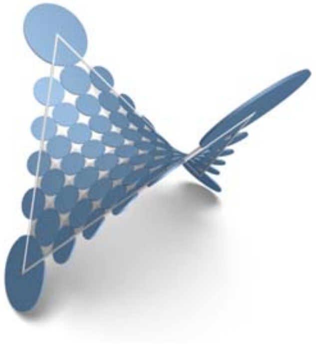
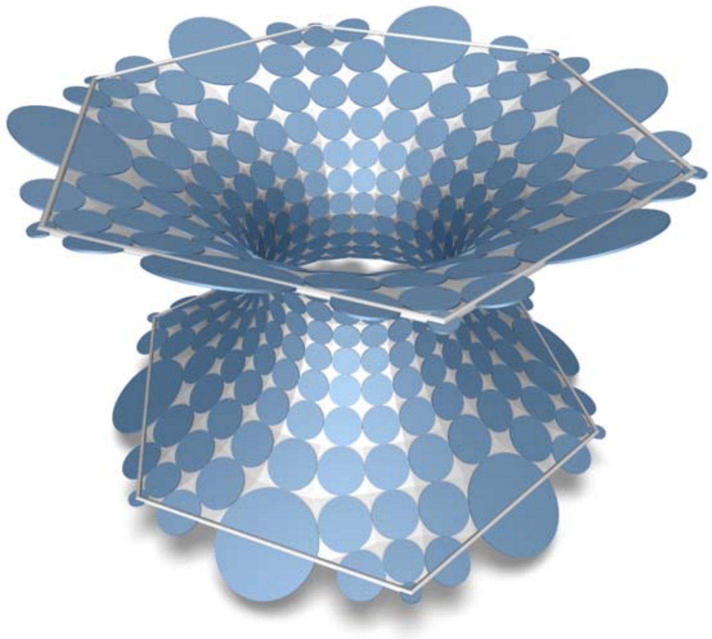
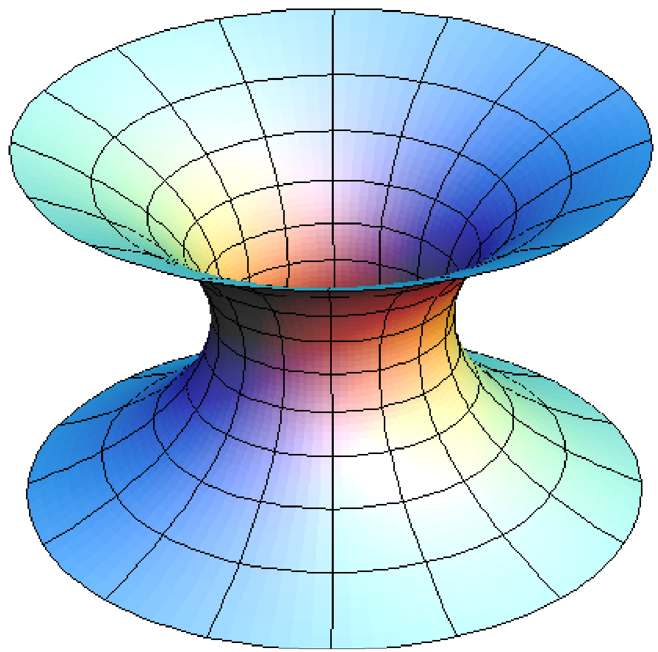


(Discrete) Differential Geometry



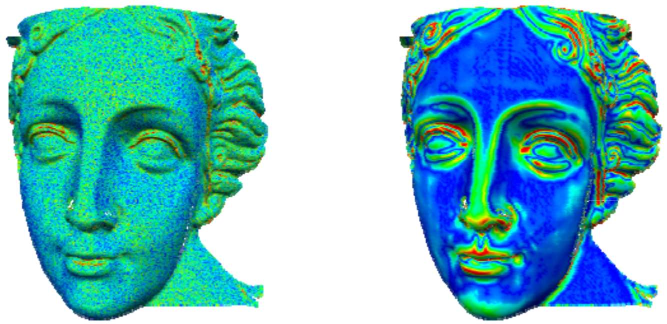


Motivation

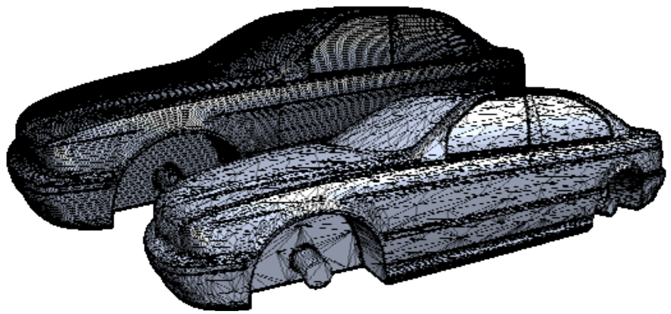
- Understand the structure of the surface
  - Properties: smoothness, “curviness”, important directions
- How to modify the surface to change these properties
- What properties are preserved for different modifications
- The math behind the scenes for many geometry processing applications

Motivation

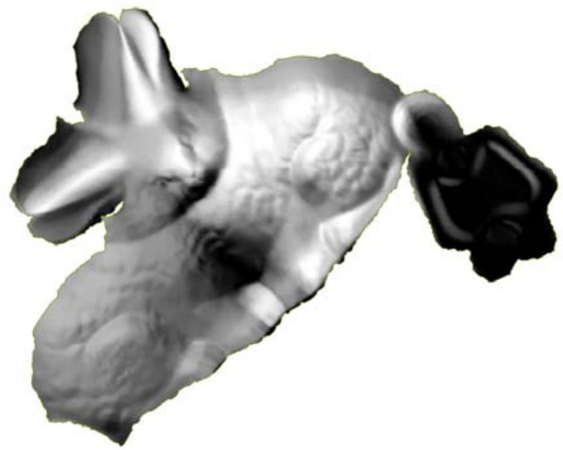
- Smoothness
  - ➡ Mesh smoothing



- Curvature
  - ➡ Adaptive simplification

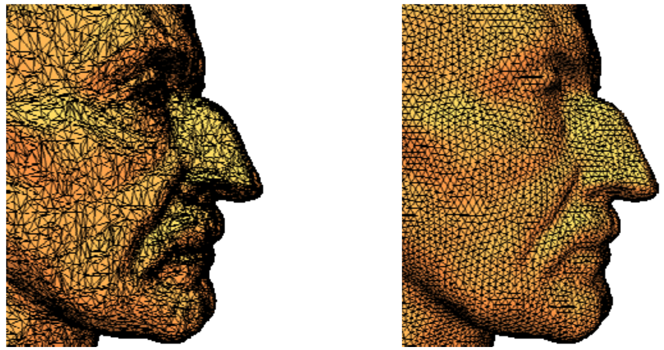


  - ➡ Parameterization

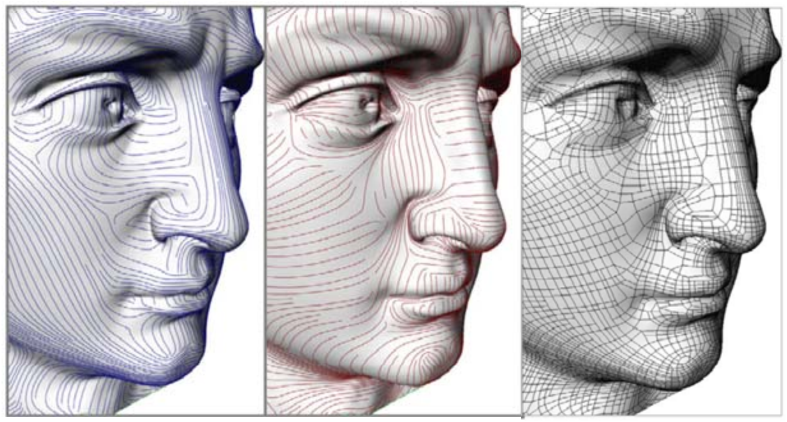


Motivation

- Triangle shape
  - ➡ Remeshing

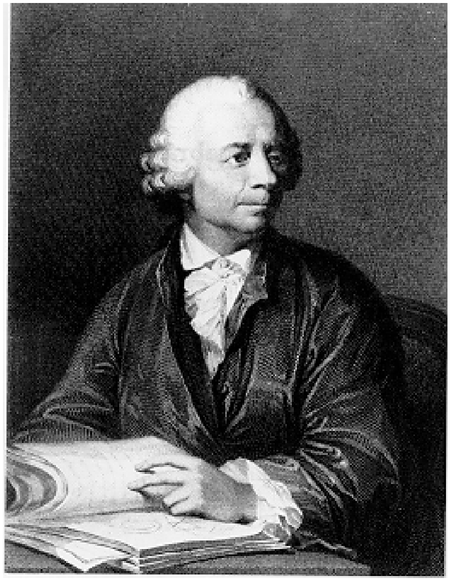


- Principal directions
  - ➡ Quad remeshing



Differential Geometry

- M.P. do Carmo: ***Differential Geometry of Curves and Surfaces***, Prentice Hall, 1976



Leonard Euler (1707 - 1783)

Carl Friedrich Gauss (1777 - 1855)

