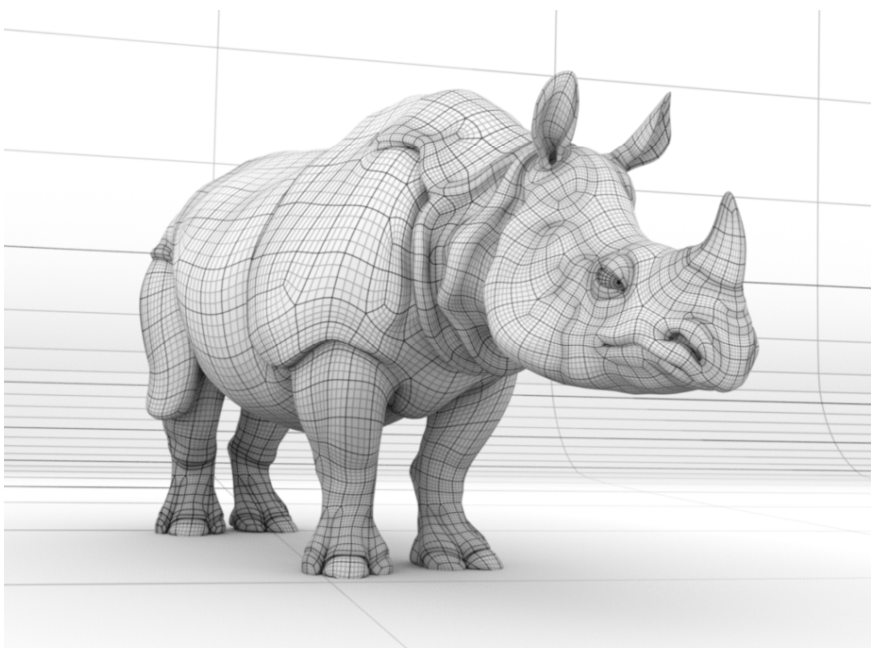
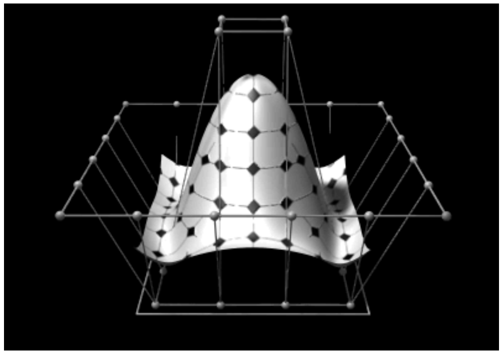
Subdivision Surfaces

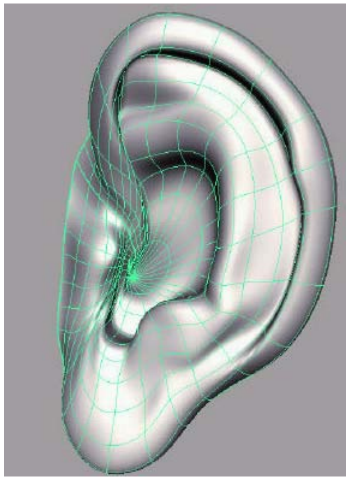


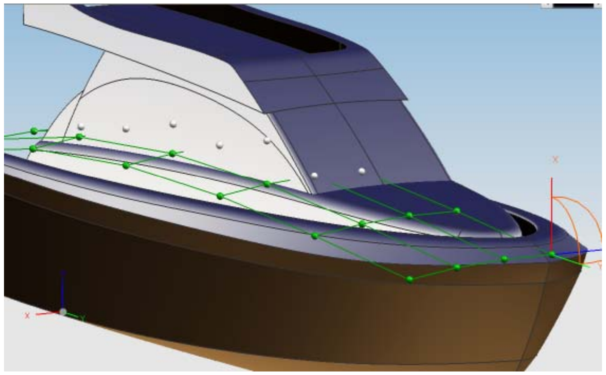
Geometric Modeling

- Sometimes need more than polygon meshes
  - Smooth surfaces



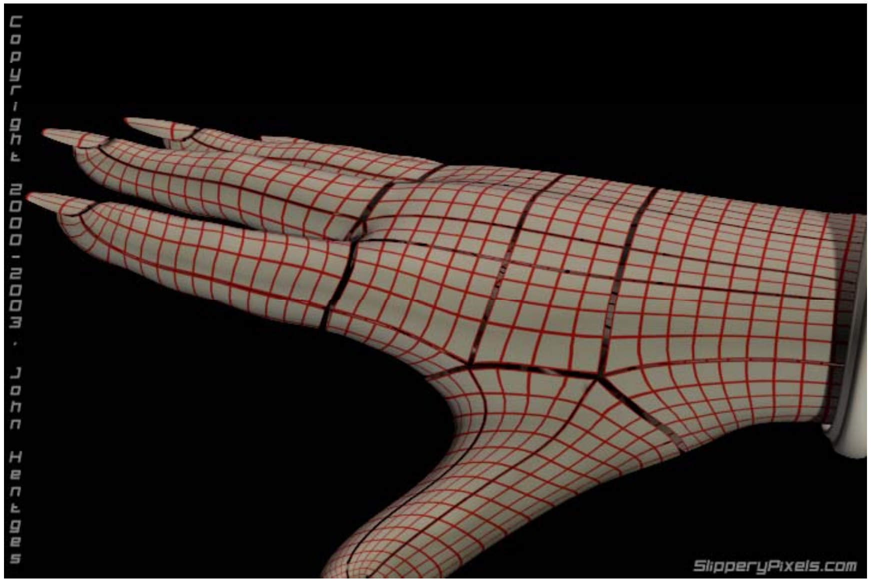
- Traditional geometric modeling used NURBS
  - Non uniform rational B-Spline
  - Demo





Problems with NURBS

- A single NURBS patch is either a topological disk, a tube or a torus

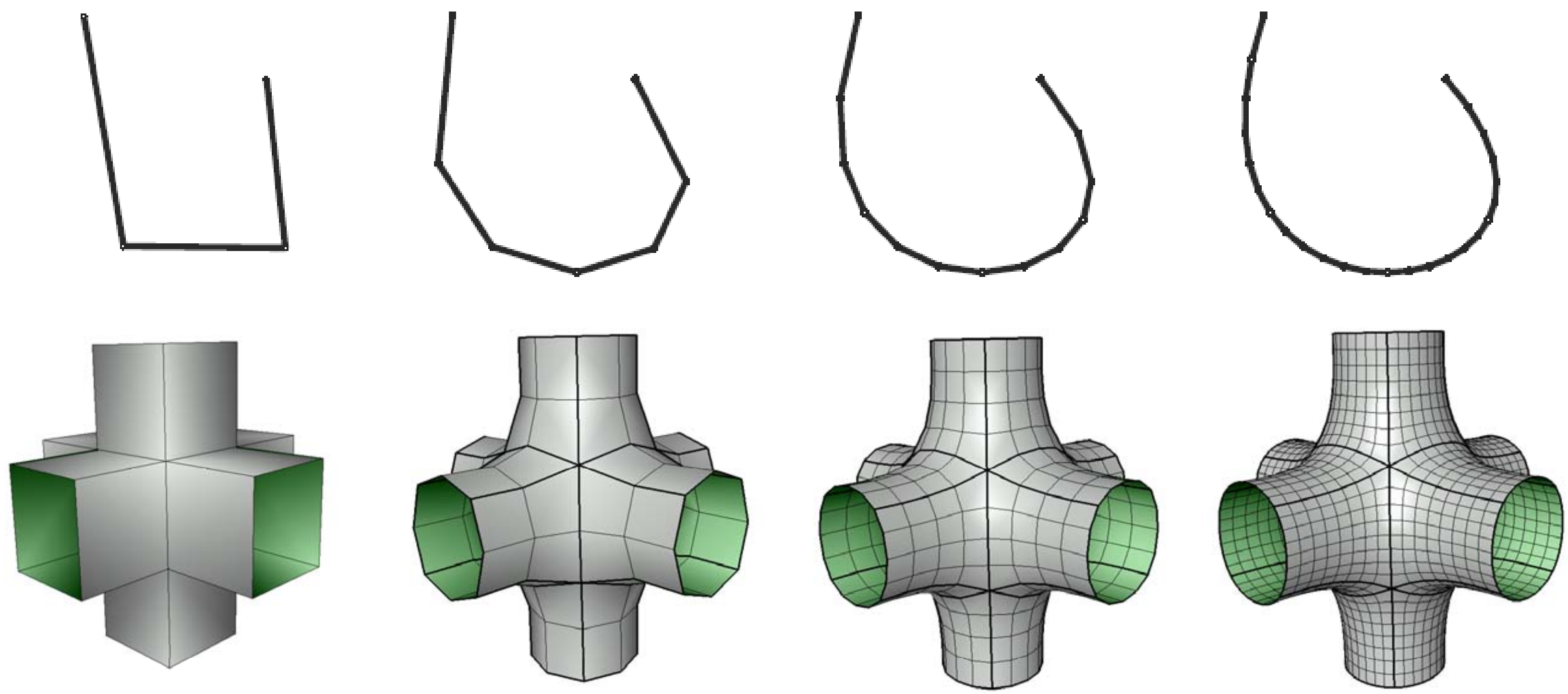


- Must use many NURBS patches to model complex geometry

- When deforming a surface made of NURBS patches, cracks arise at the seams

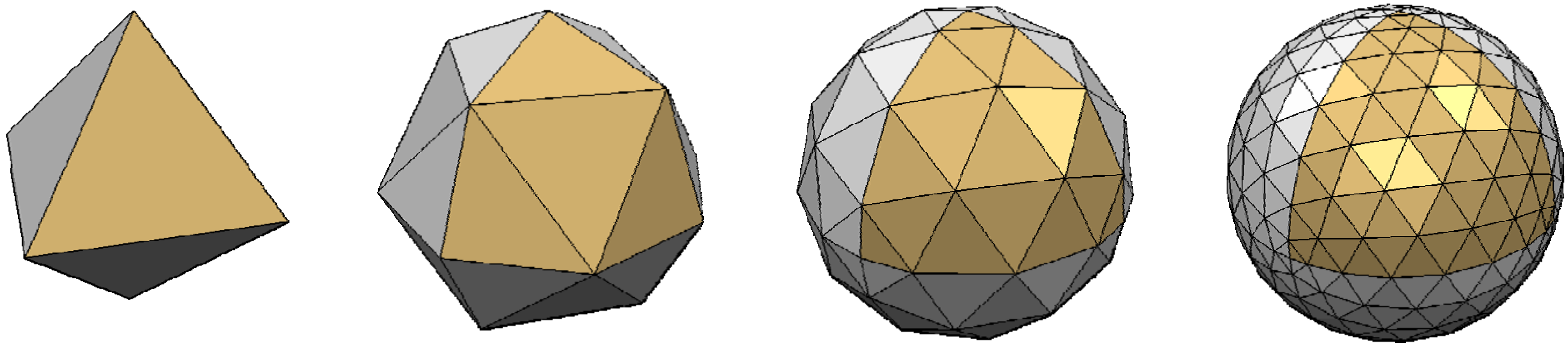
Subdivision

"Subdivision defines a smooth curve or surface as the limit of a sequence of successive refinements"



Subdivision Surfaces

- Generalization of spline curves / surfaces
  - Arbitrary control meshes
  - Successive refinement (subdivision)
  - Converges to smooth limit surface
  - Connection between splines and meshes



Subdivision Surfaces

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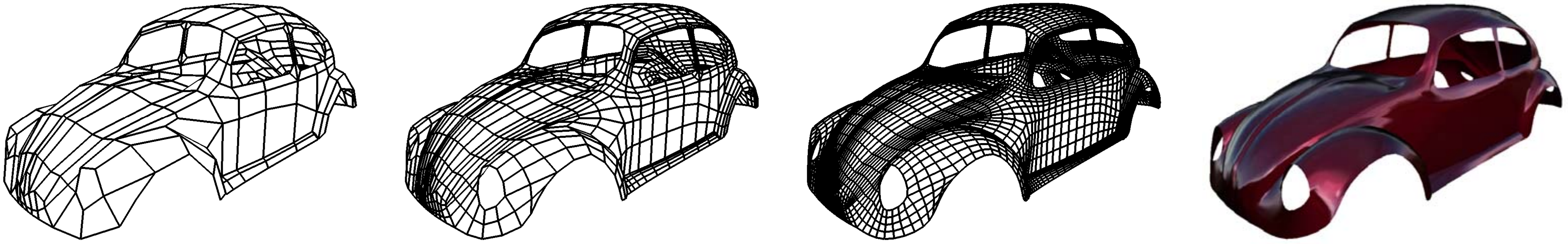
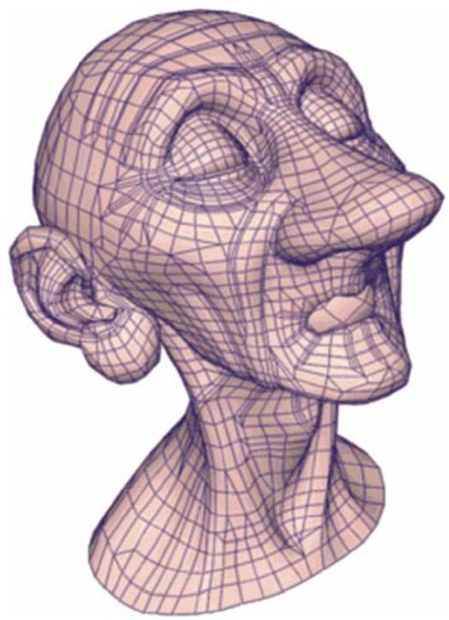


