Homework 2

Due: Thursday, November 12, 2009

Problem 1

Block decomposition. (20 points.) An *n-block* in a simplicial complex K is a pair of subcomplexes $(e, \dot{e})$, such that $\dim e = n$ and $H_n(e, \dot{e}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H_i(e, \dot{e}) \simeq 0$ for $i \neq n$. $\dot{e}$ is called the boundary of e, and the interior of e is the set of simplices in $e - \dot{e}$. A *block decomposition* of K is a set of blocks such that each simplex is in the interior of just one block, and the boundary of each n-block is in the union of the m-blocks, for $m < n$. Block decomposition together with the boundary map gives us a chain complex, which we call a *block complex*. Recall that the Block Complex Lemma tells us that the homology of the block complex is isomorphic to the homology of the simplicial complex K.

- Give a minimal block decomposition of a triangulation of the 2-sphere.
- Give a minimal block decomposition of a triangulation of the torus.
- Give a minimal block decomposition of a triangulation of the surface of genus g.
- Describe an algorithm that finds a minimal block decomposition of a simplicial complex.

Problem 2

3-torus. (20 points.) Consider the *3-dimensional torus* obtains from the unit cube by gluing opposite faces in pairs without twisting. That is, each point $(x, y, 0)$ is identified with $(x, y, 1)$, $(x, 0, z)$ with $(x, 1, z)$, and $(0, y, z)$ with $(1, y, z)$. Show that the Betti numbers of this space are $\beta_0 = \beta_3 = 1$ and $\beta_1 = \beta_2 = 3$.

Problem 3

Exact sequence of a triple. (20 points.) Let C be a simplicial complex with subcomplexes $A \subseteq B \subseteq C$. Prove the existance of the following *exact homology sequence of the triple*:

$$\ldots \to \mathsf{H}_p(B, A) \to \mathsf{H}_p(C, A) \to \mathsf{H}_p(C, B) \to \mathsf{H}_{p-1}(B, A) \to \ldots$$

Problem 4

2-manifold with boundary. (10 points.) Let $\mathbb{M}$ be a 2-manifold with genus g and b boundary circles. Use Lefschetz Duality Theorem to determine the ranks of $\mathsf{H}_p(\mathbb{M})$ and $\mathsf{H}_p(\mathbb{M}, \partial\mathbb{M})$ for $p = 0, 1, 2$. Draw the generator of the first absolute and relative homology groups.

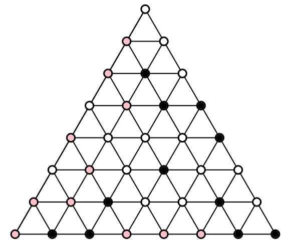

Figure 1: Each vertex receives one of three colors: white, shaded, or black.

Extra credit

Sperner Lemma. (30 points.) Let K be a triangulated triangular region. We 3-color the vertices such that

- the three corners receive three different colors;
- the vertices on each side of the region are 2-colored.

Prove that there is a triangle in K whose vertices receive three different colors.