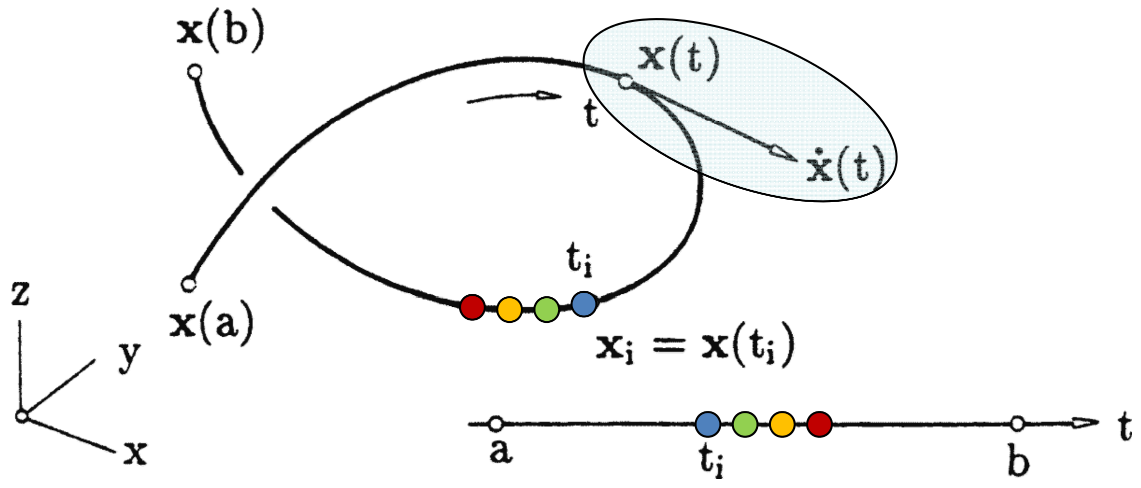
Parametric Curves

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} \neq \mathbf{0}$$

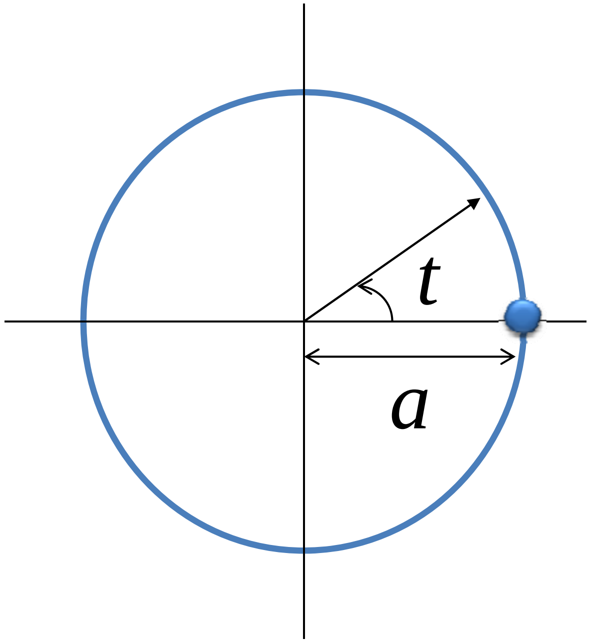
$$t \in [a, b] \subset \mathbb{R}$$

"velocity" of particle on trajectory



Parametric Curves

A Simple Example



$$\alpha_1(t) = (a\cos(t), a\sin(t))$$

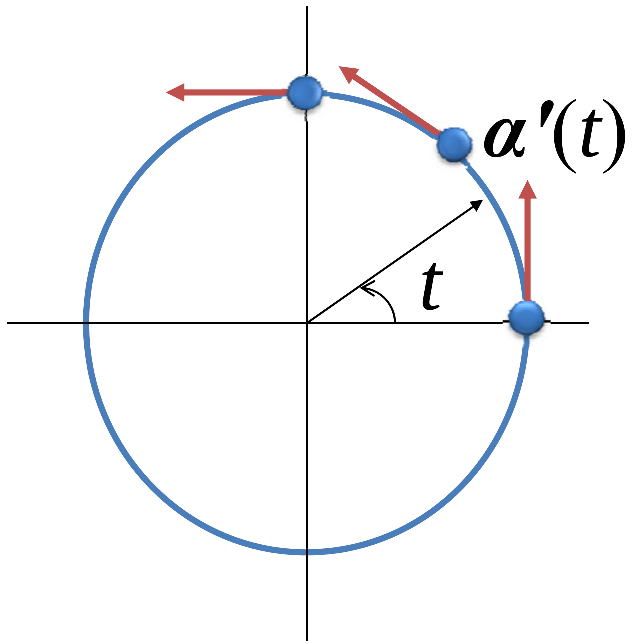
$$t \in [0, 2\pi]$$

$$\alpha_2(t) = (a\cos(2t), a\sin(2t))$$

$$t \in [0, \pi]$$

Parametric Curves

A Simple Example



$$\alpha(t) = (\cos(t), \sin(t))$$

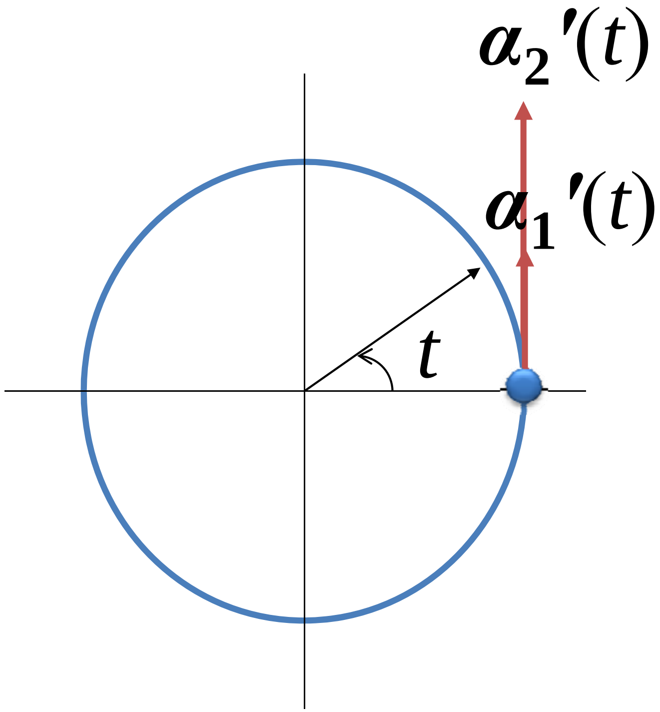
$$\alpha'(t) = (-\sin(t), \cos(t))$$

$\alpha'(t)$ - direction of movement

$|\alpha'(t)|$ - speed of movement

Parametric Curves

A Simple Example



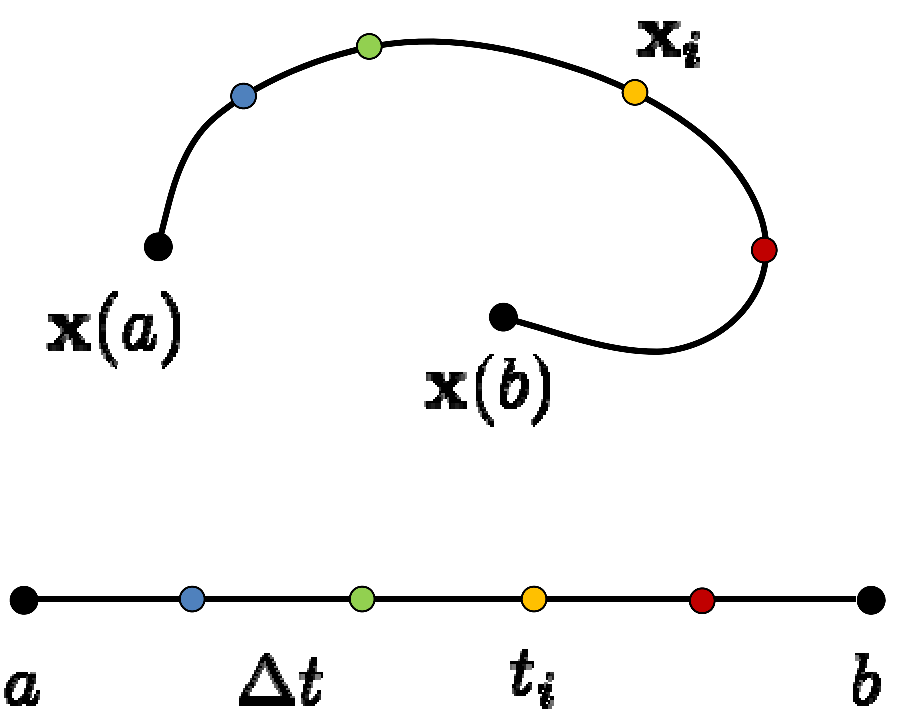
$$\alpha_1(t) = (\cos(t), \sin(t))$$

$$\alpha_2(t) = (\cos(2t), \sin(2t))$$

Same direction, different speed

Length of a Curve

- Let $t_i = a + i\Delta t$ and $\mathbf{x}_i = \mathbf{x}(t_i)$

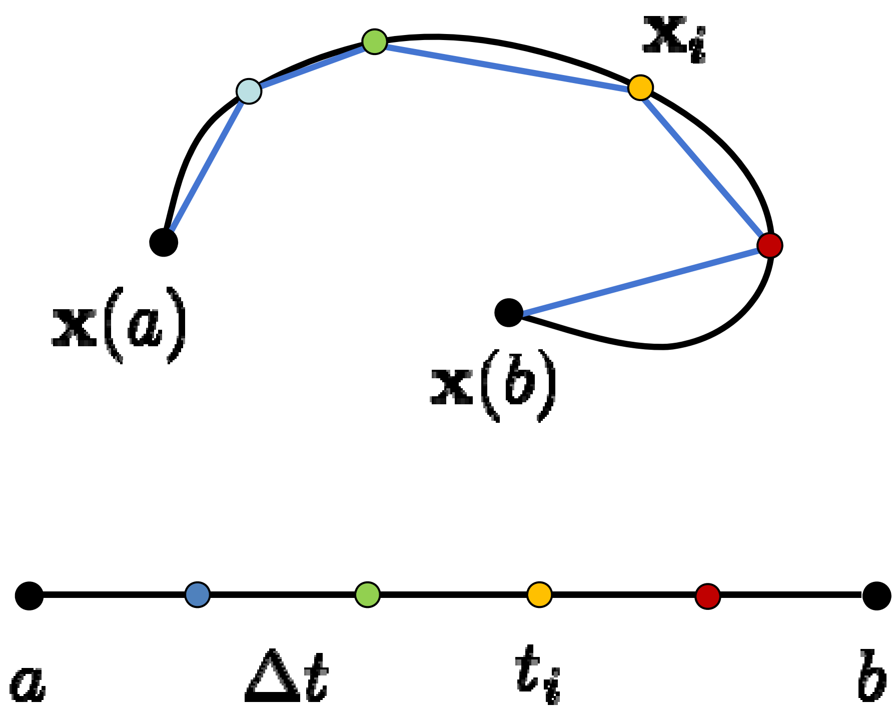


Length of a Curve

- Chord length $$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t$$

Euclidean norm

$$\Delta \mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$$

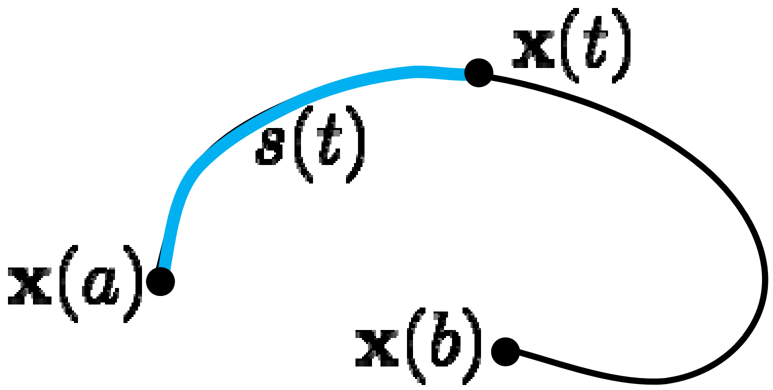


Length of a Curve

- Chord length $$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \| \frac{\Delta \mathbf{x}_i}{\Delta t} \| \Delta t$$

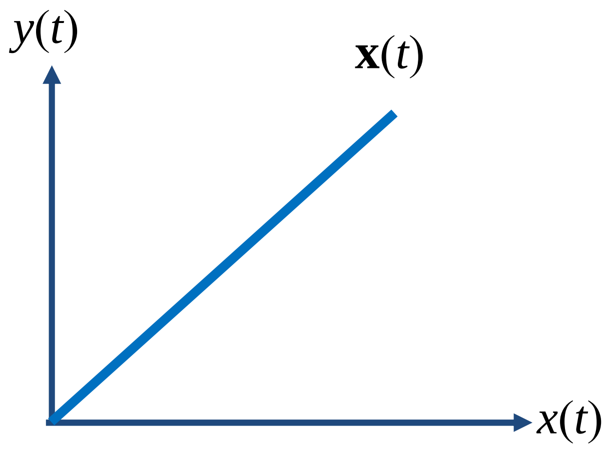
  $$\Delta \mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$$

- Arc length $$s = s(t) = \int_a^t \|\dot{\mathbf{x}}\| dt$$

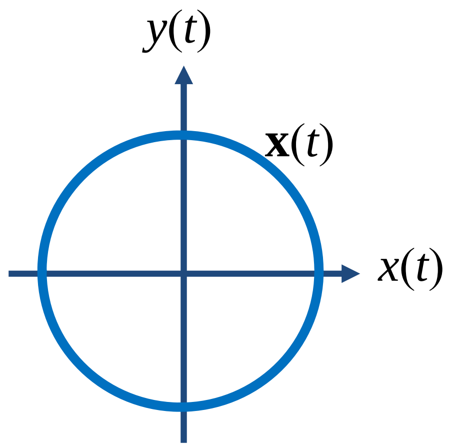


Examples

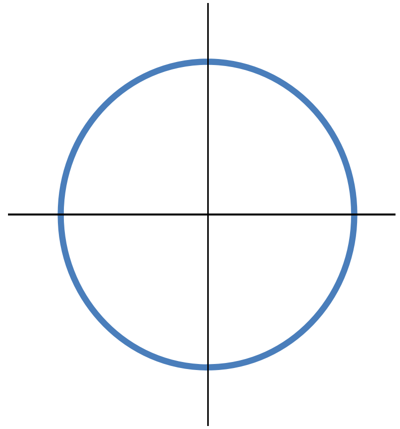
- Straight line
  - $\mathbf{x}(t) = (t,t),\ t\in[0,\infty)$
  - $\mathbf{x}(t) = (2t,2t),\ t\in[0,\infty)$
  - $\mathbf{x}(t) = (t^2,t^2),\ t\in[0,\infty)$