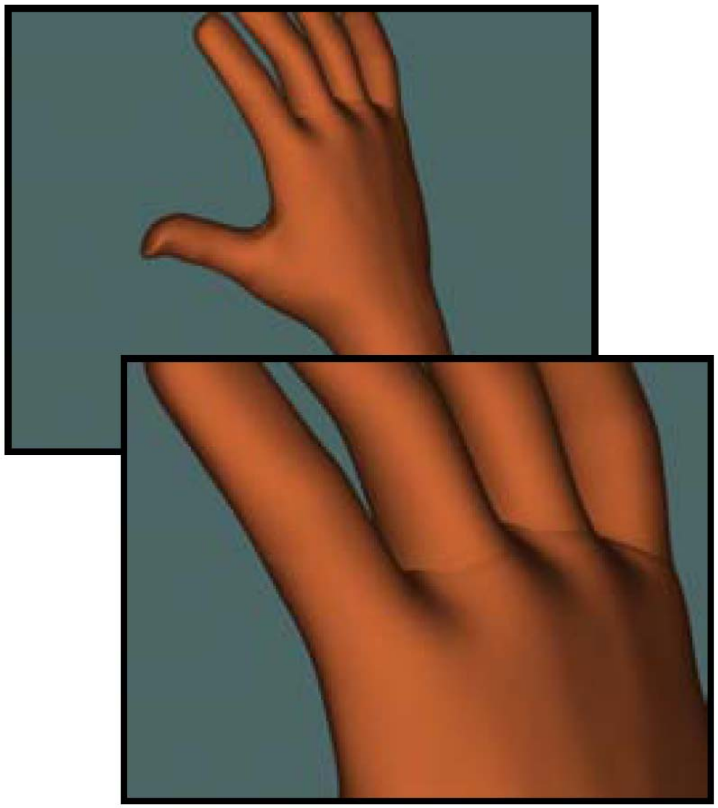
Example: Geri’s Game (Pixar)

- Subdivision used for
  - Geri’s hands and head
  - Clothing
  - Tie and shoes

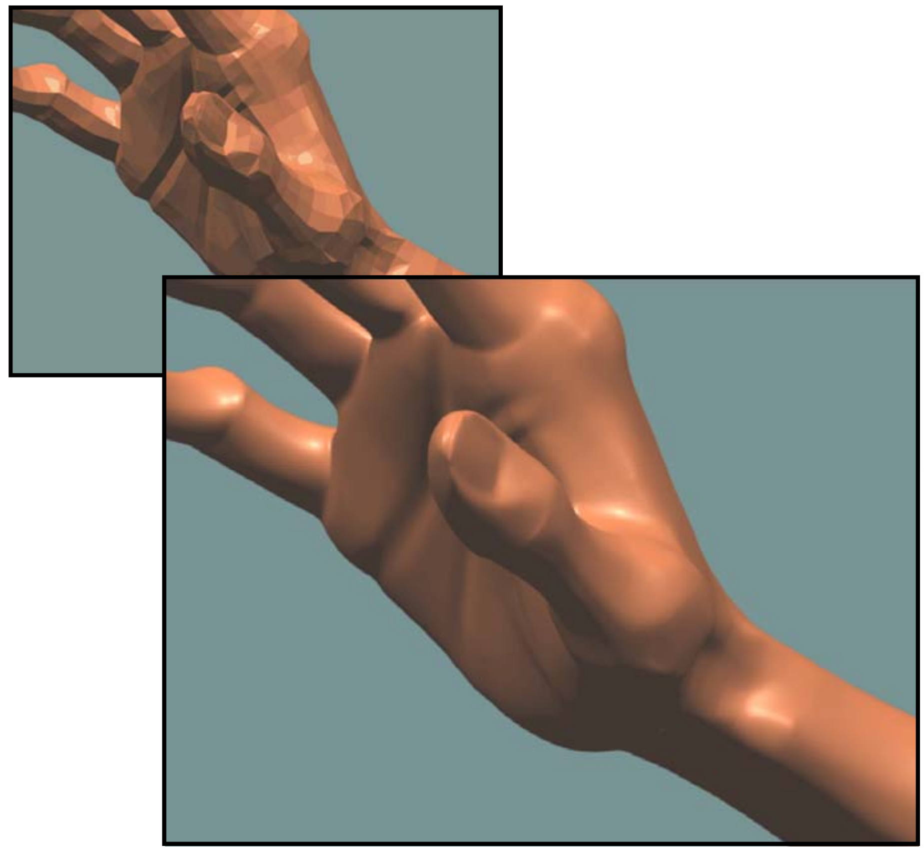


Example: Geri's Game (Pixar)

Woody's hand (NURBS)



Geri's hand (subdivision)

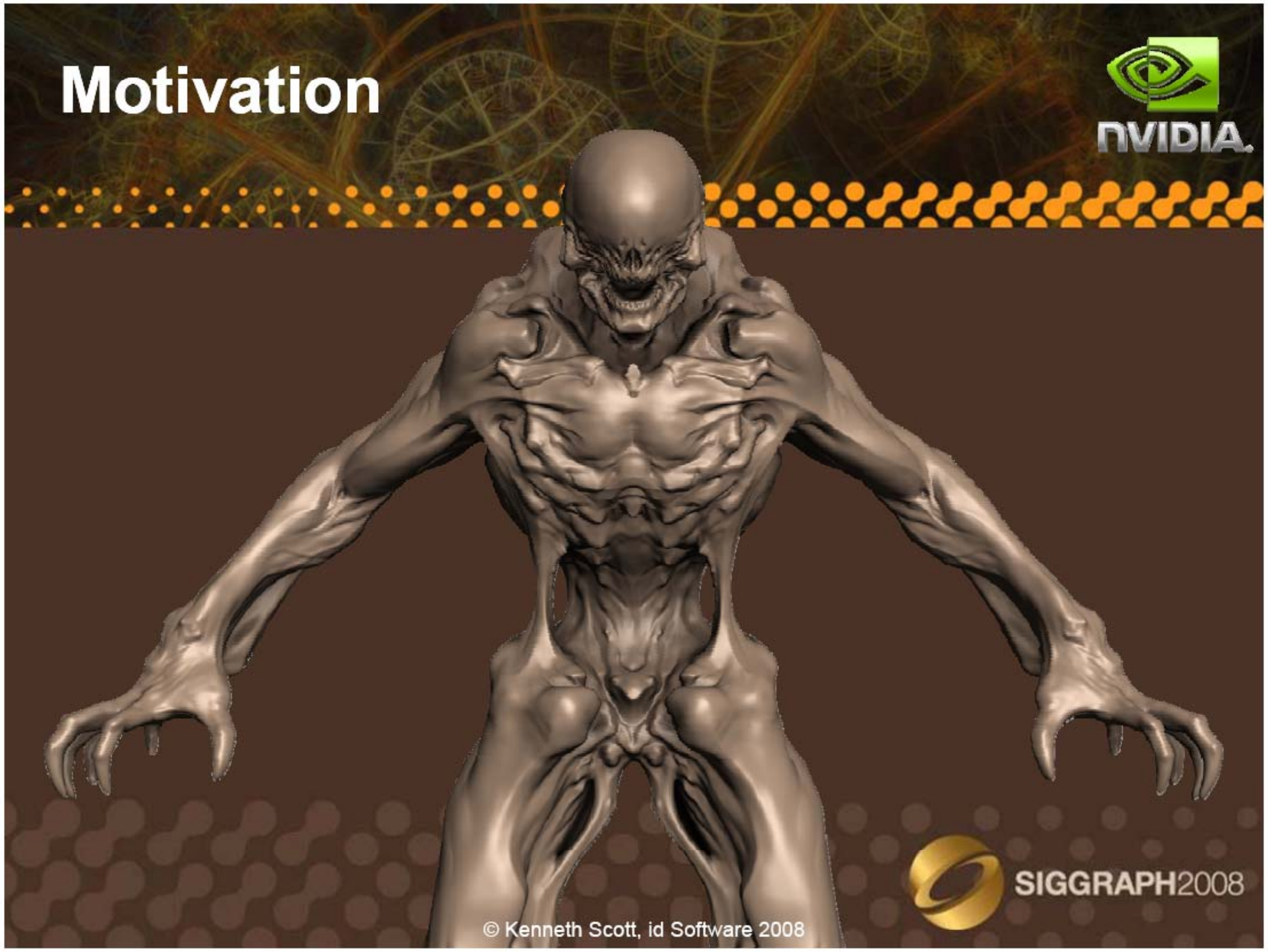


Example: Geri's Game (Pixar)

- Sharp and semi-sharp features

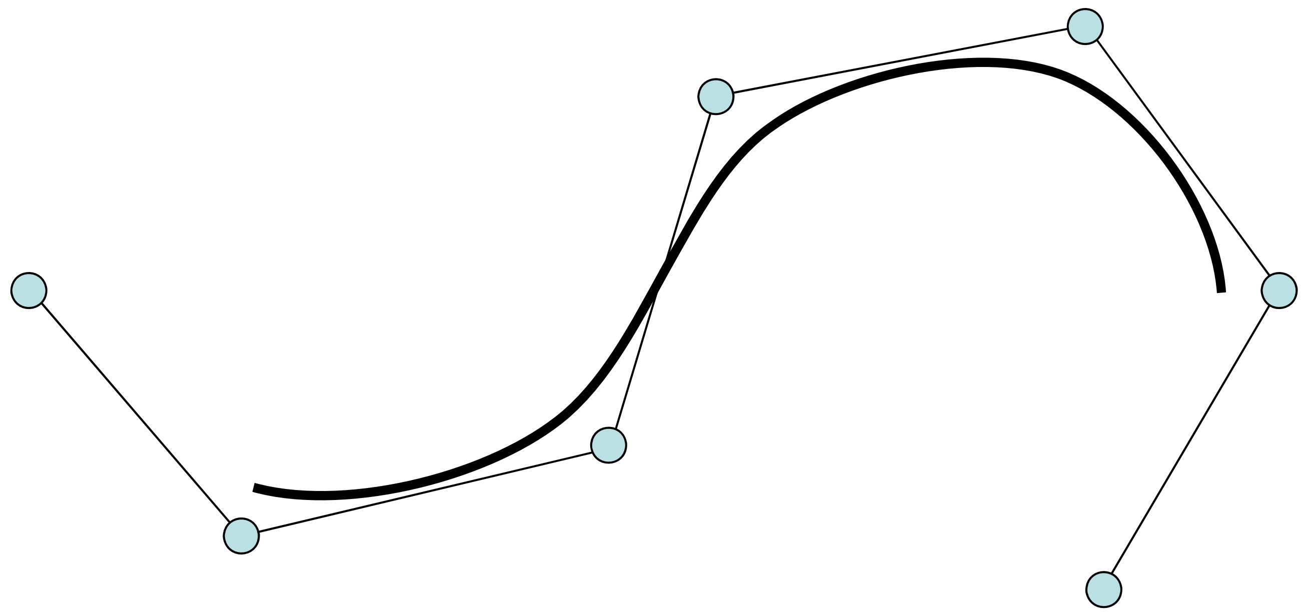
Example: Games

Supported in hardware in DirectX 11



Subdivision Curves

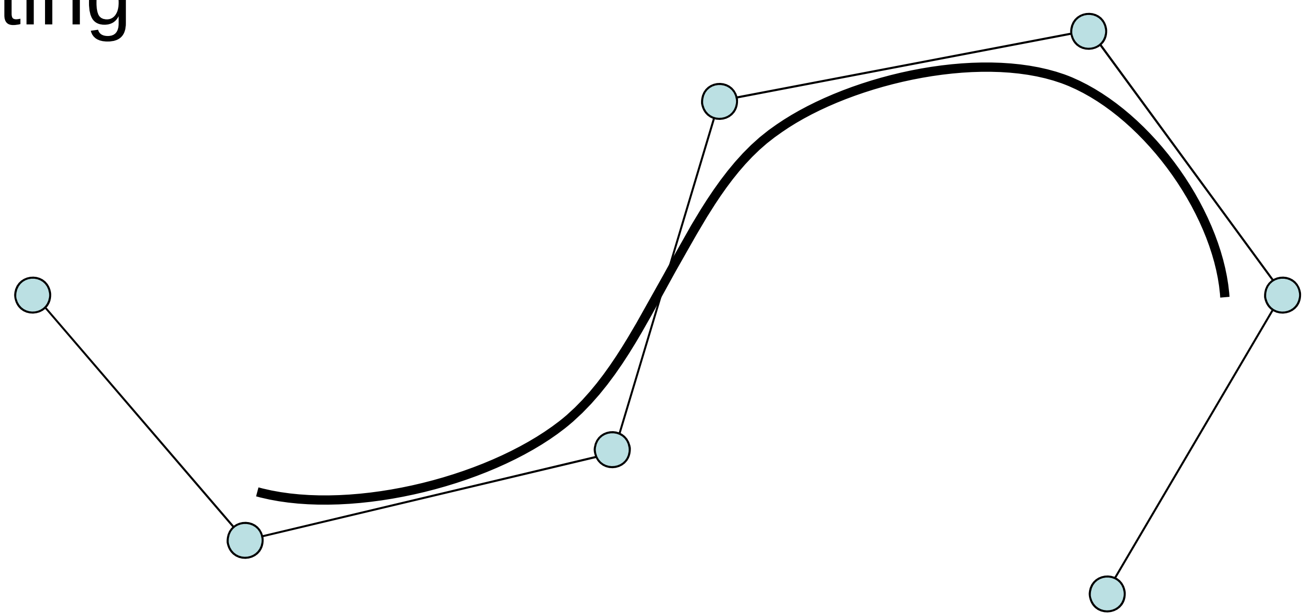
Given a control polygon...



