family of curves $\vec{\gamma}_{(\vec{l},\vec{b})}$ is used. In general, any continuous curve $\vec{\gamma}_{\vec{\alpha}} : \mathbb{R} \to \mathbb{R}^k$ such that (i) $\vec{\gamma}_{\vec{\alpha}}(s) \prec \vec{\gamma}_{\vec{\alpha}}(t)$ for $s < t$, (ii) for every $(\vec{x}, \vec{y})$ in Δ^+ there is a unique $\vec{\gamma}_{\vec{\alpha}}$ passing through. A possible research direction would be to investigate multidimensional size functions using other foliation schemes.

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