- Circle
  - $\mathbf{x}(t) = (\cos(t),\sin(t)),\ t\in[0,2\pi)$
  - $\mathbf{x}(t) = \left(\dfrac{t^2-1}{t^2+1},\dfrac{2t}{t^2+1}\right) \quad t\in(-\infty,+\infty)$



Examples



$\boldsymbol{\alpha}(t) = (a\cos(t), a\sin(t)),\ t \in [0,2\pi]$

$\boldsymbol{\alpha}'(t) = (-a\sin(t), a\cos(t))$

$$L(\alpha) = \int_0^{2\pi} |\alpha'(t)|\,dt$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + a^2\cos^2(t)}\,dt$$

$$= a\int_0^{2\pi} dt = 2\pi a$$

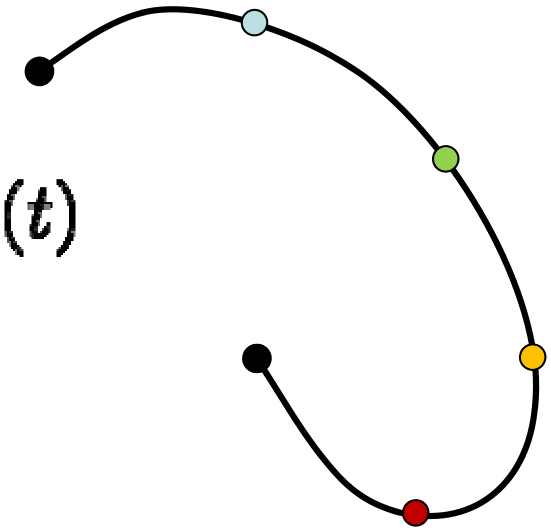
Many possible parameterizations

Length of the curve does not depend on parameterization!

Arc Length Parameterization

- Re-parameterization $\mathbf{x}(u(t))$

$$\frac{d\mathbf{x}(u(t))}{dt} = \frac{d\mathbf{x}}{du}\frac{du}{dt} = \dot{\mathbf{x}}(u(t))\dot{u}(t)$$



- Arc length parameterization

$$s = s(t) = \int_a^t \|\dot{\mathbf{x}}\| dt \qquad ds = \|\dot{\mathbf{x}}\| dt$$

  – parameter value s for $\mathbf{x}(s)$ equals length of curve from $\mathbf{x}(a)$ to $\mathbf{x}(s)$

$$\boxed{\|\dot{\mathbf{x}}(s)\| = 1 \rightarrow \dot{\mathbf{x}}(s) \cdot \ddot{\mathbf{x}}(s) = 0}$$

Curvature

$\mathbf{x}(t)$ a curve parameterized by arc length

The ***curvature*** of $\mathbf{x}$ at t: $\boxed{\kappa = \left\| \ddot{\mathbf{x}}(t) \right\|}$

$\dot{\mathbf{x}}(t)$ – the tangent vector at t

$\ddot{\mathbf{x}}(t)$ – the *change* in the tangent vector at t

$R(t) = 1/\kappa(t)$ is the ***radius of curvature*** at t

Examples

Straight line

$$\boldsymbol{\alpha}(s) = us + v,\ \ u,v \in R^2$$

$$\boldsymbol{\alpha}'(s) = u$$

$$\boldsymbol{\alpha}''(s) = \mathbf{0} \quad \rightarrow \quad |\boldsymbol{\alpha}''(s)| = 0$$

Circle

$$\boldsymbol{\alpha}(s) = (a\cos(s/a),\ a\sin(s/a)),\ s \in [0,2\pi a]$$

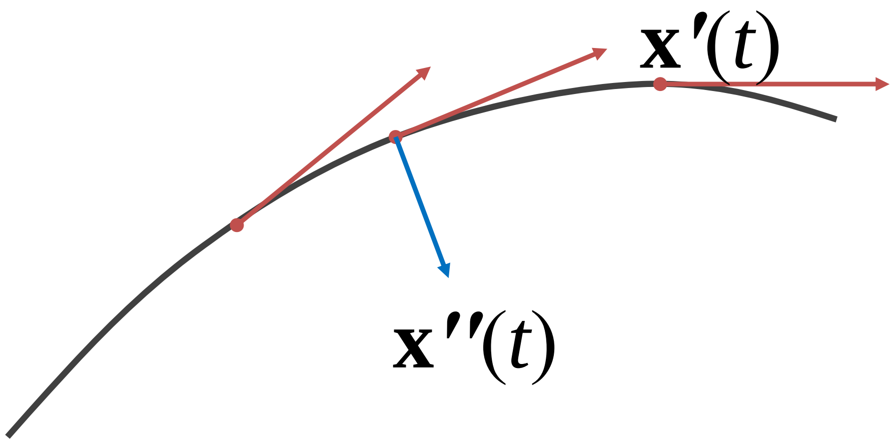
$$\boldsymbol{\alpha}'(s) = (-\sin(s/a),\ \cos(s/a))$$

$$\boldsymbol{\alpha}''(s) = (-\cos(s/a)/a,\ -\sin(s/a)/a) \rightarrow \ |\boldsymbol{\alpha}''(s)| = 1/a$$

The Normal Vector

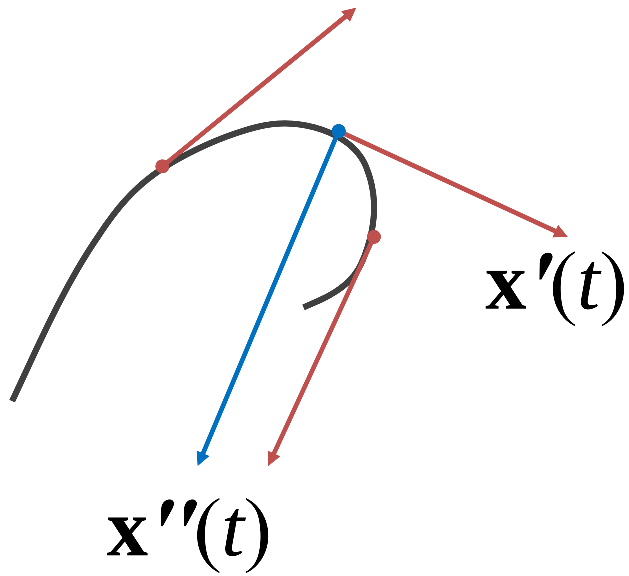
$\mathbf{x}'(t) = \boldsymbol{T}(t)$ - tangent vector

$|\mathbf{x}'(t)|$ - arc length



$\mathbf{x}''(t) = \boldsymbol{T}'(t)$ - normal direction

$|\mathbf{x}''(t)|$ - curvature

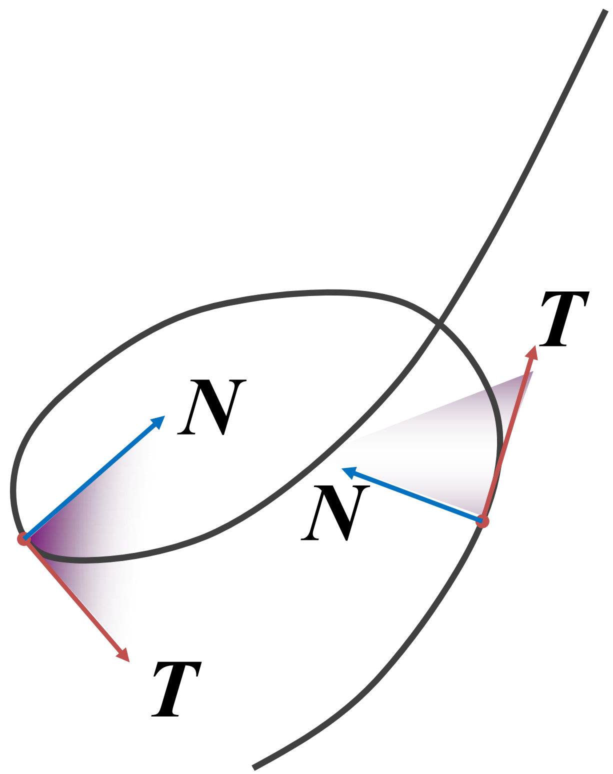


If $|\mathbf{x}''(t)| \neq \mathbf{0}$, define $N(t) = \boldsymbol{T}'(t)/|\boldsymbol{T}'(t)|$

Then $\mathbf{x}''(t) = \boldsymbol{T}'(t) = \kappa(t)N(t)$

The Osculating Plane

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the *osculating plane* at s

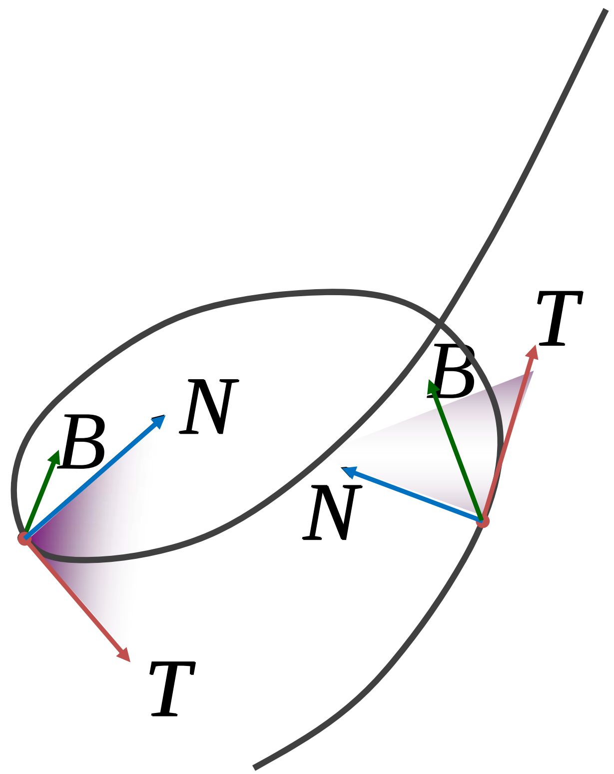


The Binormal Vector

For points s, s.t. $\kappa(s) \neq 0$, the *binormal vector* $\boldsymbol{B}(s)$ is defined as:

$$\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$$

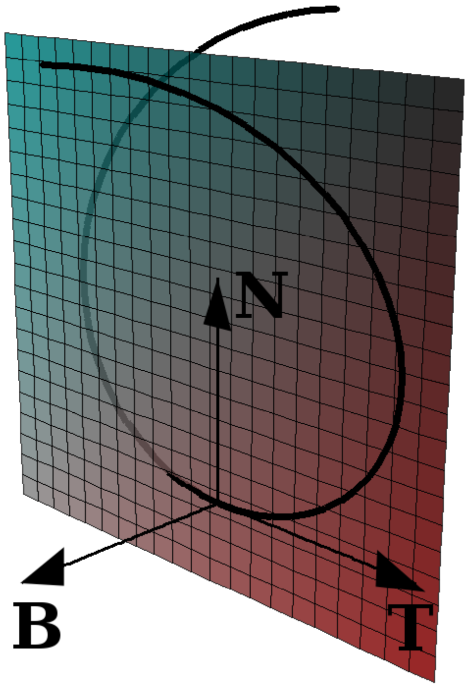
The binormal vector defines the osculating plane