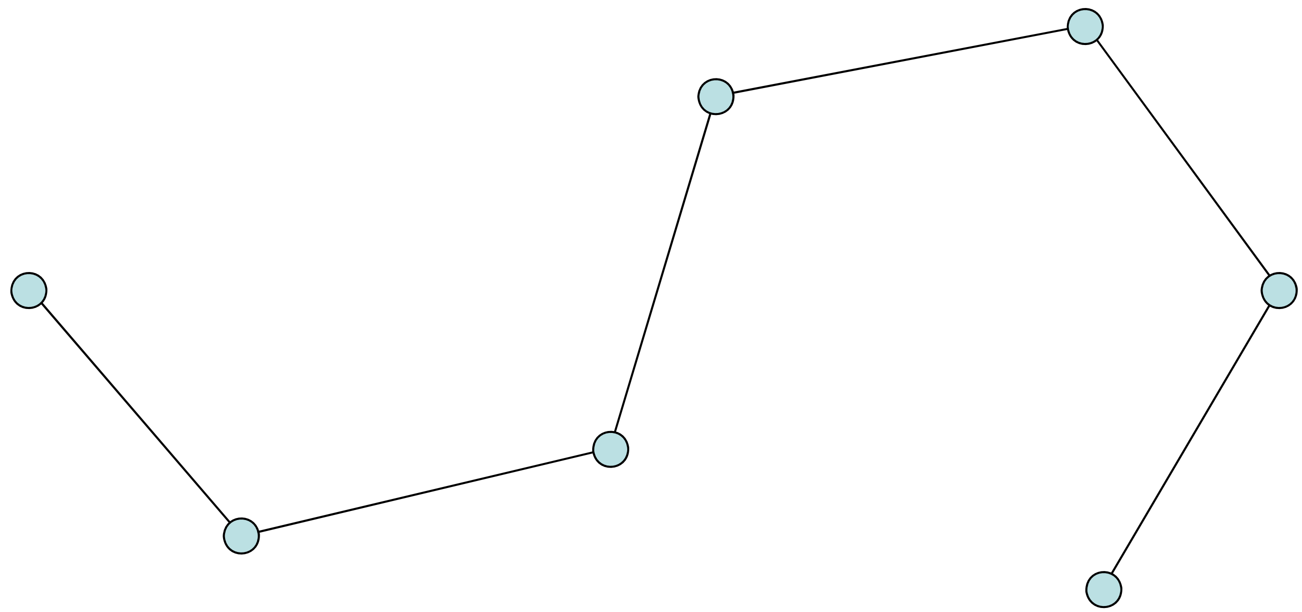
...find a smooth curve related to that polygon.

Subdivision Curve Types

- Approximating
- Interpolating
- Corner Cutting

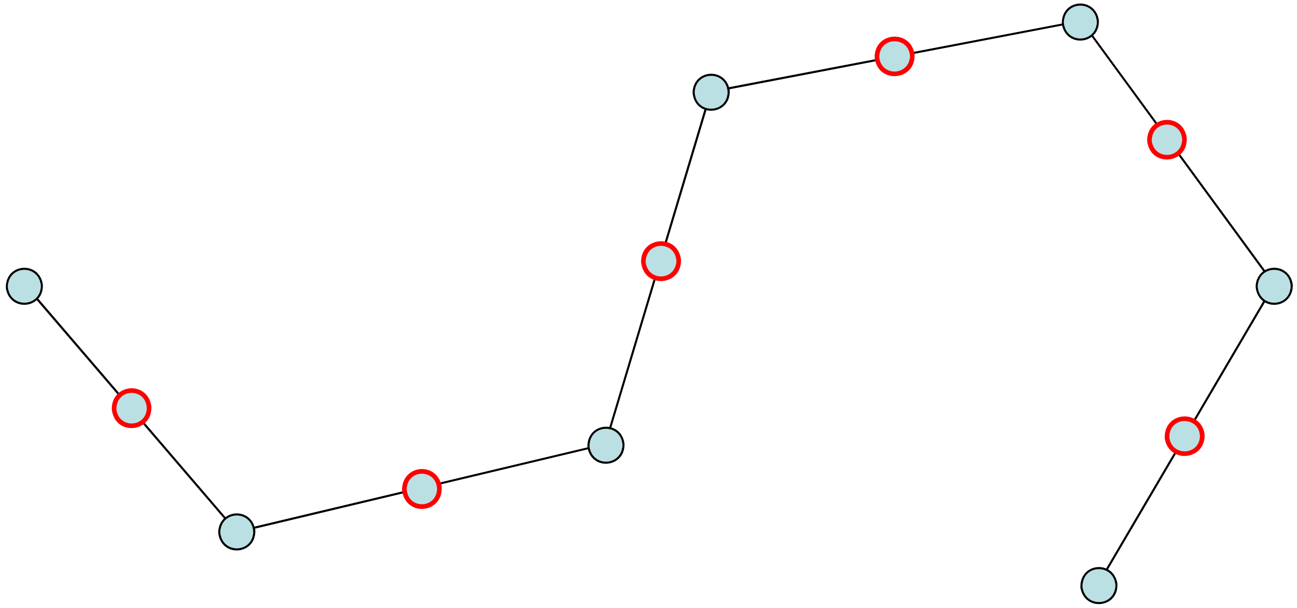


Approximating



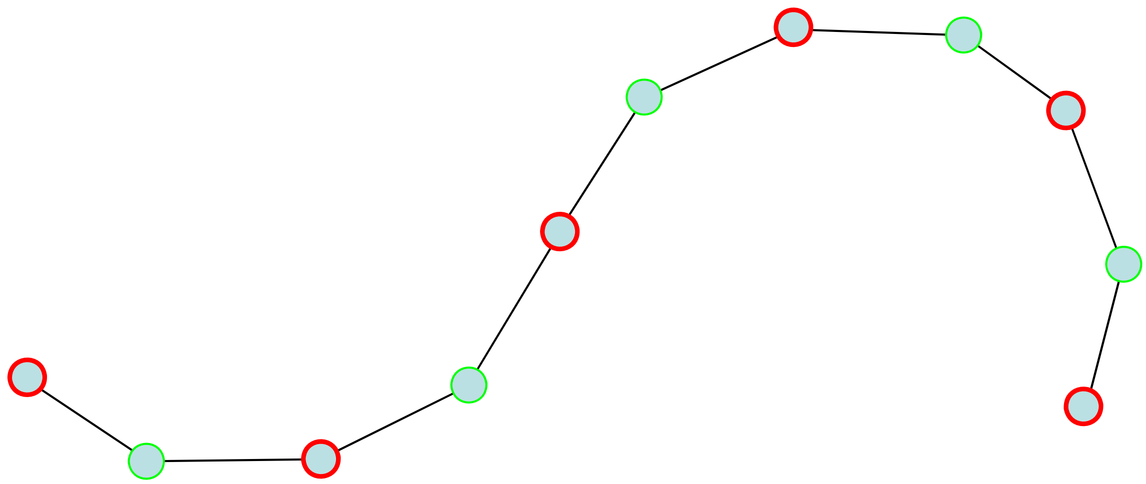
Approximating

Splitting step: split each edge in two



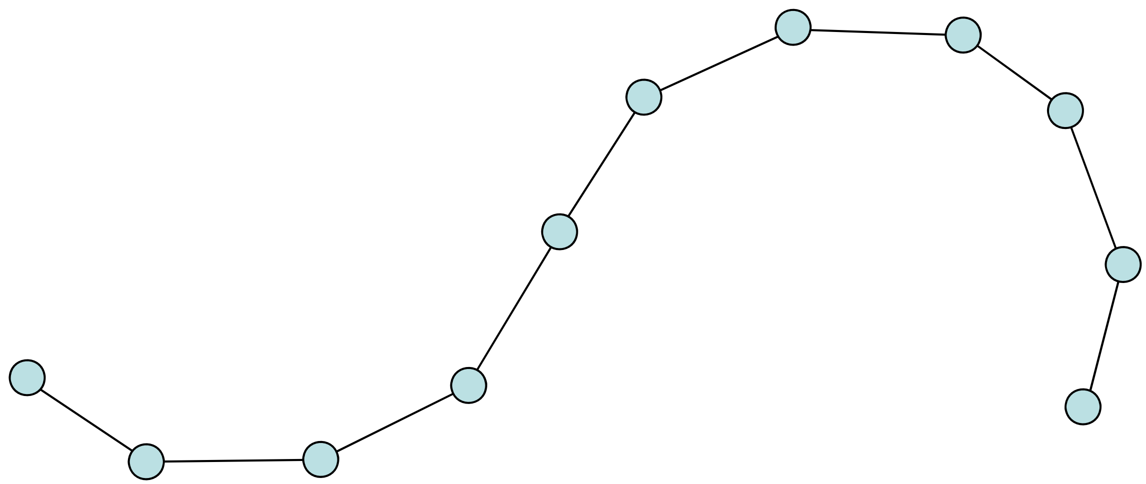
Approximating

Averaging step: relocate each (original) vertex according to some (simple) rule...



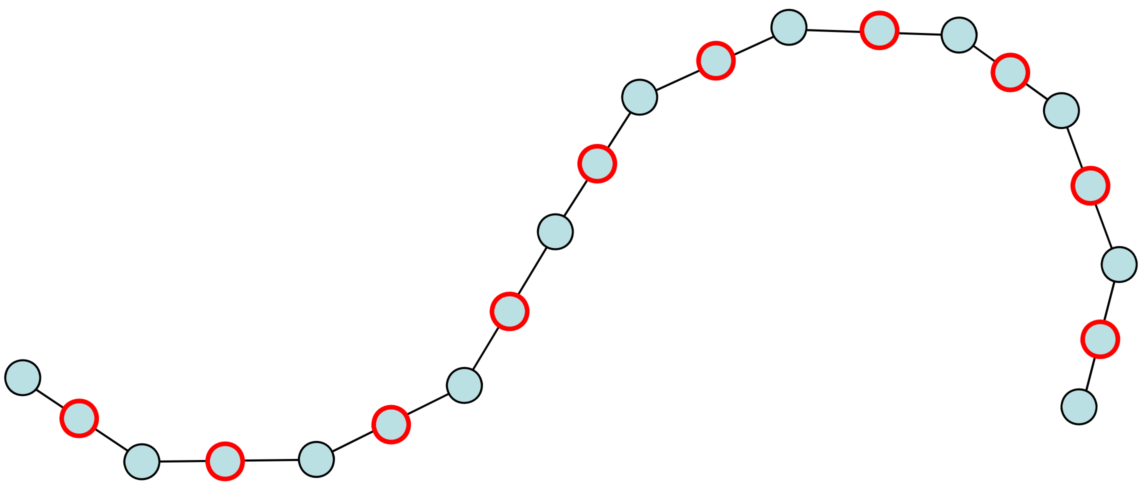
Approximating

Start over ...



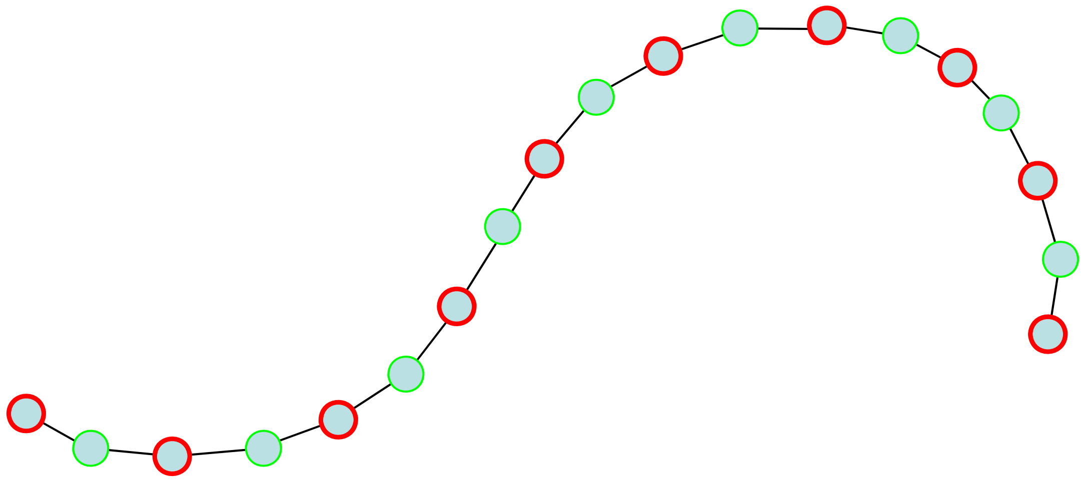
Approximating

...splitting...