The Frenet Frame

$$T = \frac{\dot{x}}{\|\dot{x}\|} \qquad N = \frac{\ddot{x}}{\|\ddot{x}\|} \qquad B = T \times N$$

tangent **normal** **binormal**

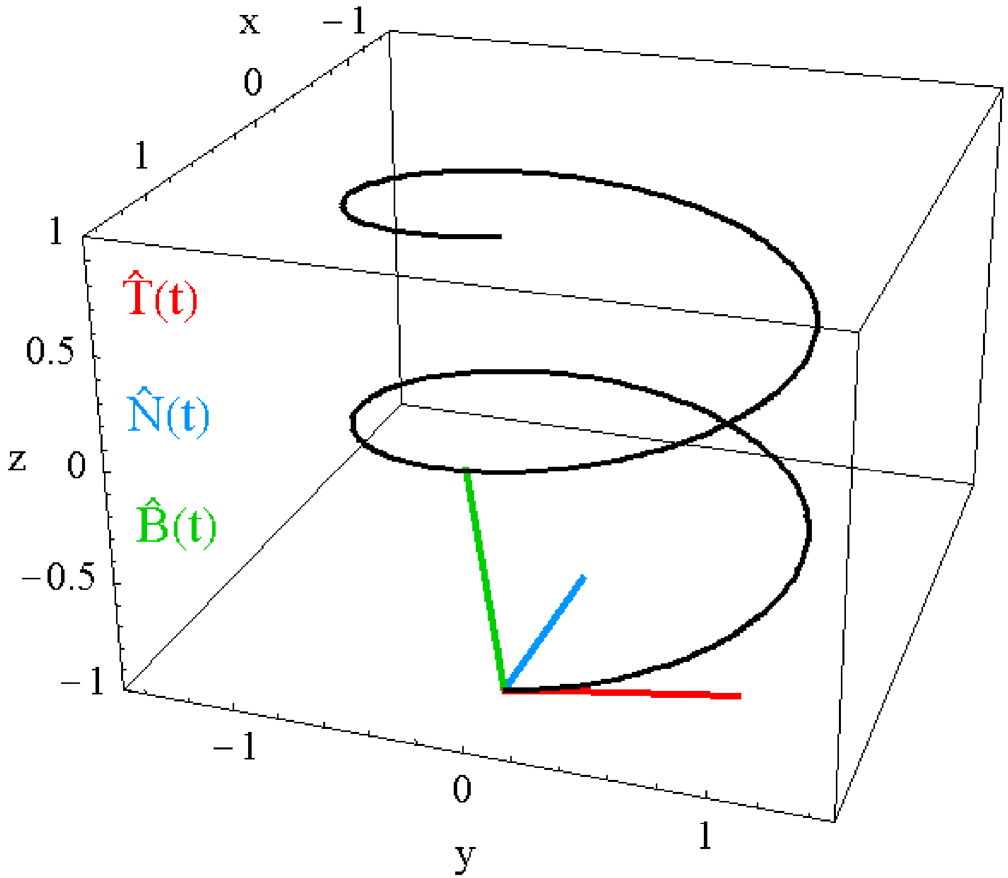


The Frenet Frame

$\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ form an orthonormal basis for R^3 called the *Frenet frame*

How does the frame change when the particle moves?

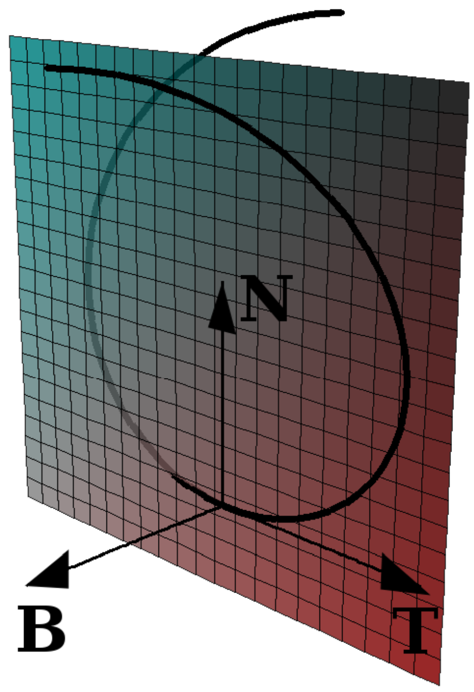
What are $\boldsymbol{T}', \boldsymbol{N}', \boldsymbol{B}'$ in terms of $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$?



The Frenet Frame

- Frenet-Serret formulas

$$\begin{aligned}
\dot{T} &= + \kappa N \\
\dot{N} &= -\kappa T + \tau B \\
\dot{B} &= - \tau N
\end{aligned}$$



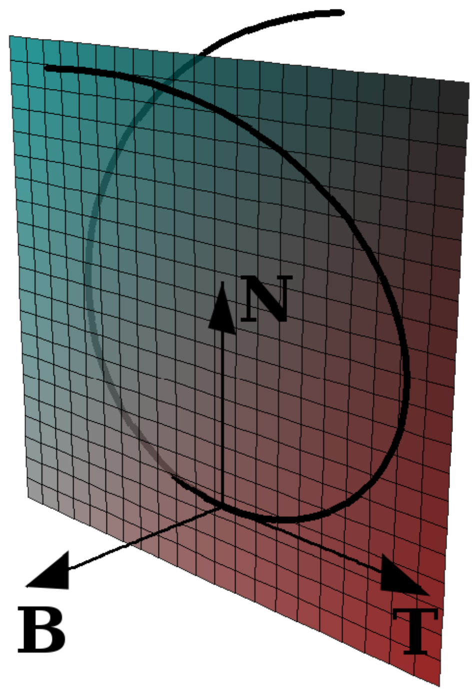
- curvature $\kappa = \left\| \ddot{x} \right\|$

- torsion $\tau = \dfrac{1}{\kappa^2} \det\left[\dot{x}, \ddot{x}, \dddot{x}\right]$

(arc-length parameterization)

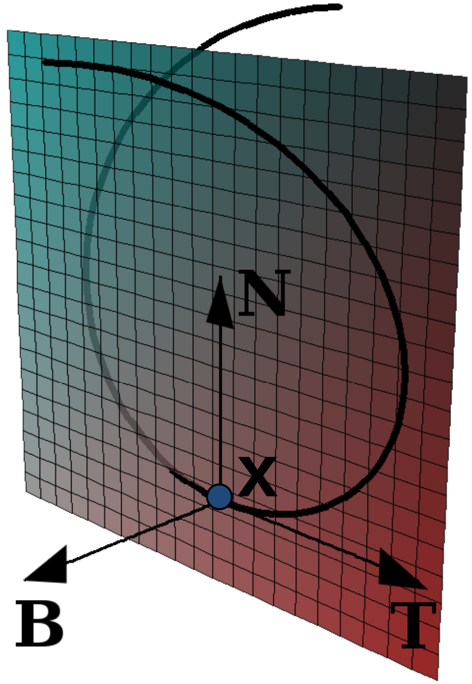
Curvature and Torsion

- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
  - intrinsic properties of the curve
- Invariant under rigid (translation+rotation) motion
- Define curve uniquely up to rigid motion

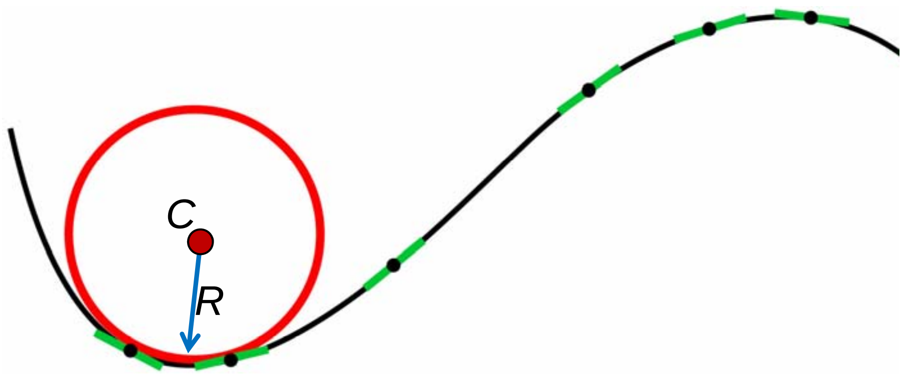


Curvature and Torsion

- Planes defined by $\mathbf{x}$ and two vectors:
  - ***osculating plane***: vectors $\mathbf{T}$ and $\mathbf{N}$
  - ***normal plane***: vectors $\mathbf{N}$ and $\mathbf{B}$
  - ***rectifying plane***: vectors $\mathbf{T}$ and $\mathbf{B}$

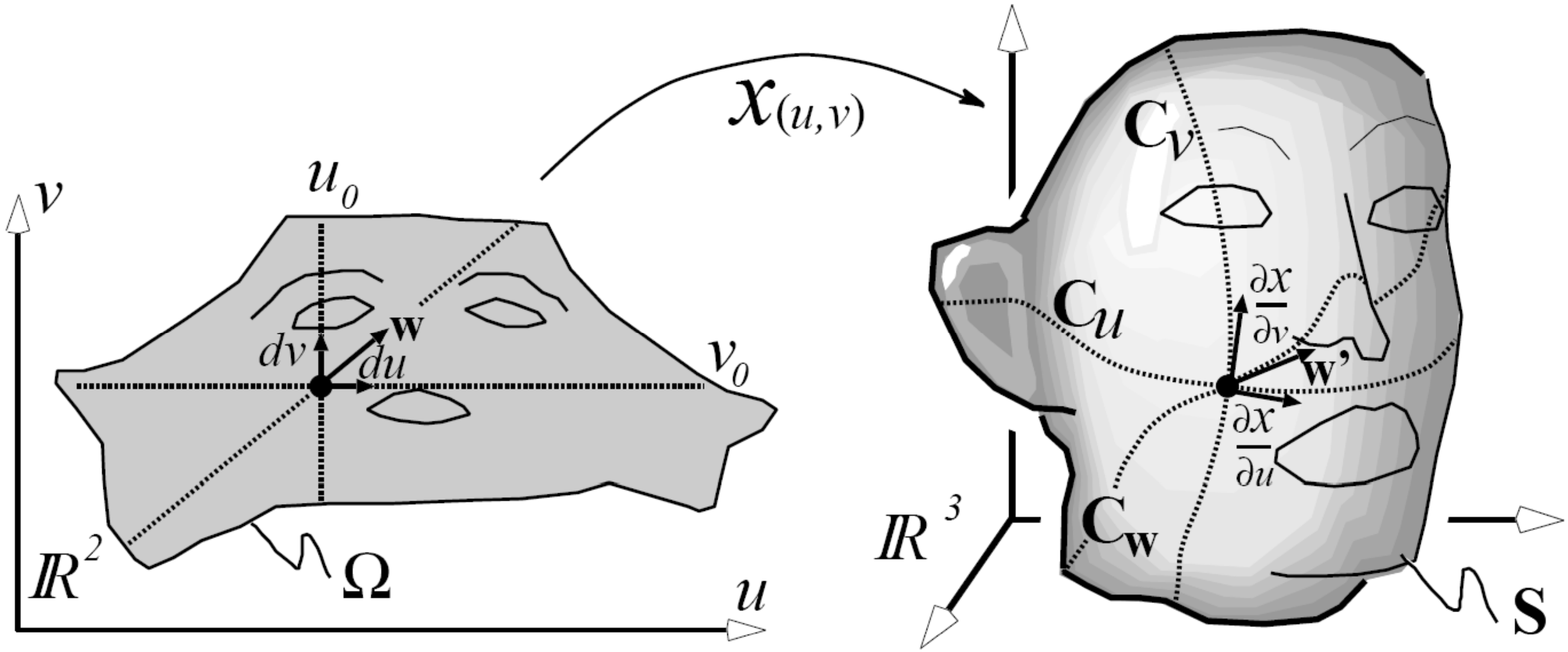


- Osculating circle
  - second order contact with curve
  - center $C = x + \kappa^{-1} N$
  - radius $R = \kappa^{-1}$