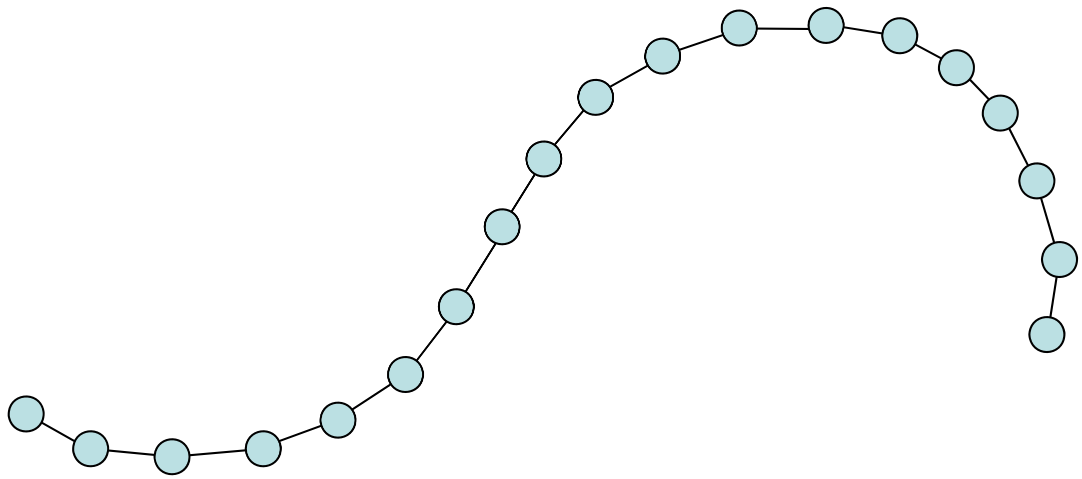
Approximating

...averaging...



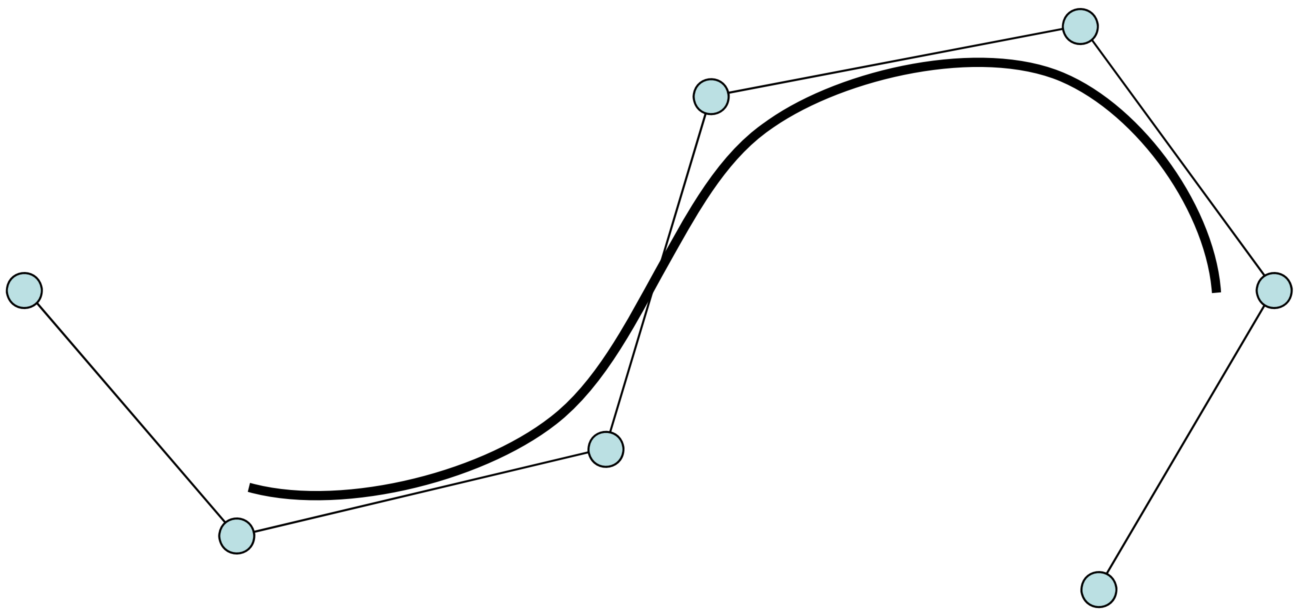
Approximating

...and so on...



Approximating

If the rule is designed carefully...



... the control polygons will converge to a smooth limit curve!

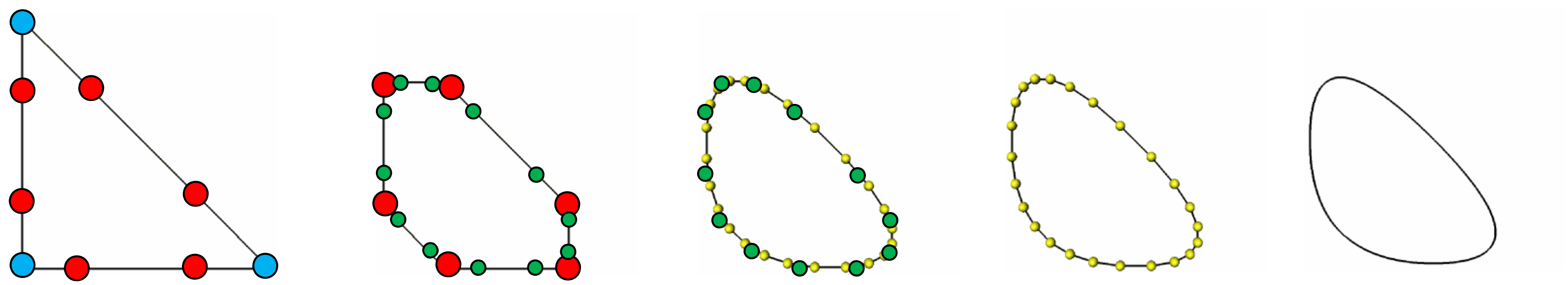
Equivalent to …

- Insert *single* new point at mid-edge
- *Filter* entire set of points.

Catmull-Clark rule: Filter with (1/8, 6/8, 1/8)

Corner Cutting

- Subdivision rule:
  - Insert *two* new vertices at ¼ and ¾ of each edge
  - *Remove* the old vertices
  - Connect the new vertices



B-Spline Curves

- Piecewise polynomial of degree n

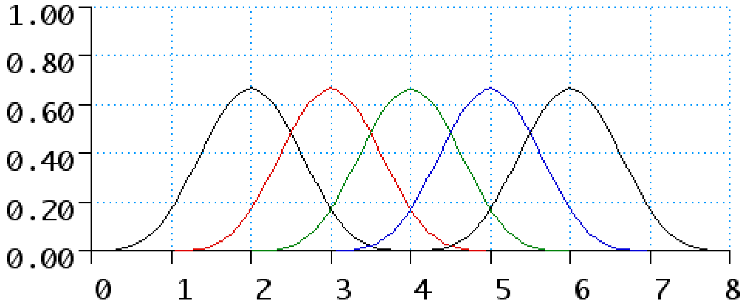
B-spline curve

control points

$$\mathbf{s}(u) = \sum_{i=0}^{k} \mathbf{d}_i N_i^n(u)$$

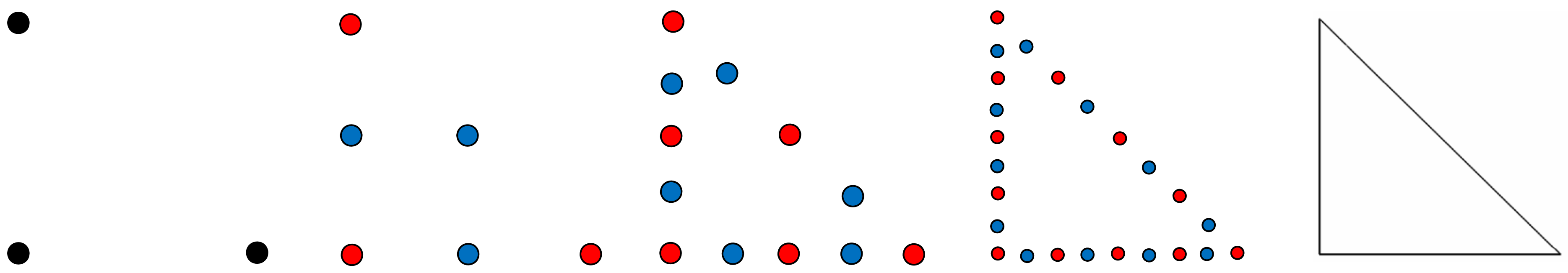
parameter value

basis functions



B-Spline Curves

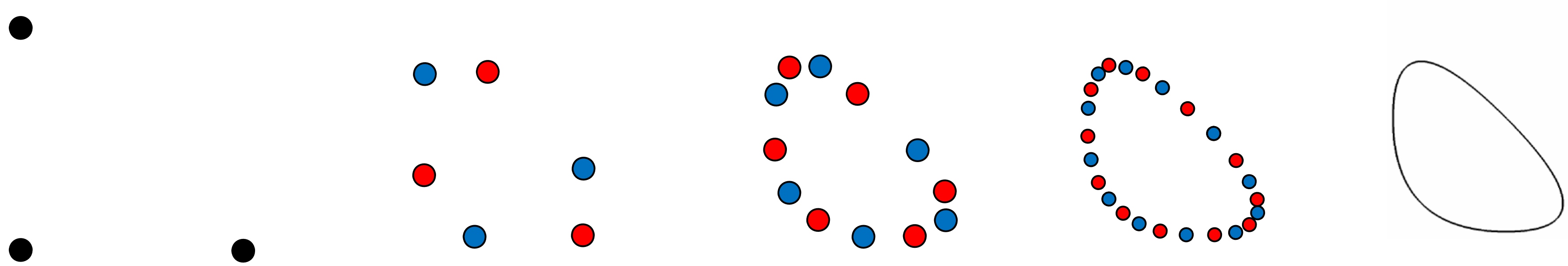
- Distinguish between odd and even points
- **Linear B-spline**
  - Odd coefficients (1/2, 1/2)
  - Even coefficient (1)



B-Spline Curves

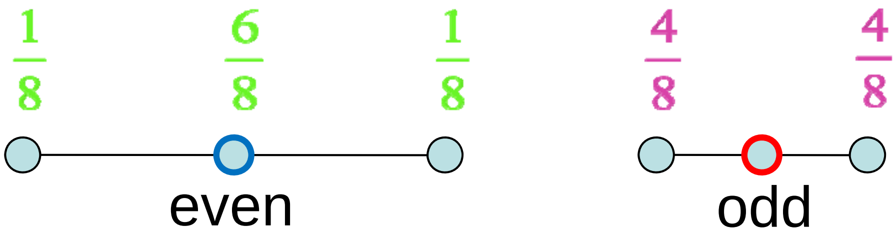
- **Quadratic B-Spline (Chaikin)**
  - Odd coefficients (¼, ¾)
  - Even coefficients (¾ , ¼)

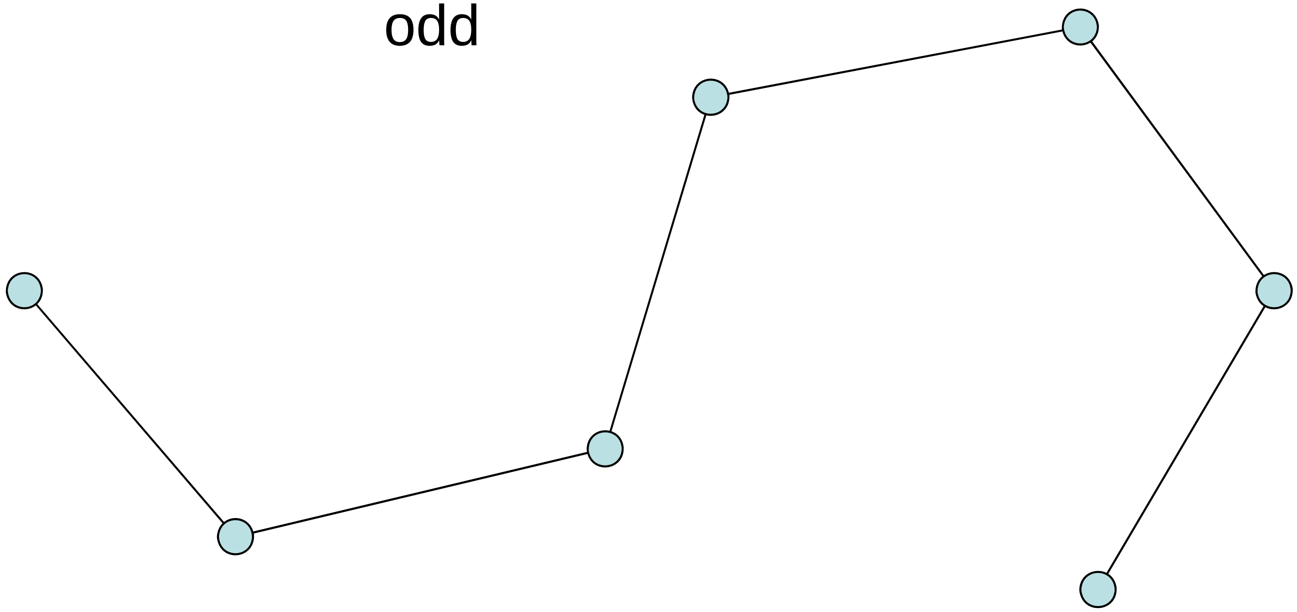
demo



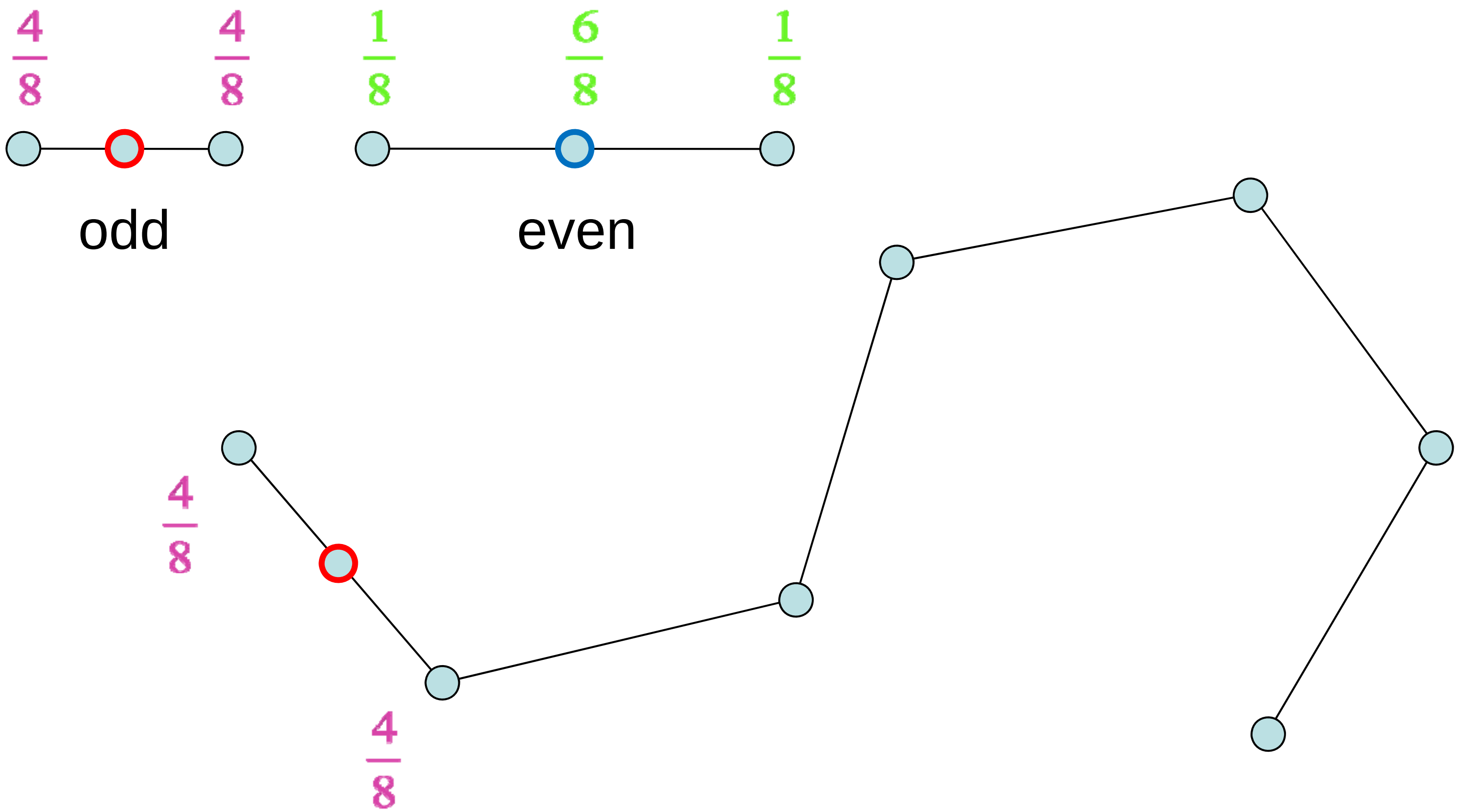
- **Cubic B-Spline (Catmull-Clark)**
  - Odd coefficients (4/8, 4/8)
  - Even coefficients (1/8, 6/8, 1/8)

Cubic B-Spline





Cubic B-Spline



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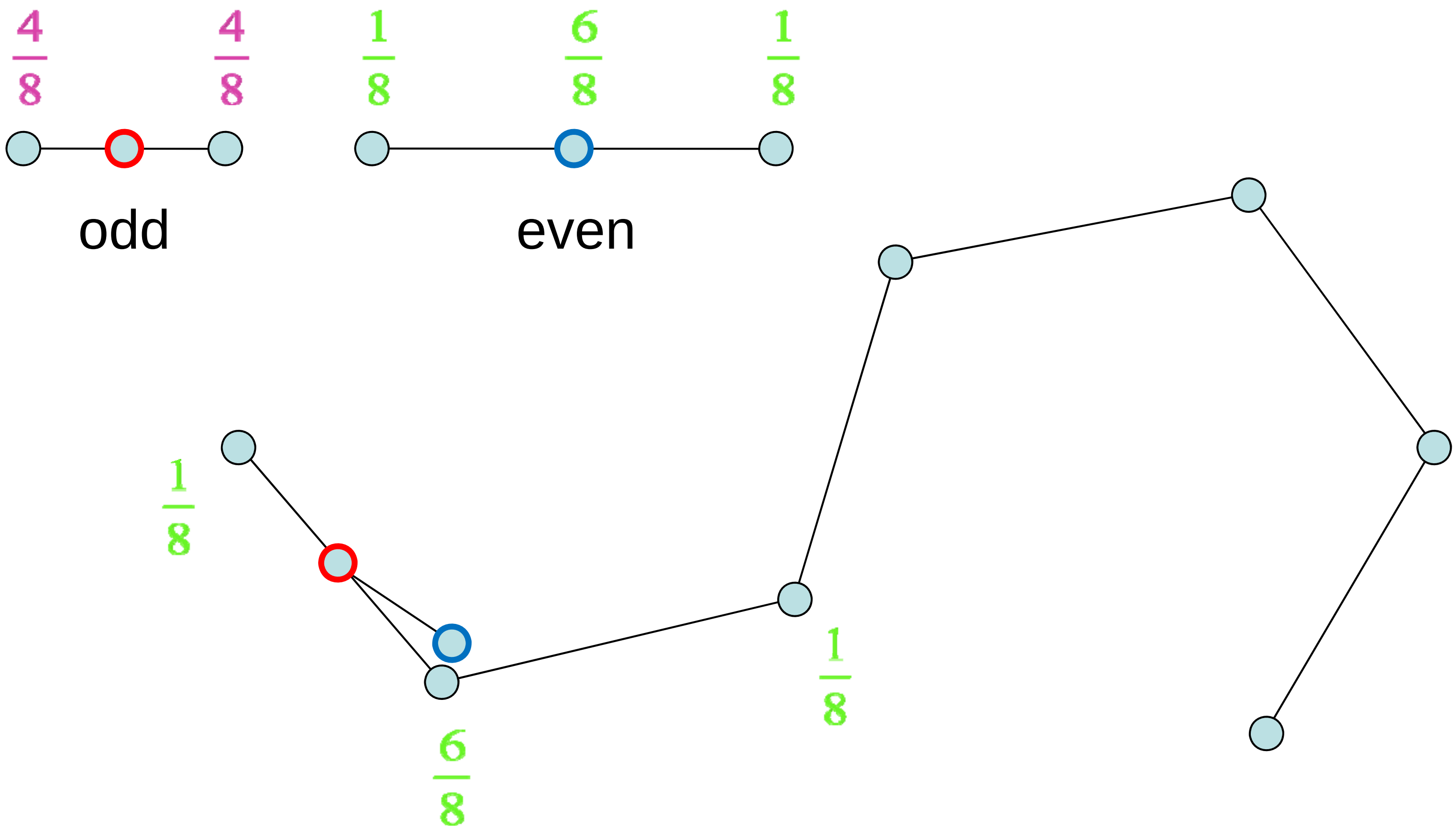

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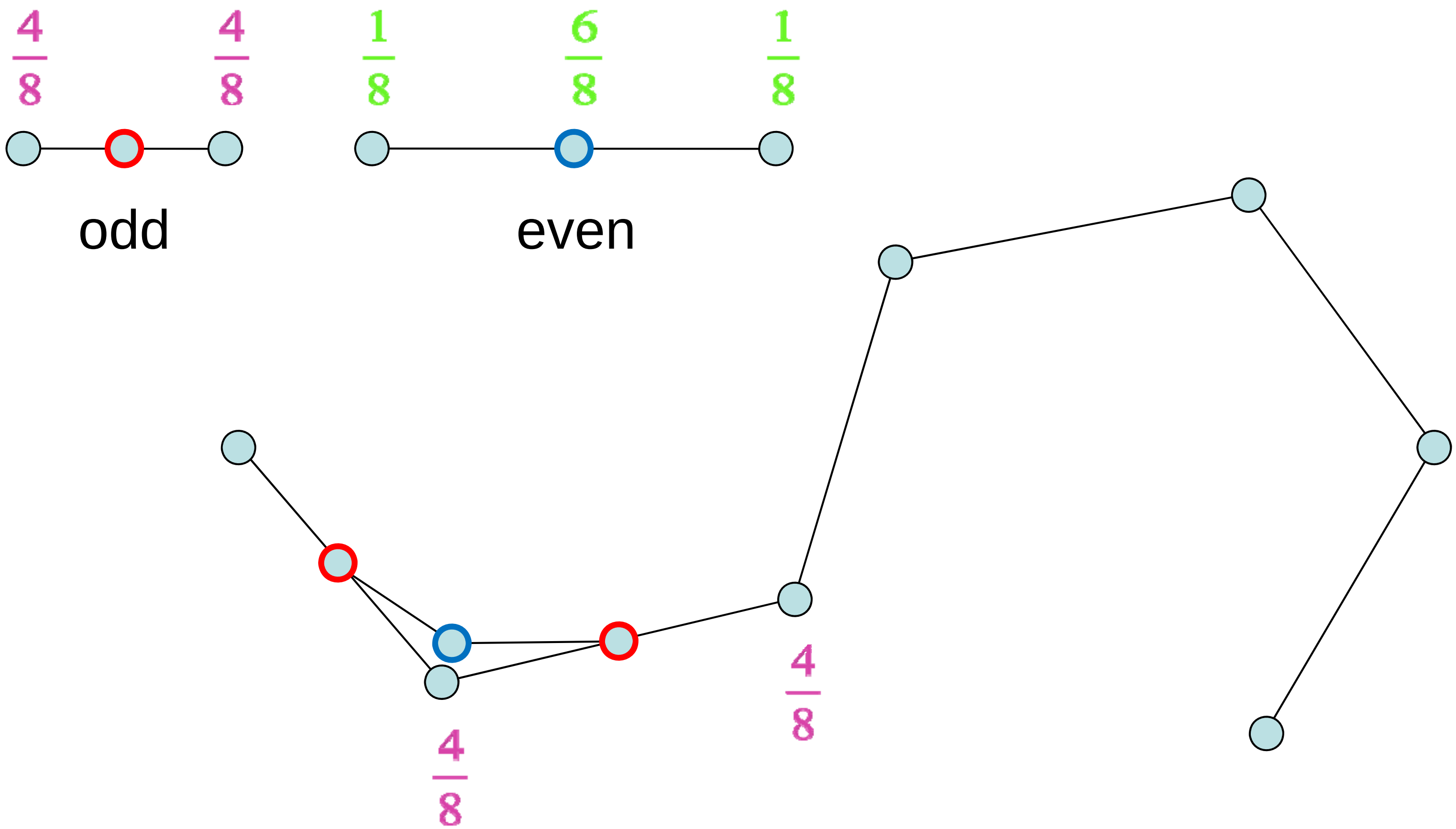
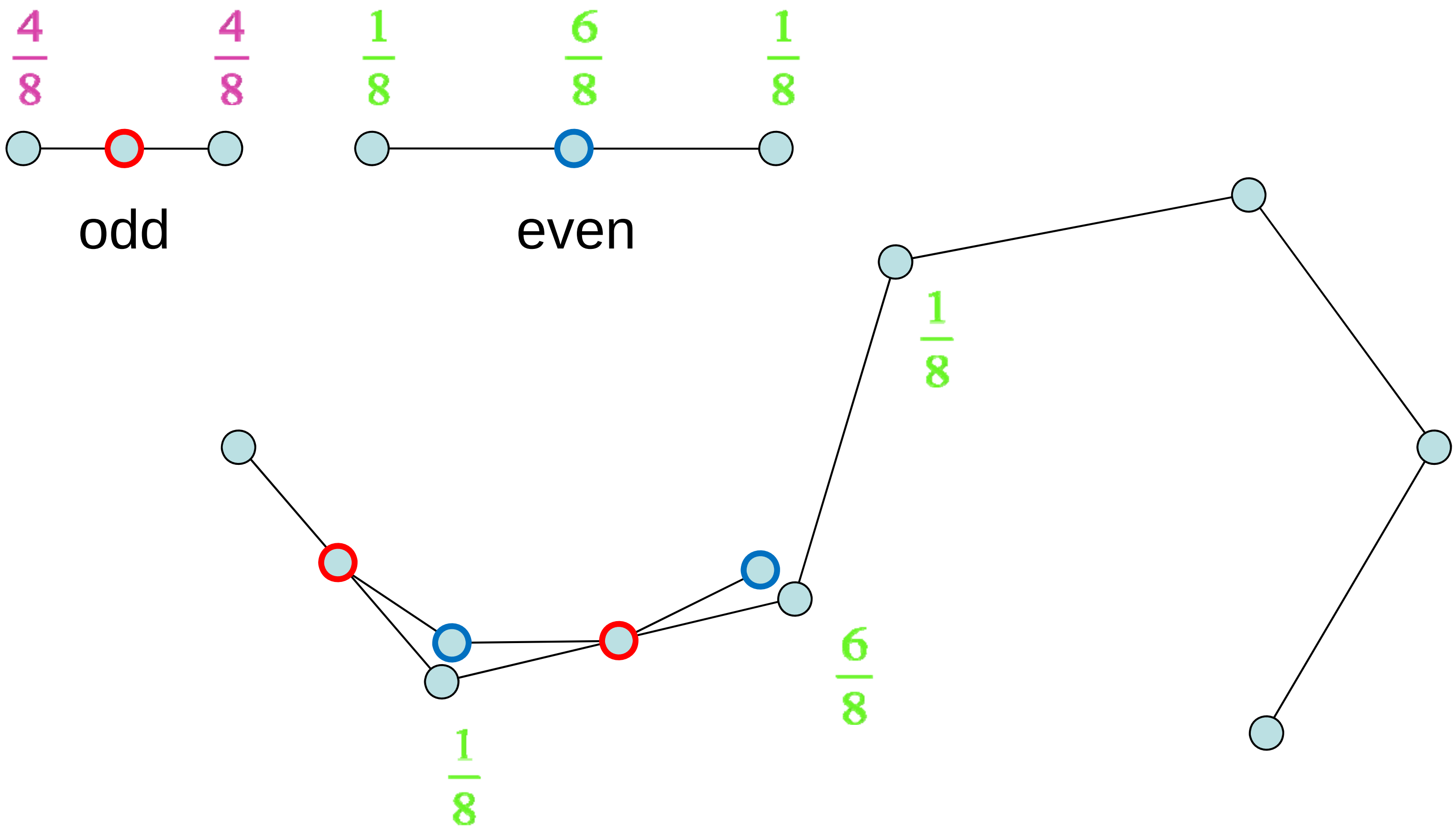