Differential Geometry: Surfaces

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$

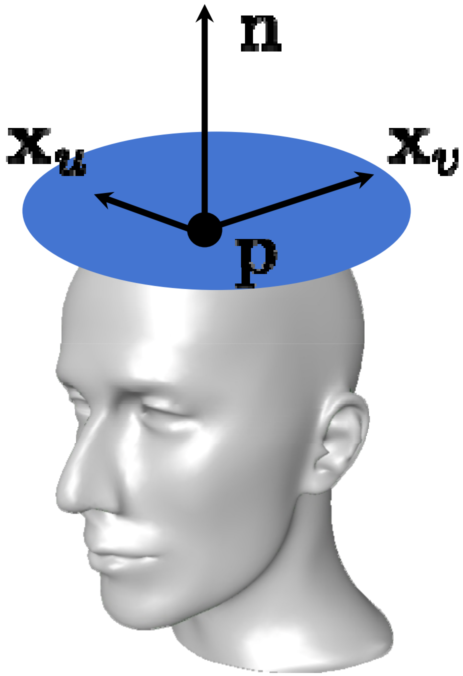


Differential Geometry: Surfaces

- Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$

- Normal vector

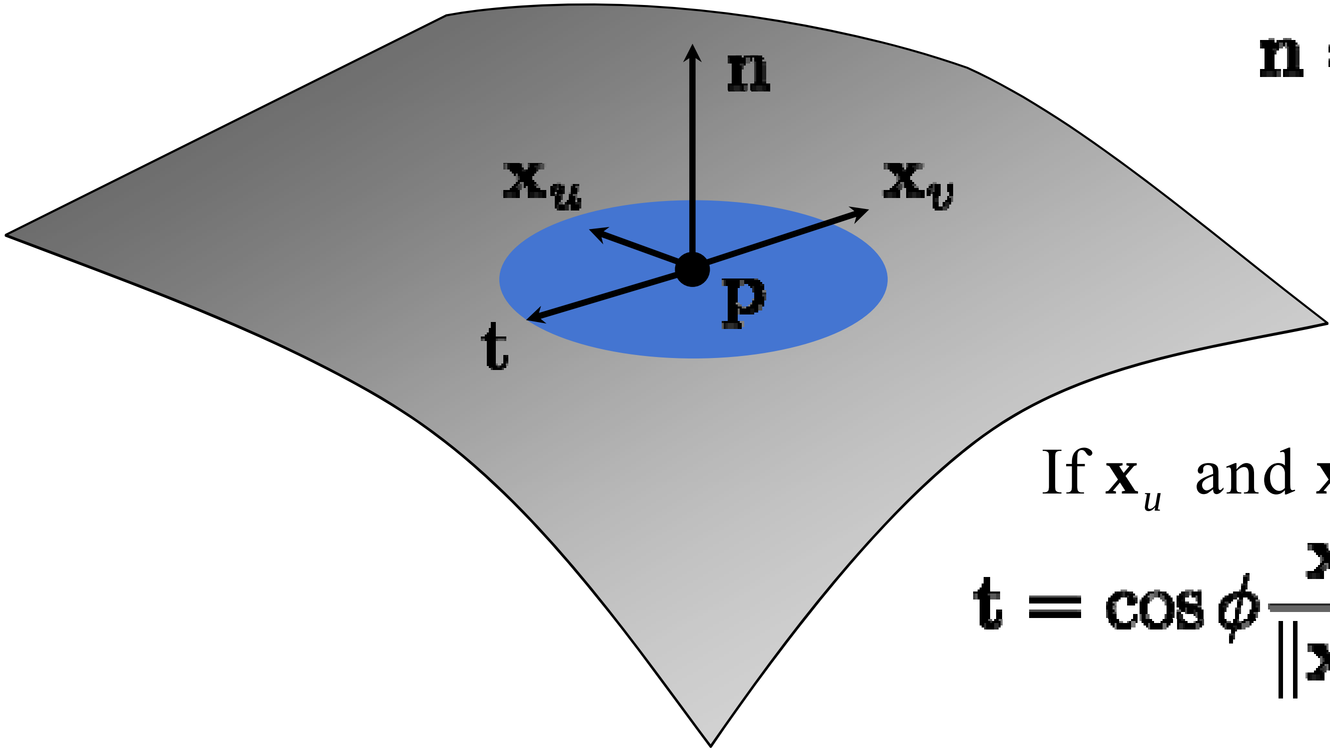


$$\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|$$

  – assuming regular parameterization, i.e.

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

Normal Curvature

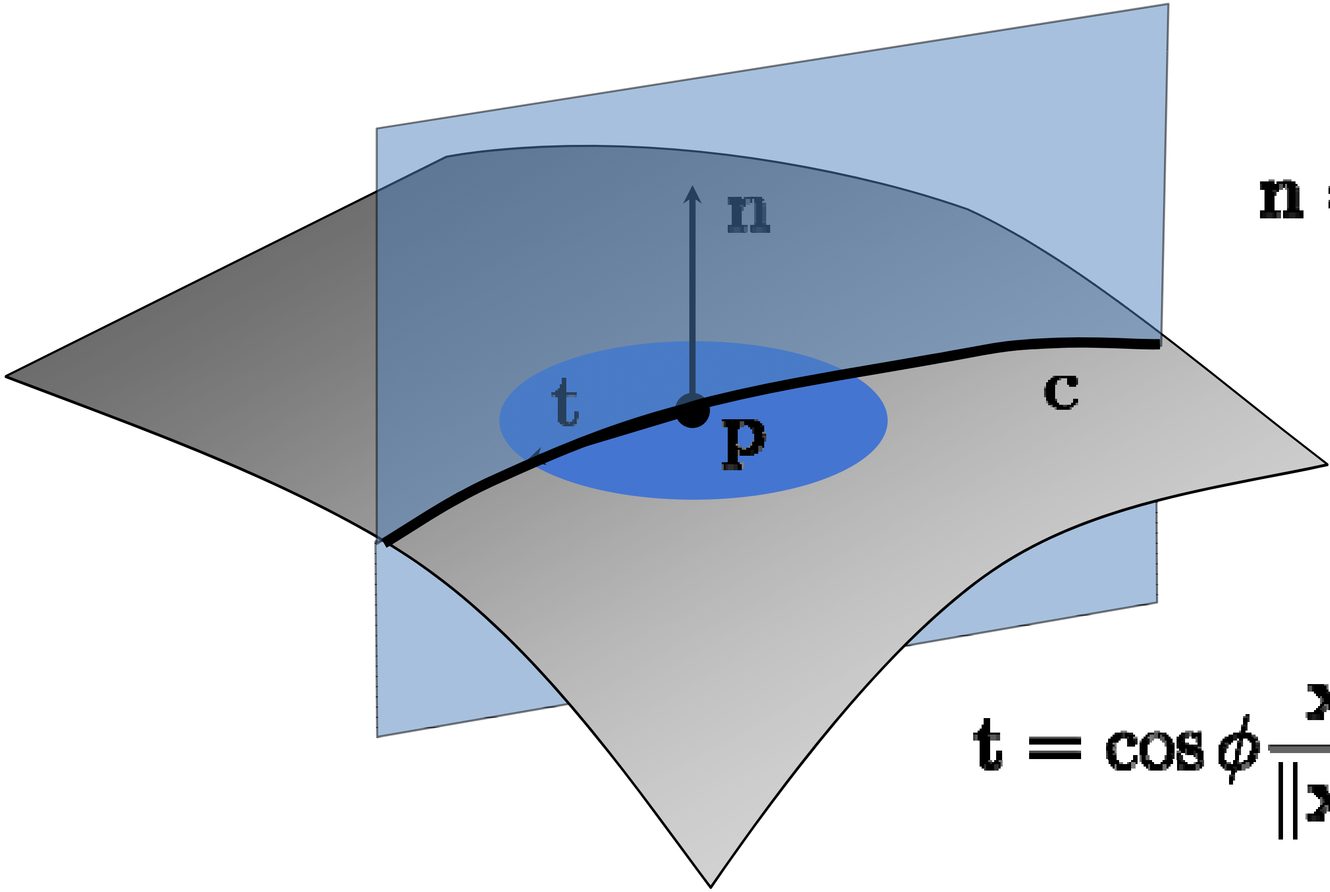


$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

If $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal:

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

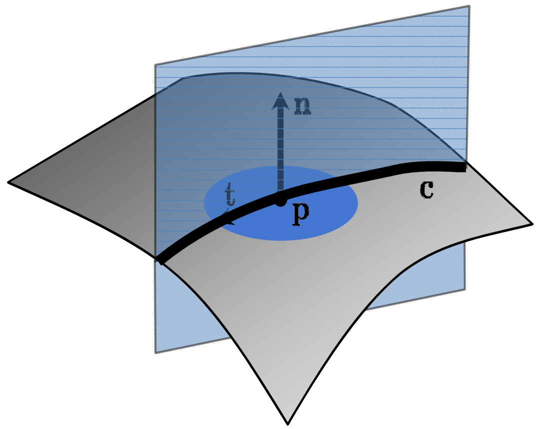


$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

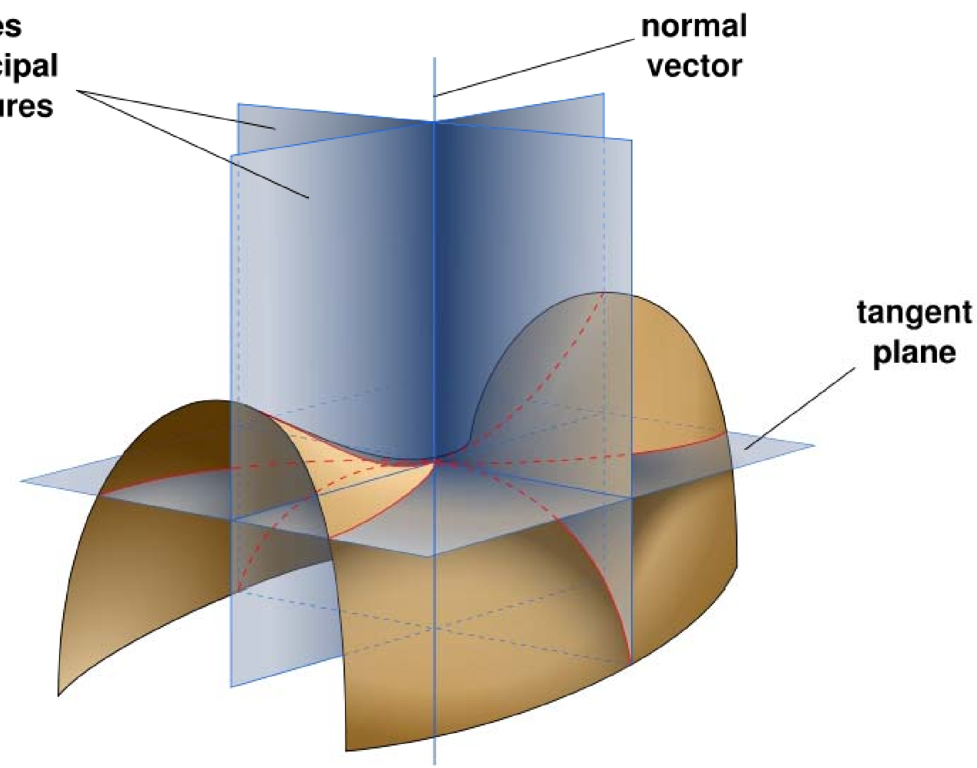
Surface Curvature

- Principal Curvatures
  - maximum curvature $\kappa_1 = \max_\phi \kappa_n(\phi)$
  - minimum curvature $\kappa_2 = \min_\phi \kappa_n(\phi)$



- Mean Curvature $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\phi) d\phi$
- Gaussian Curvature $K = \kappa_1 \cdot \kappa_2$