Cubic B-Spline



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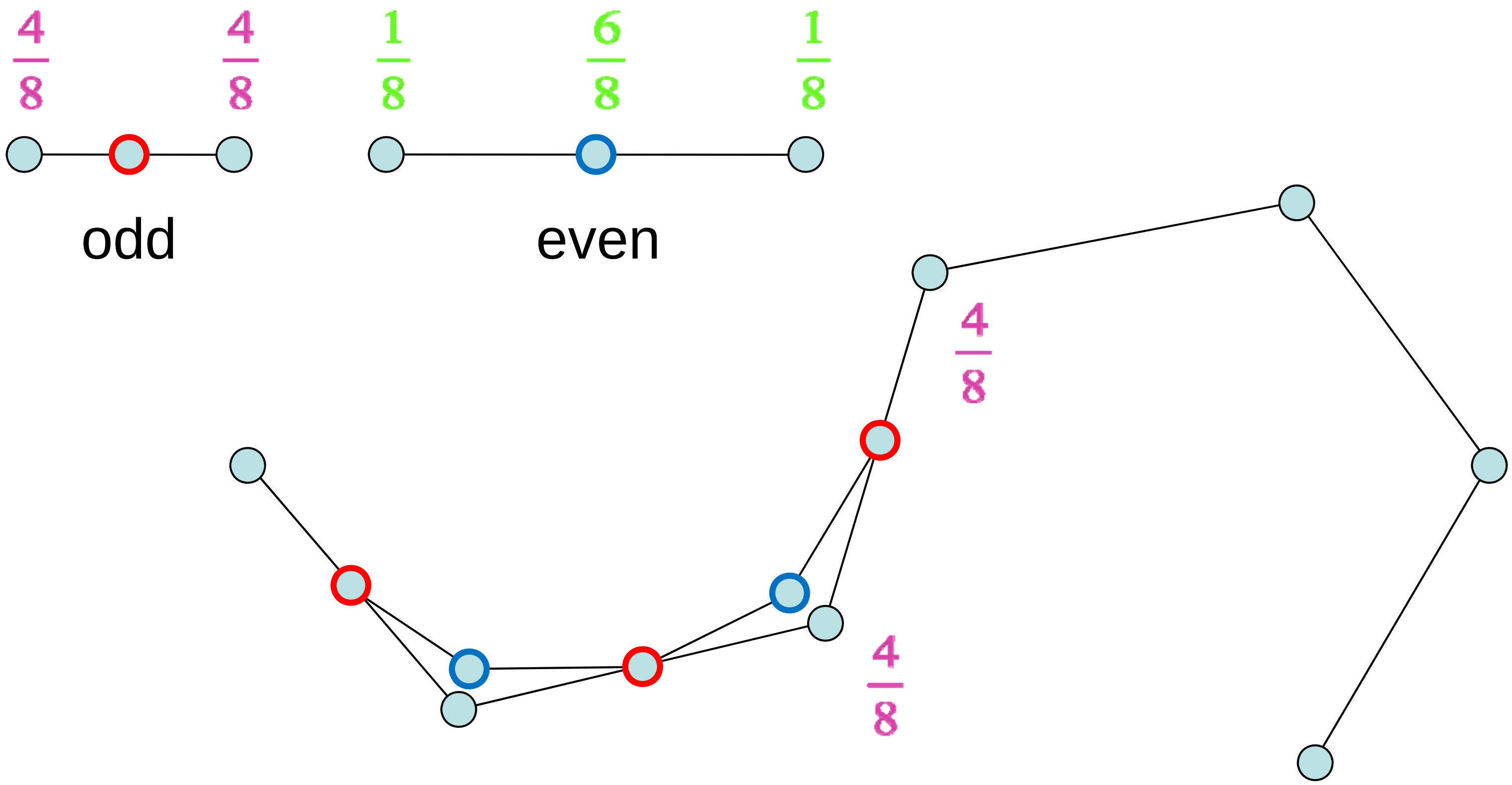
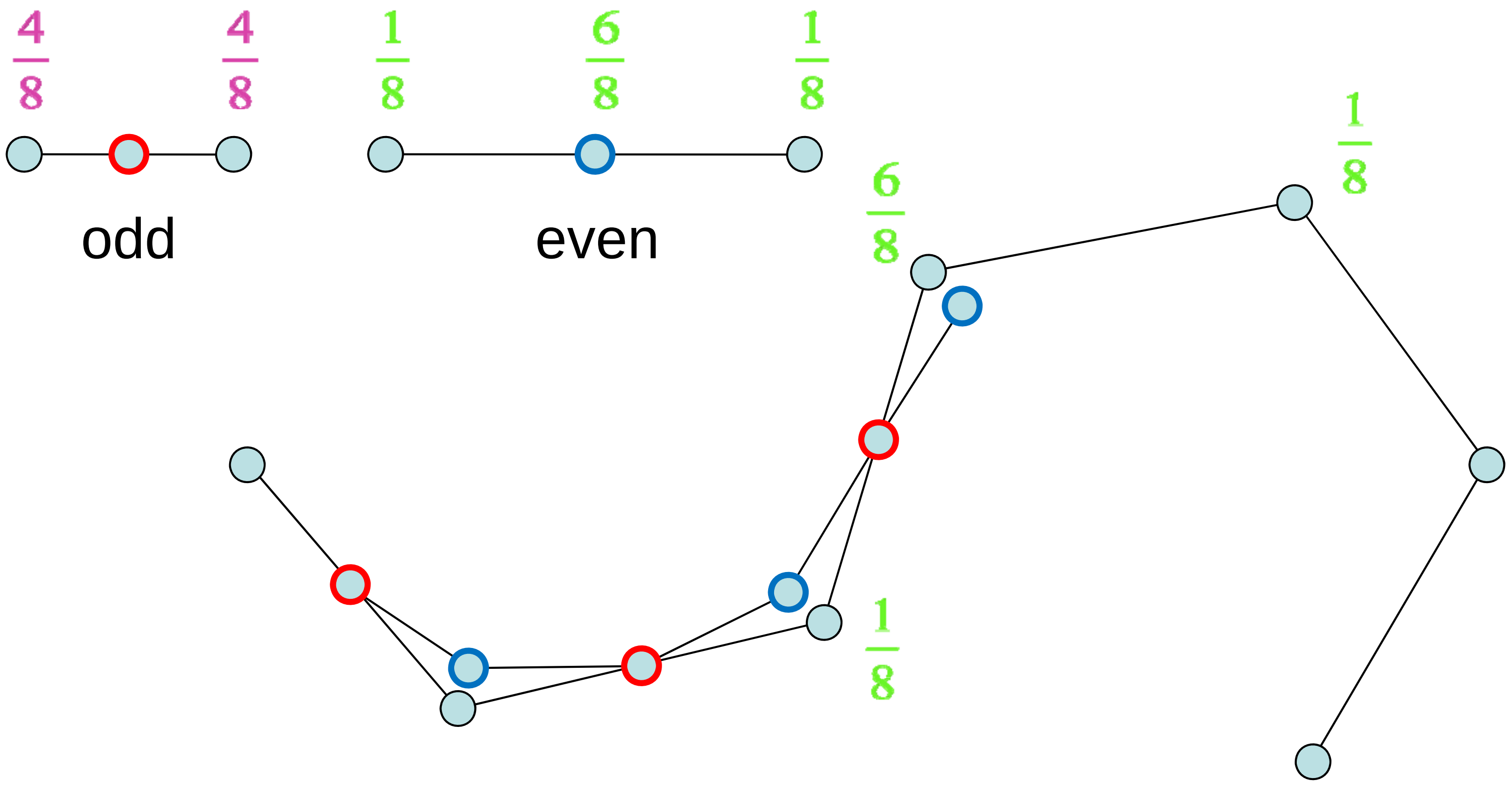


Cubic B-Spline



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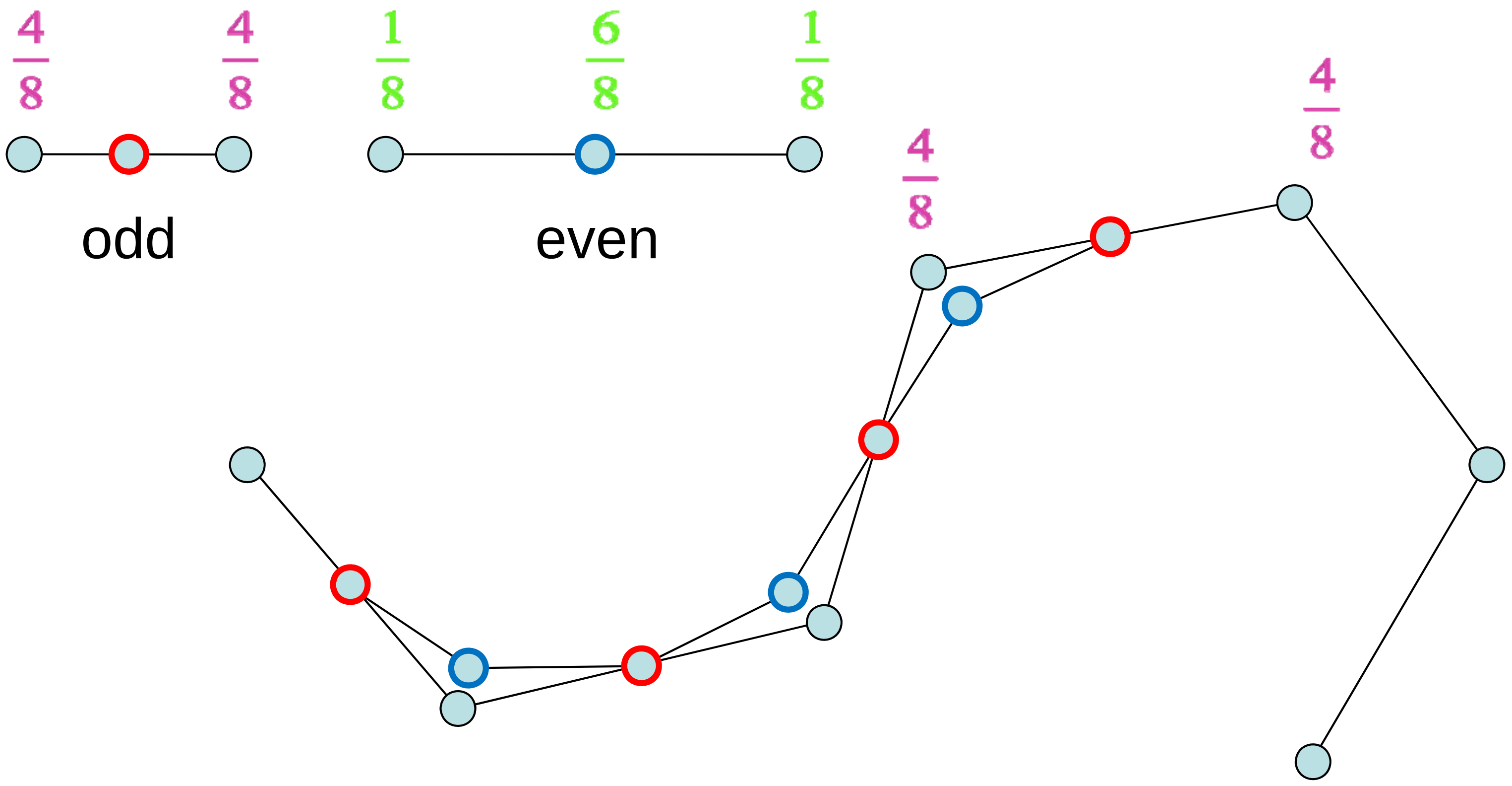

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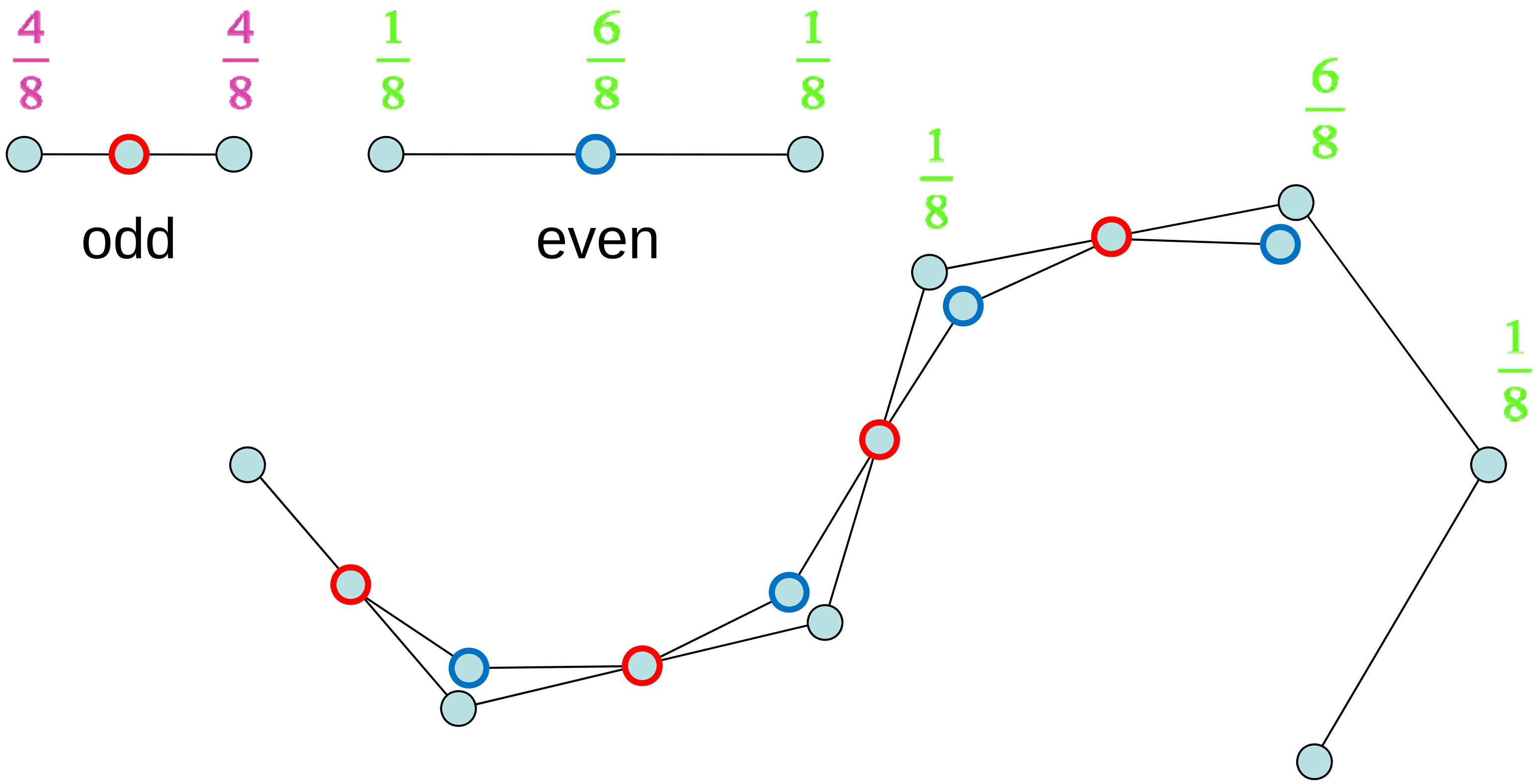
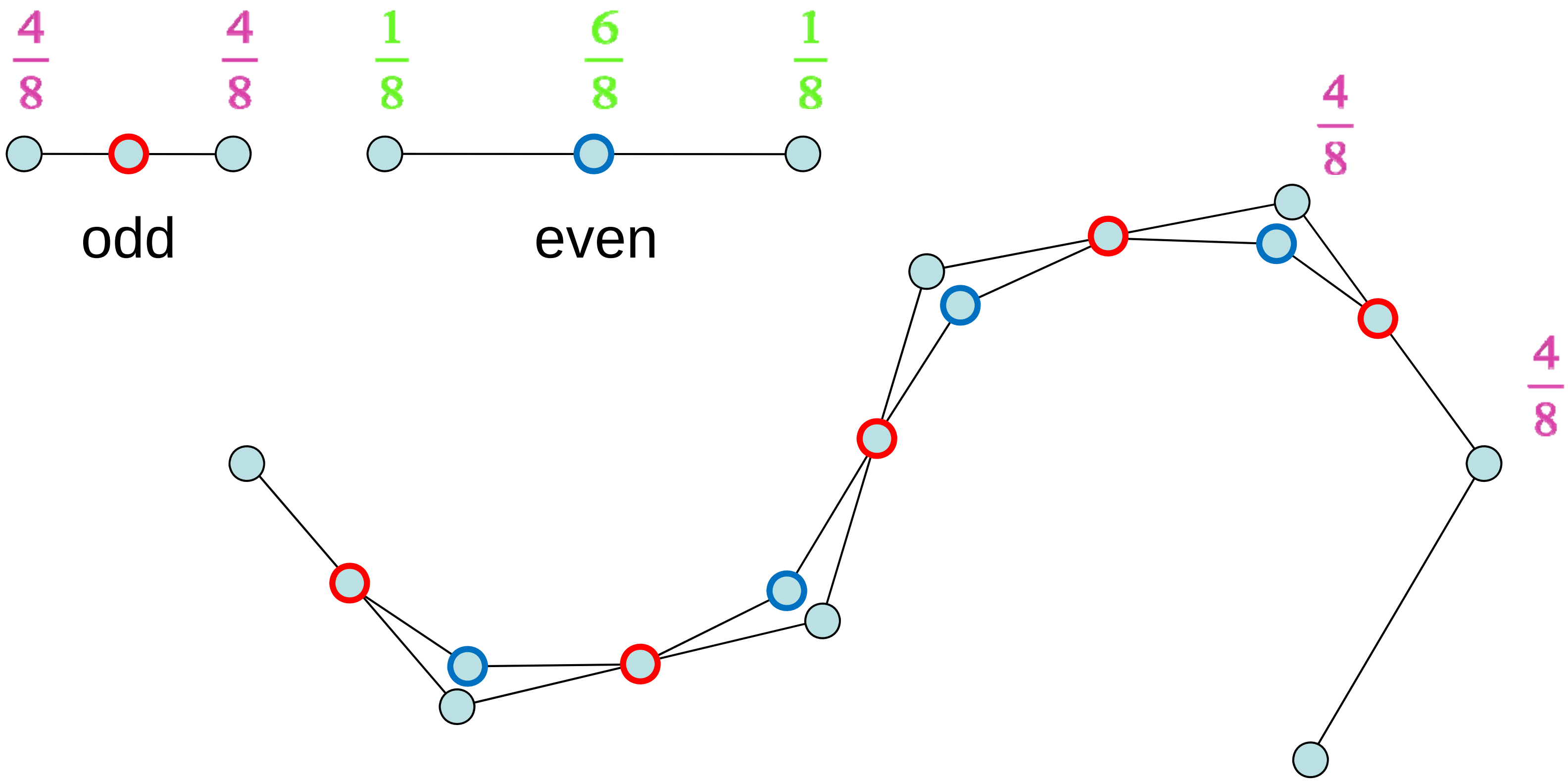


Cubic B-Spline



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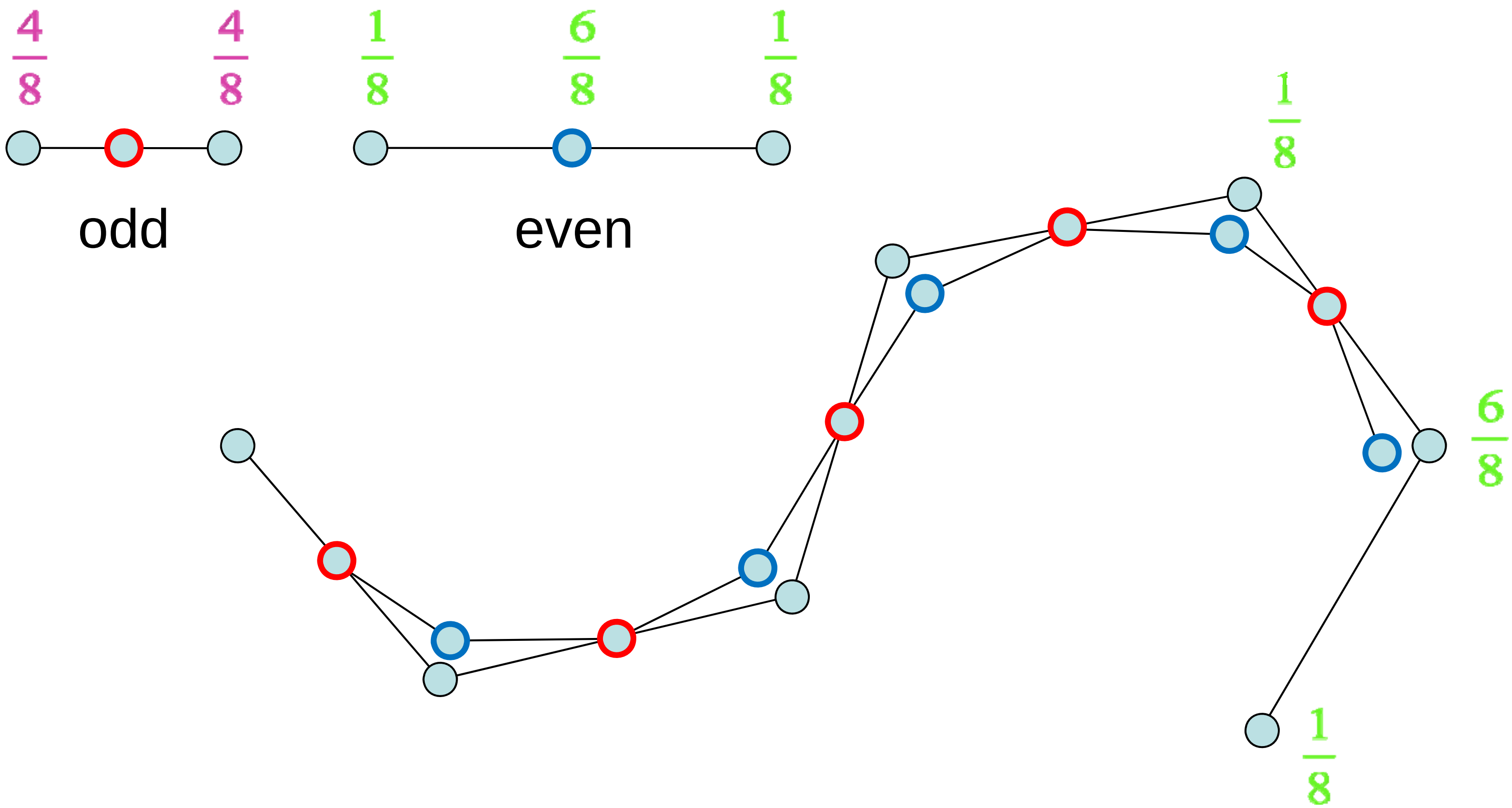

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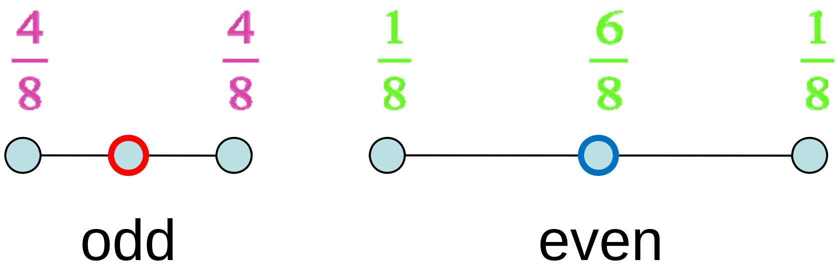

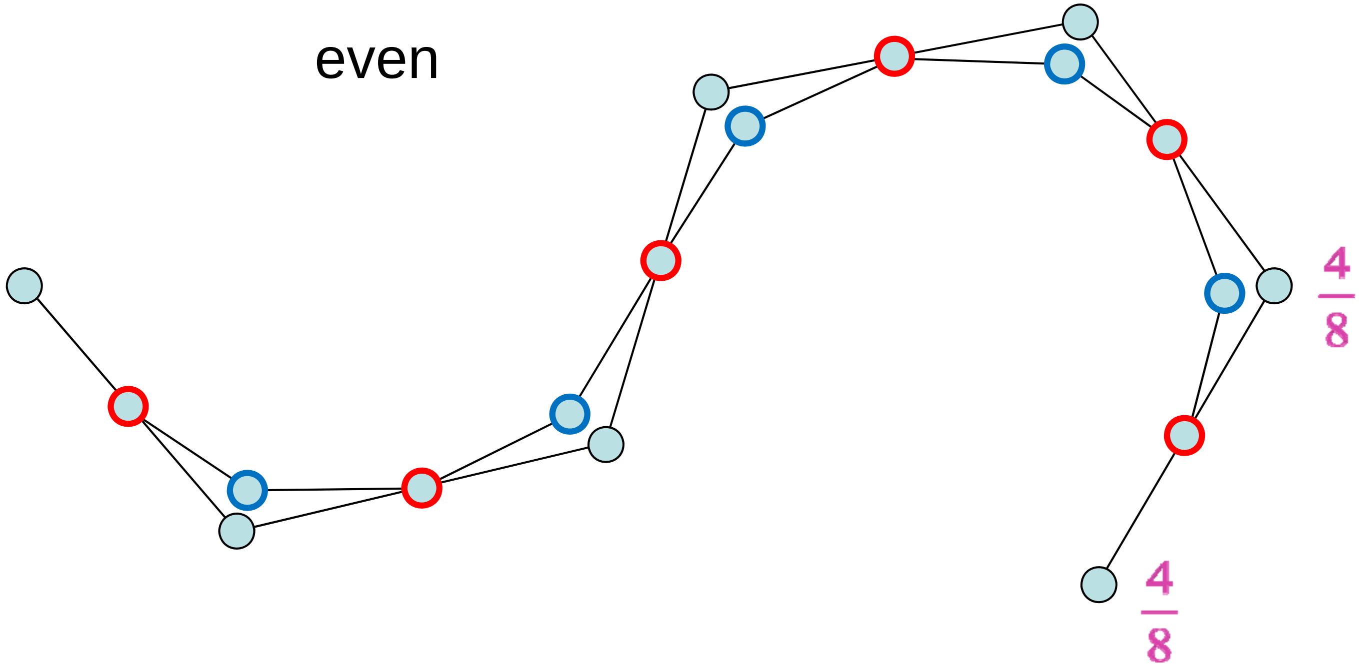

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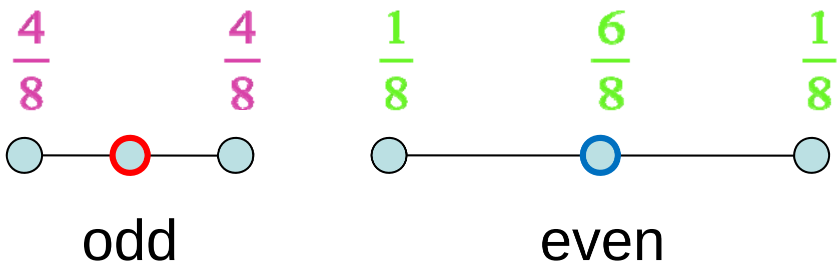