Principal Curvature

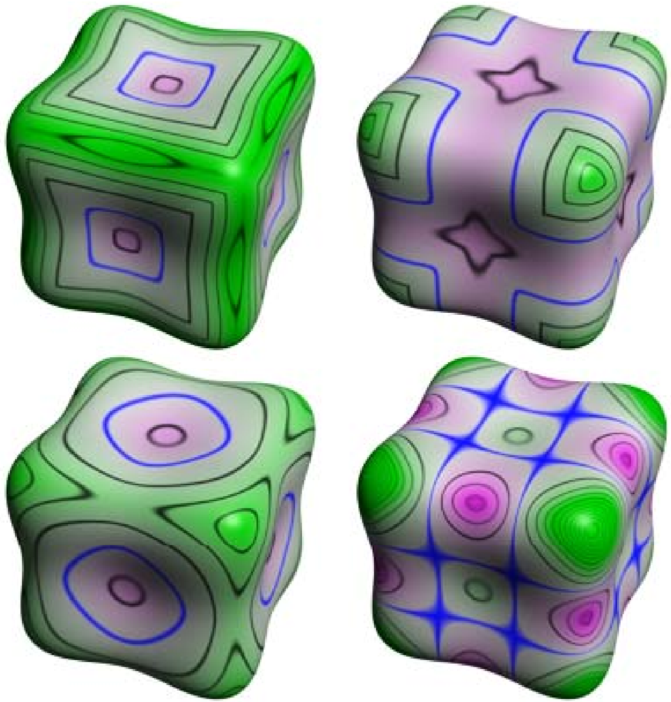


Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \qquad \theta = \text{angle with } \kappa_1$$

Curvature

$$\kappa_1 = \max_{\phi} \kappa_n(\phi)$$



$$\kappa_2 = \min_{\phi} \kappa_n(\phi)$$

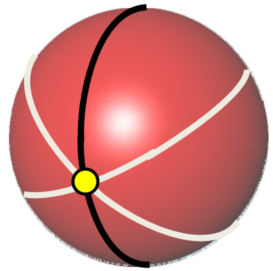
$$H = \frac{1}{2}(\kappa_1 + \kappa_2)$$

$$K = \kappa_1 \cdot \kappa_2$$

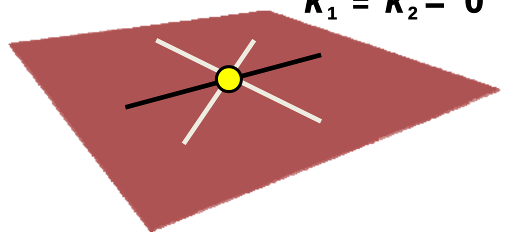
Surface Classification

Isotropic

Equal in all directions



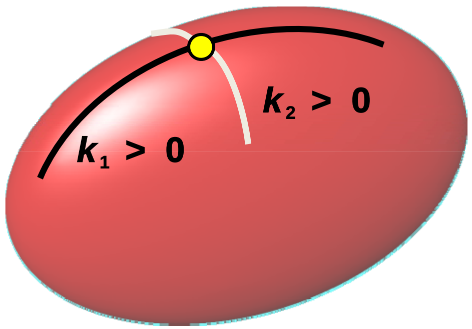
spherical



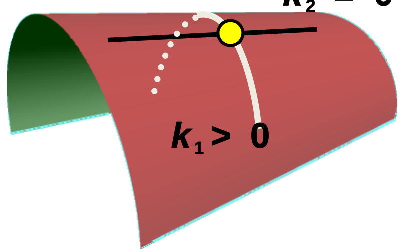
planar

Anisotropic

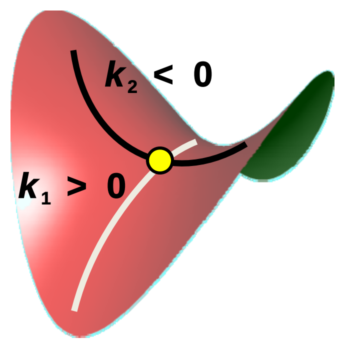
Distinct principal directions



elliptic
$K > 0$

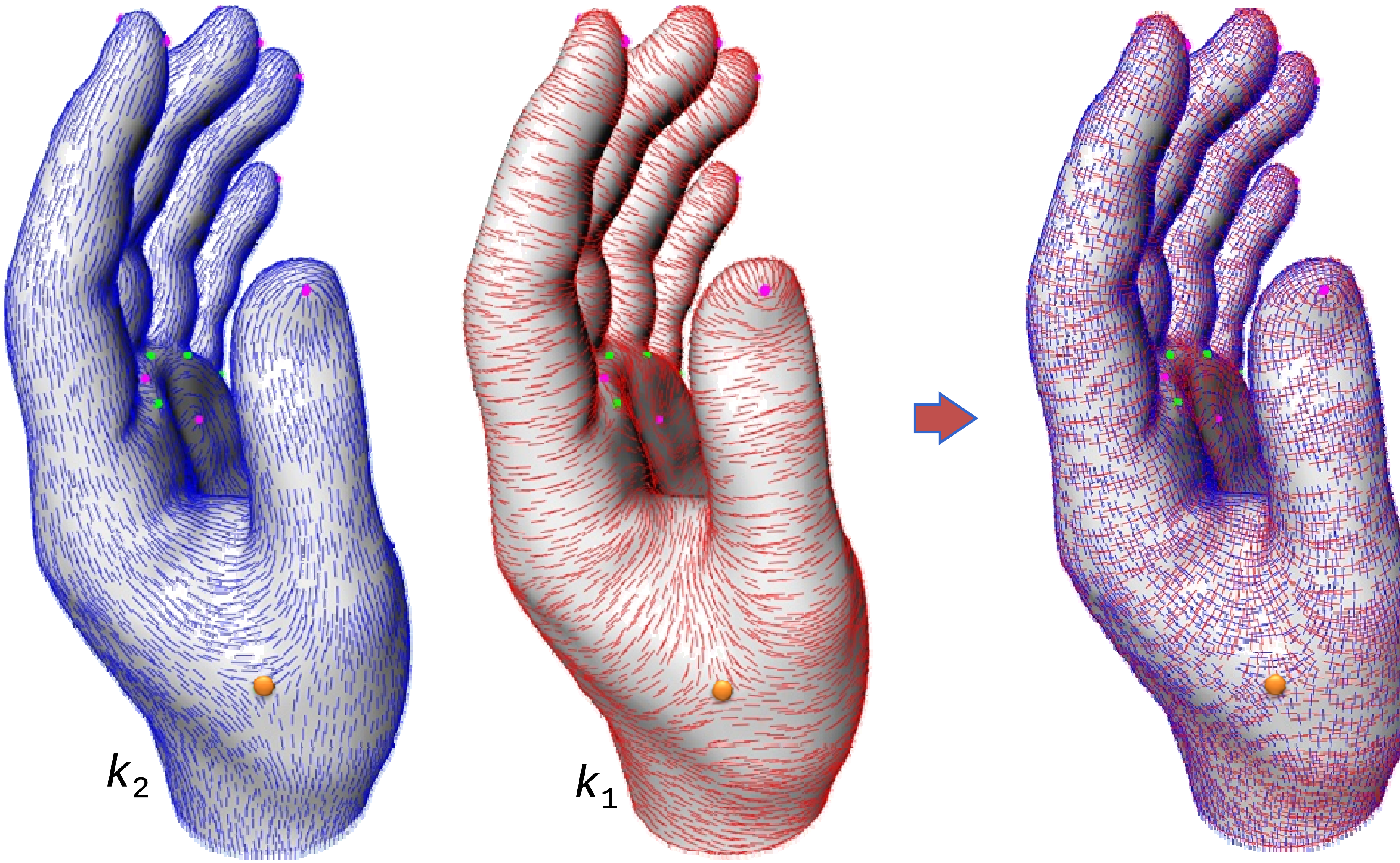


parabolic
$K = 0$
developable



hyperbolic
$K < 0$

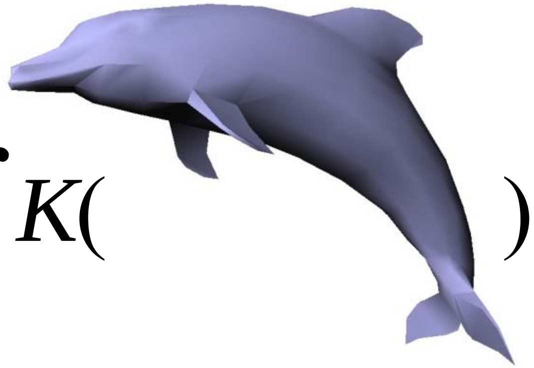
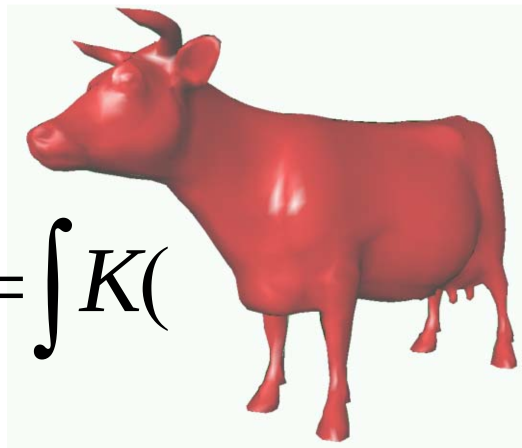
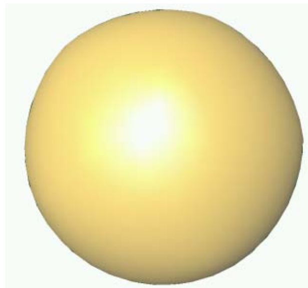
Principal Directions



Gauss-Bonnet Theorem

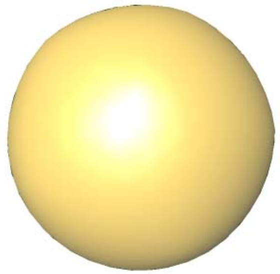
For ANY closed manifold surface with Euler number χ=2-2g:

$$\int K = 2\pi\chi$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

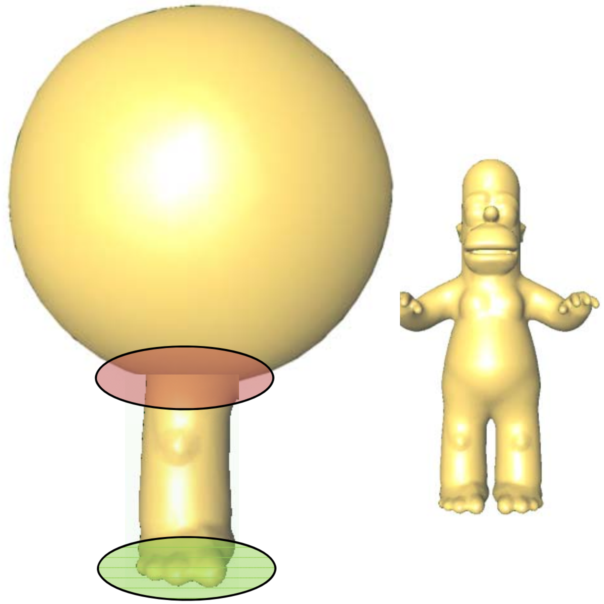
Gauss-Bonnet Theorem
Example

- Sphere
  - $k_1 = k_2 = 1/r$
  - $K = k_1 k_2 = 1/r^2$



$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$

- Manipulate sphere
  - New positive + negative curvature
  - Cancel out!



Fundamental Forms

- First fundamental form

$$\mathbf{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{bmatrix}$$

- Second fundamental form

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$

Fundamental Forms

- $\mathbf{I}$ and $\mathbf{II}$ allow to measure
  - length, angles, area, curvature
  - arc element

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

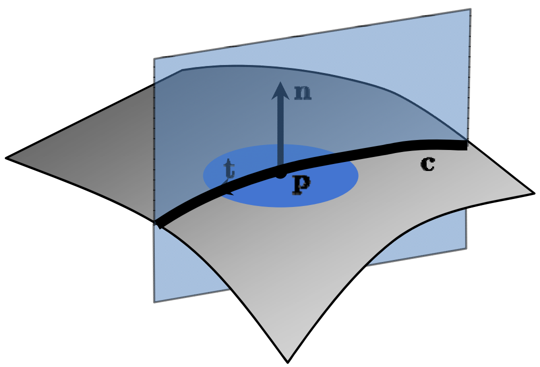
  - area element

$$dA = \sqrt{EG - F^2}dudv$$

Fundamental Forms

- Normal curvature = curvature of the normal curve at point $\mathbf{c} \in \mathbf{x}(u,v)$ $\quad \mathbf{p} \in \mathbf{c}$
- Can be expressed in terms of fundamental forms as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I}\, \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$



$$\mathbf{t} = a\mathbf{x}_u + b\mathbf{x}_v$$

$$\bar{\mathbf{t}} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Intrinsic Geometry

- Properties of the surface that only depend on the first fundamental form
  - length
  - angles
  - Gaussian curvature (Theorema Egregium)

Laplace Operator

$$f : R^3 \to R \qquad \Delta f : R^3 \to R$$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2} \quad \in R$$

scalar function in Euclidean space

divergence operator

Cartesian coordinates

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Laplace-Beltrami Operator

- Extension to functions on manifolds

$$f : S \to R$$

$$\underbrace{\Delta_S}_{\text{Laplace-Beltrami}} \underbrace{f}_{\text{scalar function on manifold } S} = \underbrace{\mathrm{div}_S}_{\text{divergence operator}} \underbrace{\nabla_S}_{\text{gradient operator}} f \quad \in R$$

Laplace-Beltrami Operator

- For coordinate function(s)

$$f(x,y,z)=\mathbf{x}$$

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_S \boxed{\mathbf{x}} = \mathrm{div}_S \nabla_S \boxed{\mathbf{x}} = -2H\mathbf{n} \quad \in R^3$$

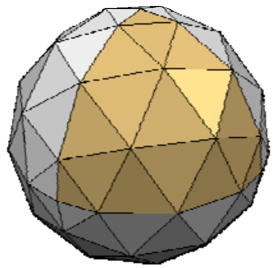
coordinate function

divergence operator

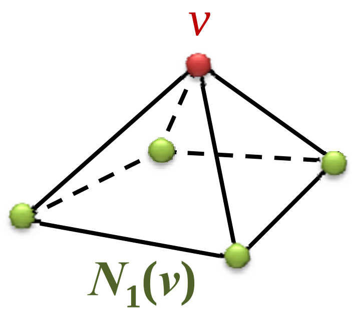
surface normal

Discrete Differential Operators

- **Assumption:** Meshes are piecewise linear approximations of smooth surfaces



- **Approach:** Approximate differential properties at point v as finite differences over local mesh neighborhood $N(v)$
  - v = mesh vertex
  - $N_d(v)$ = d-ring neighborhood

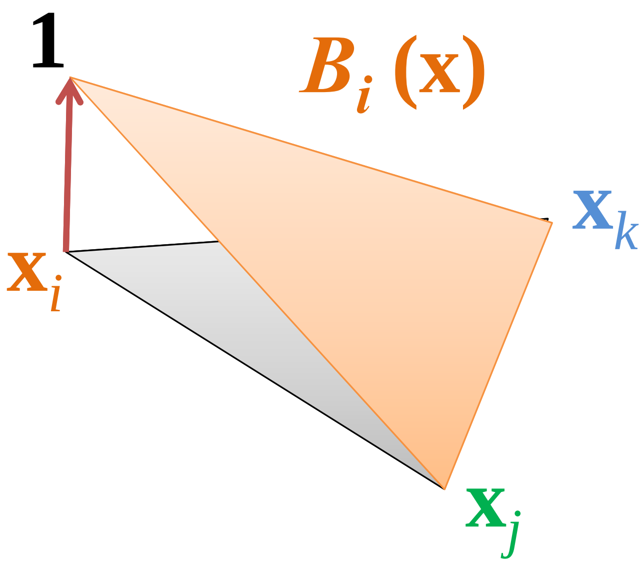


- **Disclaimer:** many possible discretizations, none is "perfect"

Functions on Meshes

- Function f given at mesh vertices $f(v_i) = f(\mathbf{x}_i) = f_i$
- Linear interpolation to triangle $\mathbf{x} \in \left(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k\right)$

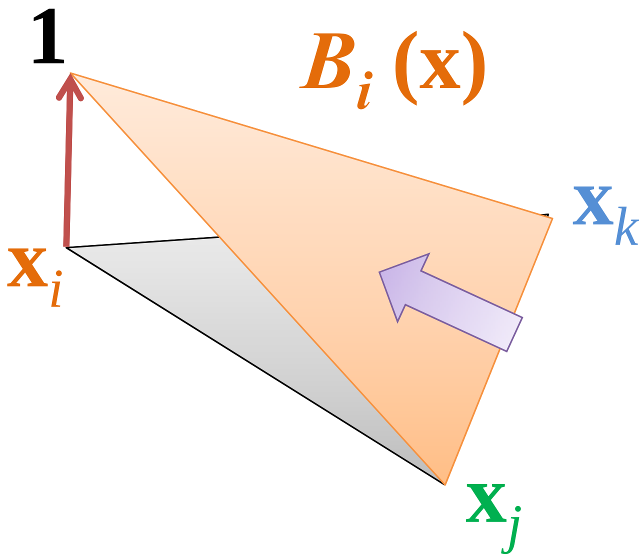
$$f(\mathbf{x}) = f_i B_i(\mathbf{x}) + f_j B_j(\mathbf{x}) + f_k B_k(\mathbf{x})$$



Gradient of a Function

$$f(\mathbf{x}) = f_i B_i(\mathbf{x}) + f_j B_j(\mathbf{x}) + f_k B_k(\mathbf{x})$$

$$\nabla f(\mathbf{x}) = f_i \nabla B_i(\mathbf{x}) + f_j \nabla B_j(\mathbf{x}) + f_k \nabla B_k(\mathbf{x})$$



Steepest ascent direction perpendicular to opposite edge

$$\nabla B_i(\mathbf{x}) = \nabla B_i = \frac{\left(\mathbf{x}_k - \mathbf{x}_j\right)^{\perp}}{2A_T}$$

Constant in the triangle

Gradient of a Function

$$B_i(\mathbf{x}) + B_j(\mathbf{x}) + B_k(\mathbf{x}) = 1$$

$$\nabla B_i + \nabla B_j + \nabla B_k = 0$$

$$\nabla f(\mathbf{x}) = \left(f_j - f_i\right)\nabla B_j(\mathbf{x}) + \left(f_k - f_i\right)\nabla B_k(\mathbf{x})$$

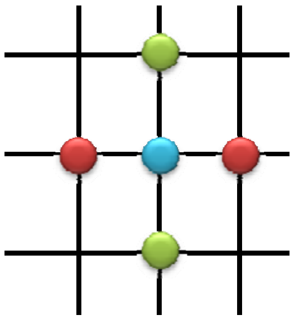
$$\boxed{\nabla f(\mathbf{x}) = \left(f_j - f_i\right)\frac{\left(\mathbf{x}_i - \mathbf{x}_k\right)^{\perp}}{2A_T} + \left(f_k - f_i\right)\frac{\left(\mathbf{x}_j - \mathbf{x}_i\right)^{\perp}}{2A_T}}$$

Discrete Laplace-Beltrami
First Approach

- Laplace operator: $\Delta f = \operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$

- In 2D: $\Delta f = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

- On a grid – finite differences discretization:

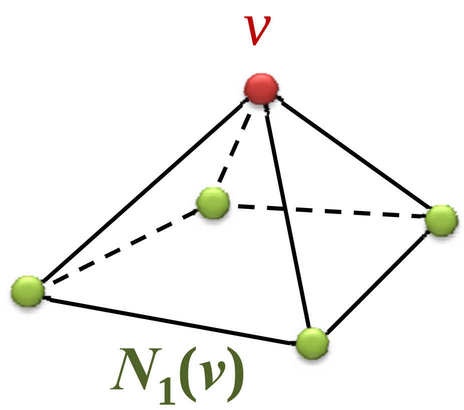


$$\Delta f(x_i, y_i) = \frac{f(x_{i+1}, y_i) - f(x_i, y_i)}{h^2} + \frac{f(x_{i-1}, y_i) - f(x_i, y_i)}{h^2} + \frac{f(x_i, y_{i+1}) - f(x_i, y_i)}{h^2} + \frac{f(x_i, y_{i-1}) - f(x_i, y_i)}{h^2}$$

Discrete Laplace-Beltrami
Uniform Discretization

$$\Delta f = \sum_{v_i \in N_1(v)} (f(v) - f(v_i)) =$$

$$= |N_1(v)| f(v) - \sum_{v_i \in N_1(v)} f(v_i)$$



Normalized: $$\Delta f = \frac{1}{|N_1(v)|} \sum_{v_i \in N_1(v)} (f(v) - f(v_i)) =$$

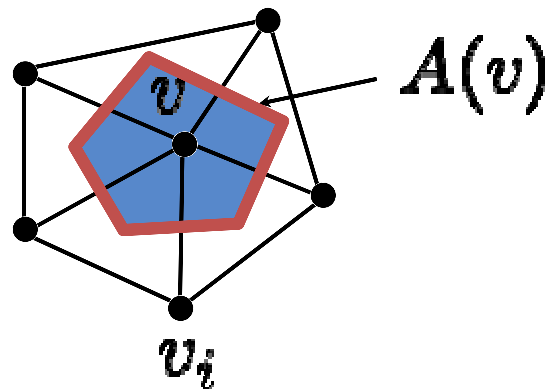
$$= \boxed{f(v) - \frac{1}{|N_1(v)|} \sum_{v_i \in N_1(v)} f(v_i)}$$

Discrete Laplace-Beltrami
Second Approach

- Laplace-Beltrami operator: $\Delta_{\mathcal{S}} f = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f$
- Compute integral around vertex

Divergence theorem

$$\int_{A(v)} \Delta f(\mathbf{u}) dA$$

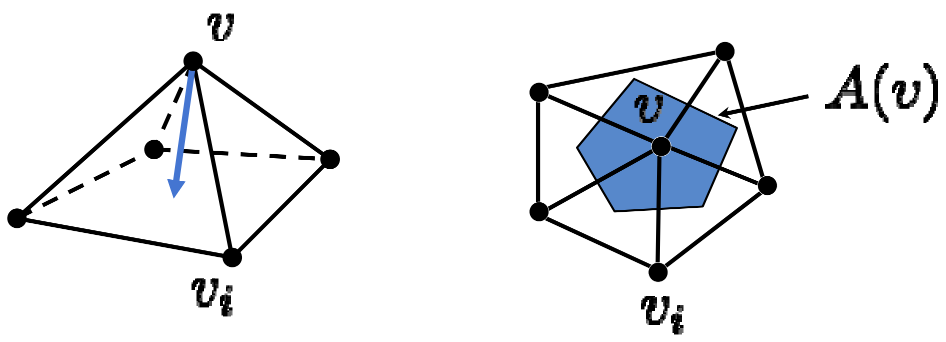


Discrete Laplace-Beltrami
Cotangent Formula

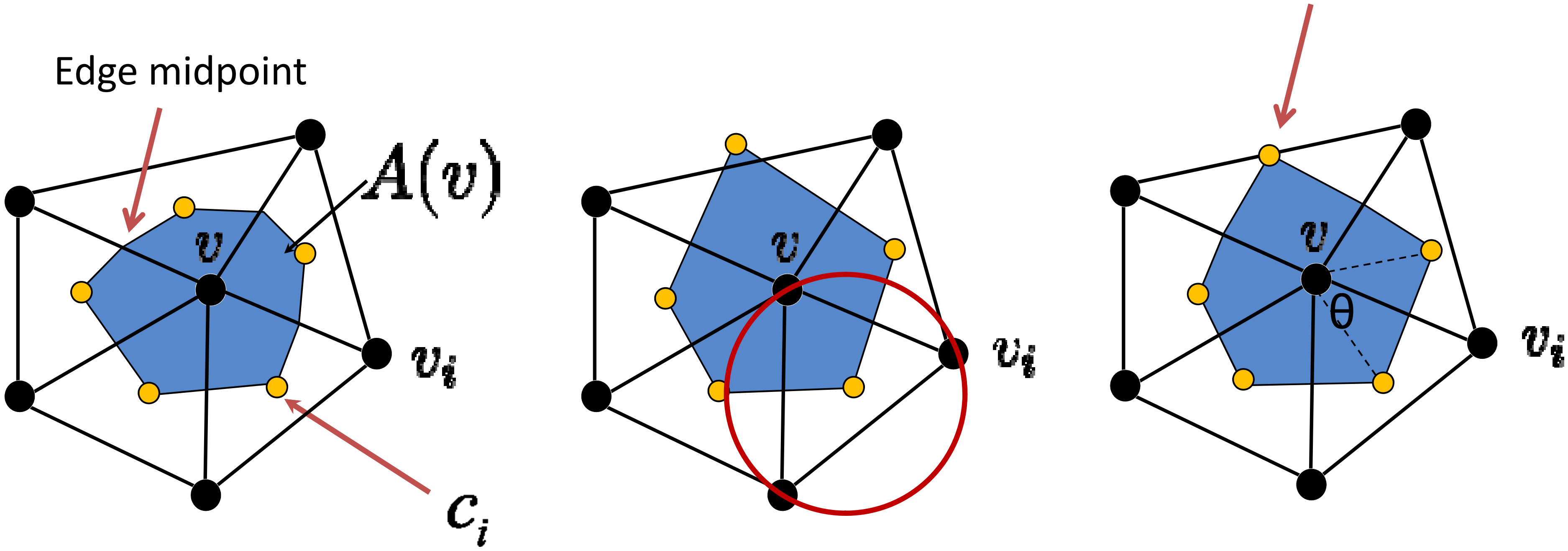
Plugging in expression for gradients gives:

$$\Delta f(v) = \sum_{v_i \in N_1(v)} w_i (f(v_i) - f(v))$$

$$= \boxed{\frac{1}{2A(v)}} \sum_{v_i \in N_1(v)} \boxed{(\cot \alpha_i + \cot \beta_i)} (f(v_i) - f(v))$$



The Averaging Region



Barycentric cell

c_i = barycenter of triangle

Voronoi cell

c_i = circumcenter of triangle

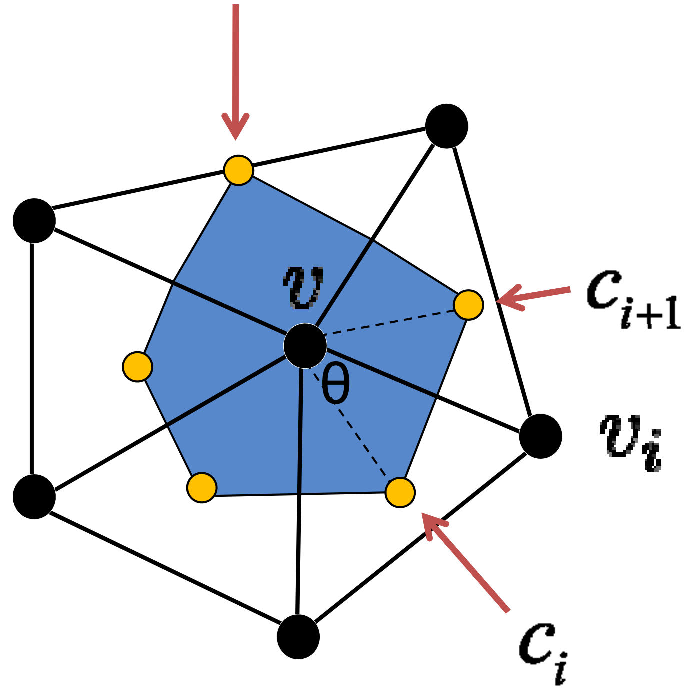
Mixed cell

The Averaging Region

Mixed Cell

If $\theta < \pi/2$, c_i is the circumcenter of the triangle $(v_i\ , v, v_{i+1}\ )$

If $\theta \geq \pi/2$, c_i is the midpoint of the edge $(v_i\ , v_{i+1}\ )$



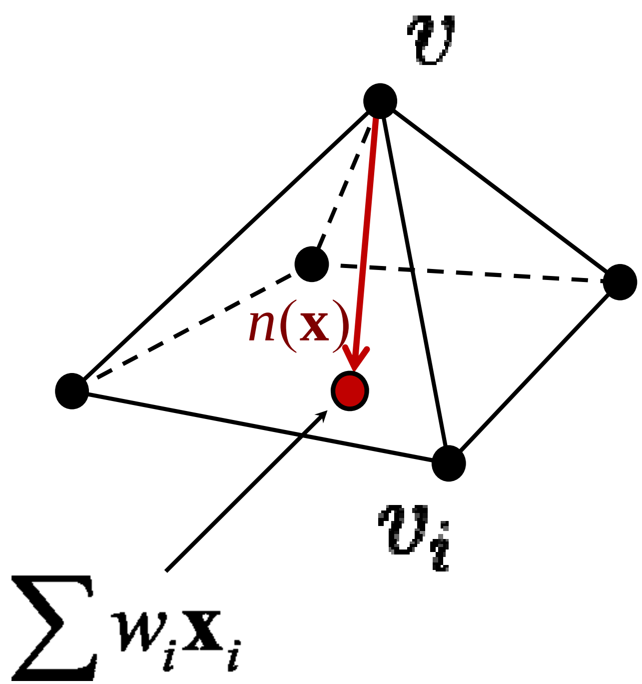
$$A(v) = \sum_{v_i \in \mathcal{N}(v)} \left(Area(c_i, v, (v+v_i)/2) + Area(c_{i+1}, v, (v+v_i)/2) \right)$$

Discrete Normal

$$\Delta_{\mathcal{S}} \mathbf{x} = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

$$n(\mathbf{x}) = \sum_{v_i \in N_1(v)} w_i (\mathbf{x}_i - \mathbf{x}) \qquad \sum_i w_i = 1$$

$$= \left(\sum_{\mathbf{x}_i \in N_1(\mathbf{x})} w_i \mathbf{x}_i \right) - \mathbf{x}$$



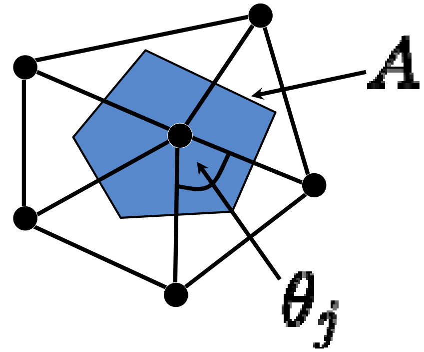
Discrete Curvatures

- Mean curvature

$$H = \|\Delta_S \mathbf{x}\|$$

- Gaussian curvature

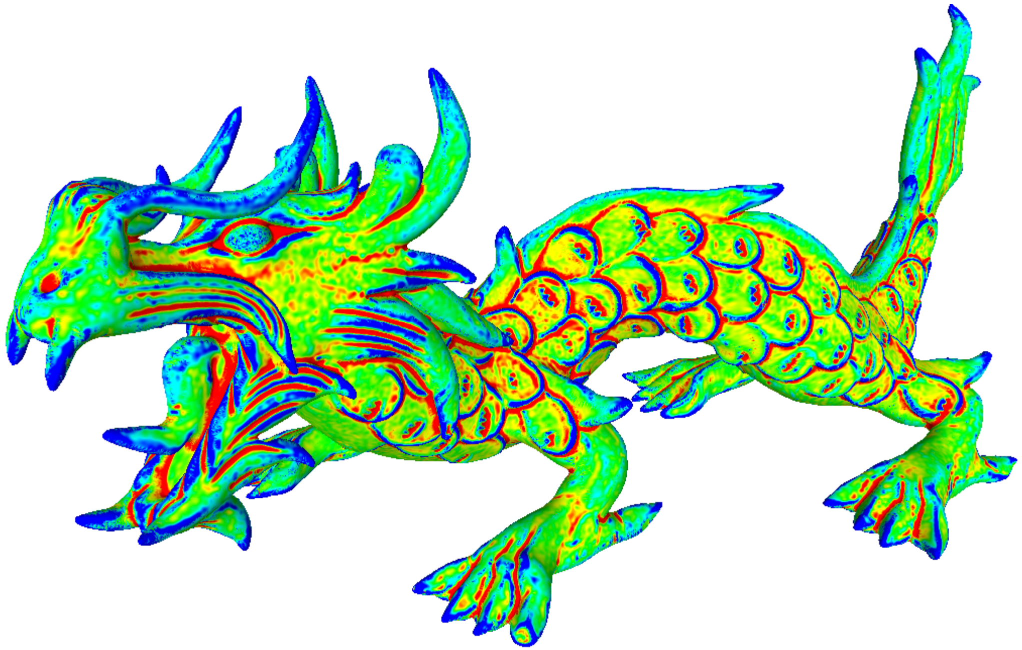
$$G = (2\pi - \sum_j \theta_j)/A$$



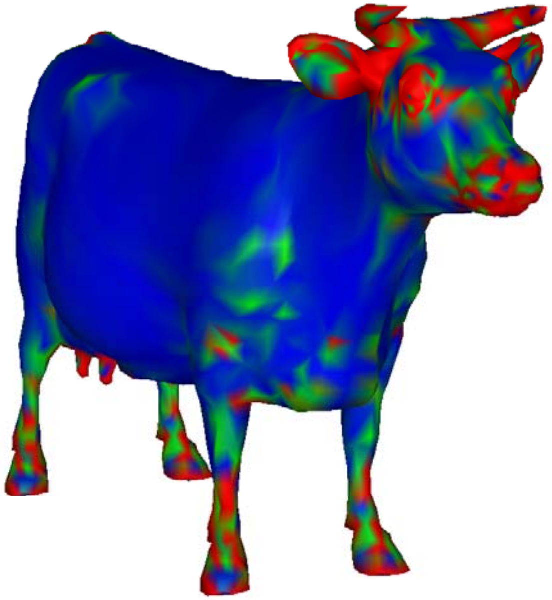
- Principal curvatures

$$\kappa_1 = H + \sqrt{H^2 - G} \qquad \kappa_2 = H - \sqrt{H^2 - G}$$

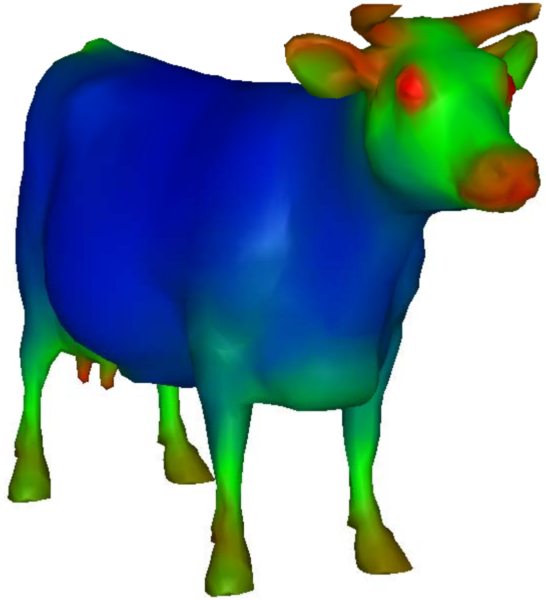
Example: Mean Curvature



Example: Gaussian Curvature



original



smoothed

Discrete Gauss-Bonnet (Descartes) theorem:

$$\sum_v K_v = \sum_v \left[2\pi - \sum_i \theta_i \right] = 2\pi\chi$$

References

- “Discrete Differential-Geometry Operators for Triangulated 2-Manifolds”, Meyer et al., ’02
- “Restricted Delaunay triangulations and normal cycle”, Cohen-Steiner et al., SoCG ‘03
- “On the convergence of metric and geometric properties of polyhedral surfaces”, Hildebrandt et al., ’06
- “Discrete Laplace operators: No free lunch”, Wardetzky et al., SGP ‘07