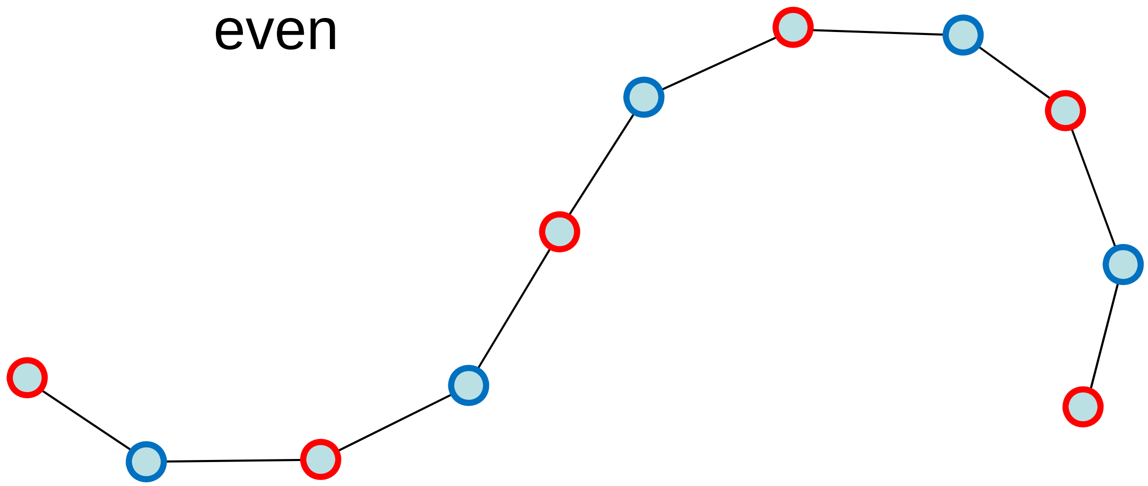

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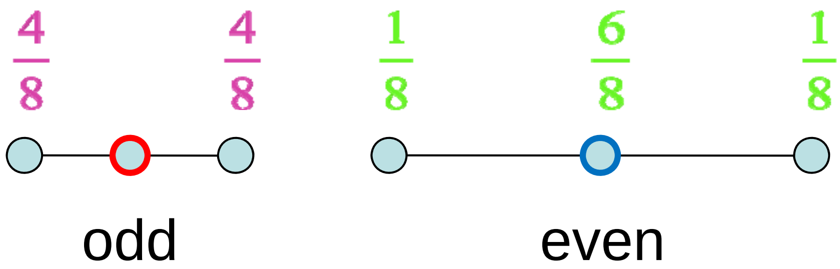
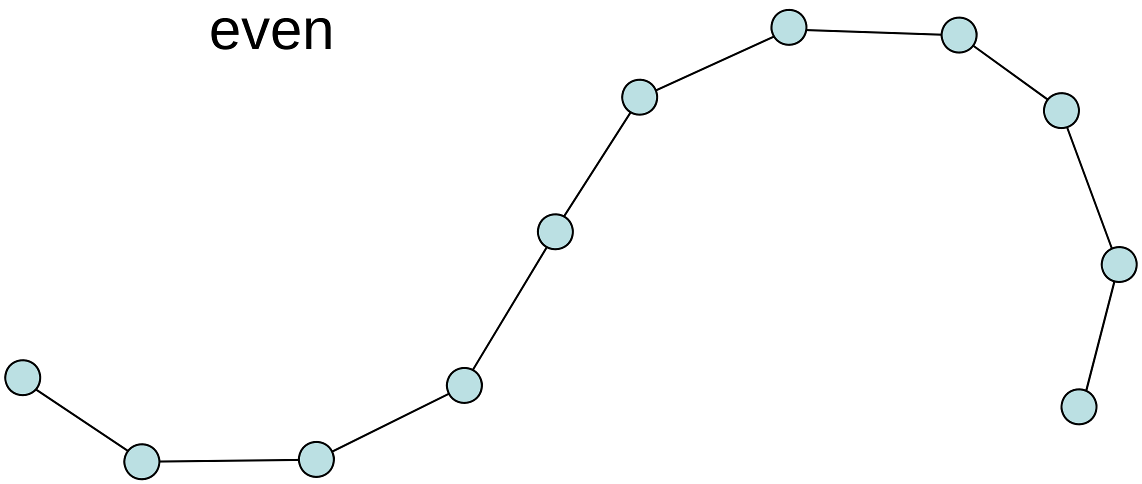




B-Spline Curves

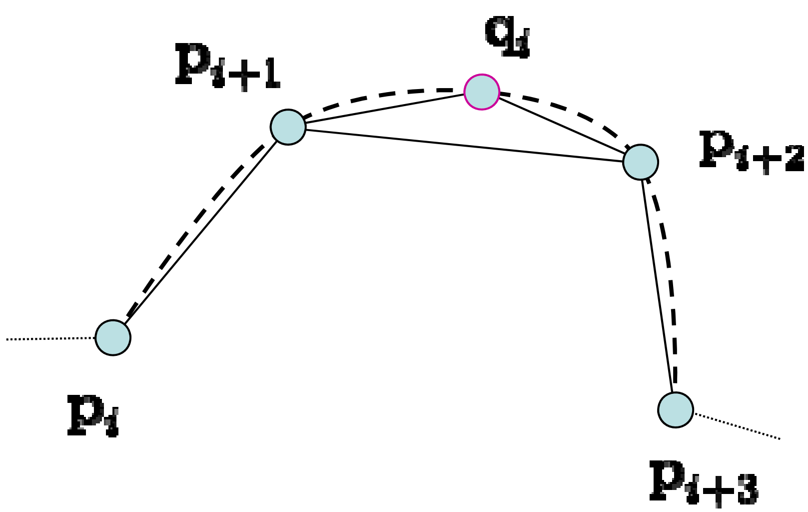
- Subdivision rules for control polygon

$$\mathbf{d}^0 \rightarrow \mathbf{d}^1 = S\mathbf{d}^0 \rightarrow \ldots \rightarrow \mathbf{d}^j = S\mathbf{d}^{j-1} = S^j\mathbf{d}^0$$

- Mask of size *n* yields C^{n-1} curve

Interpolating (4-point Scheme)

- Keep old vertices
- Generate new vertices by fitting cubic curve to old vertices
- C^1 continuous limit curve

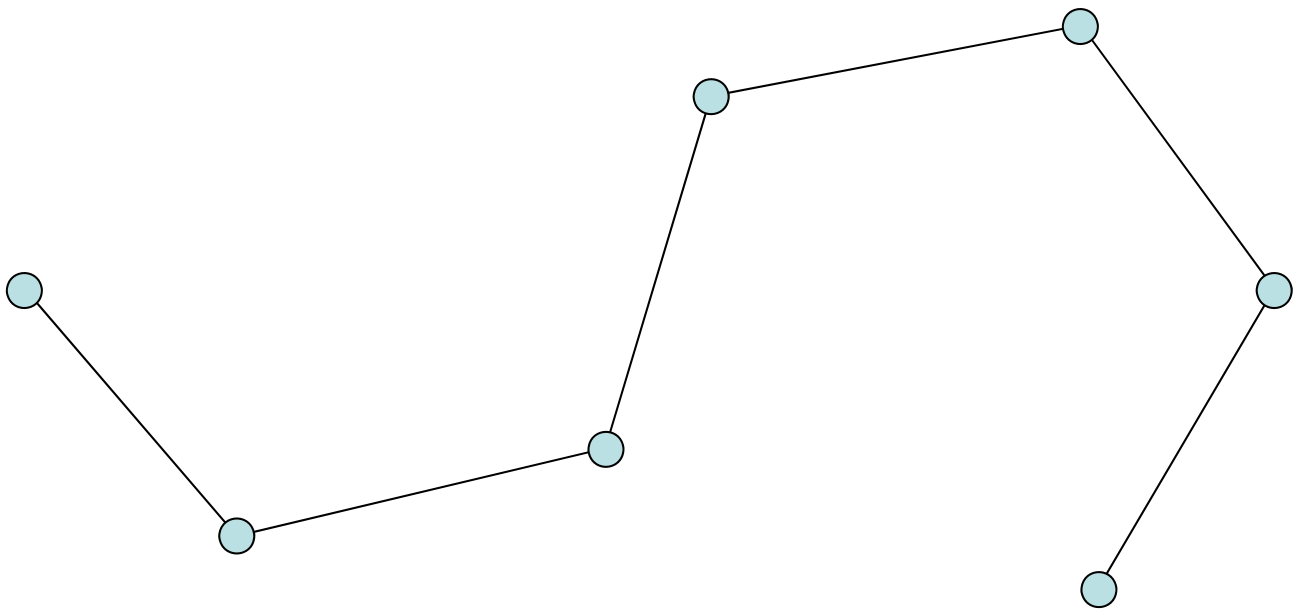


$$f(x) = ax^3 + bx^2 + cx + d$$

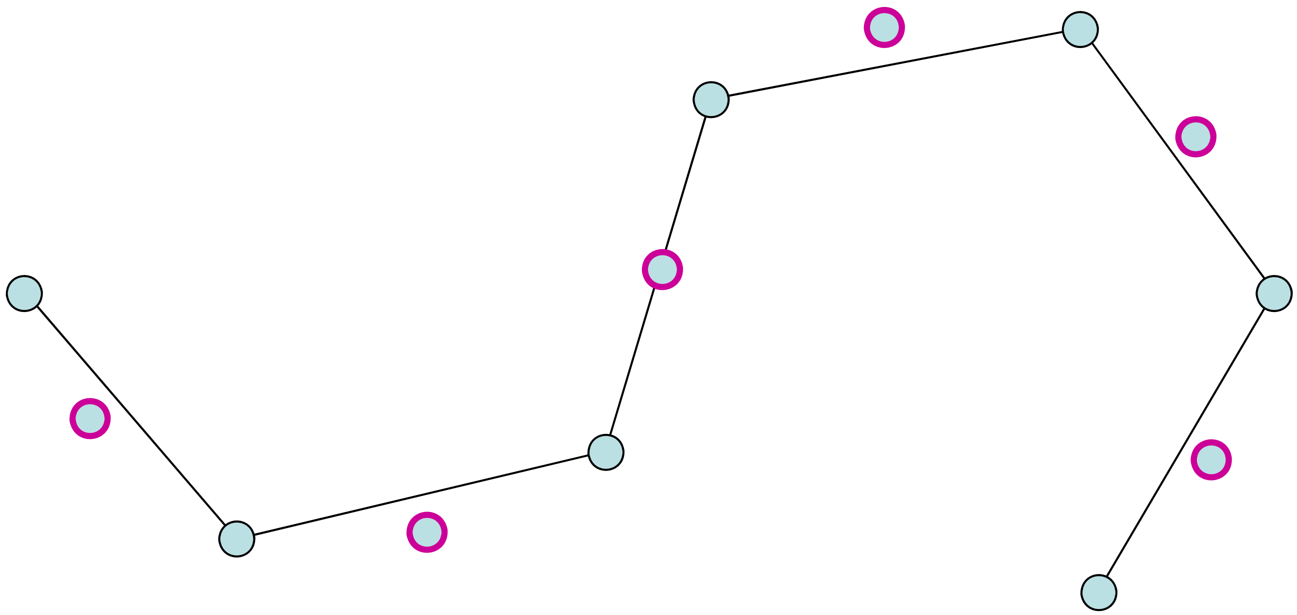
$$f(j) = \mathbf{p}_{i+j}\,, \quad j = 0, \ldots, 3$$

$$\begin{aligned} \mathbf{q}_i &= f(3/2) \\ &= \frac{1}{16}\left(-\mathbf{p}_i + 9\mathbf{p}_{i+1} + 9\mathbf{p}_{i+2} - \mathbf{p}_{i+3}\right) \end{aligned}$$

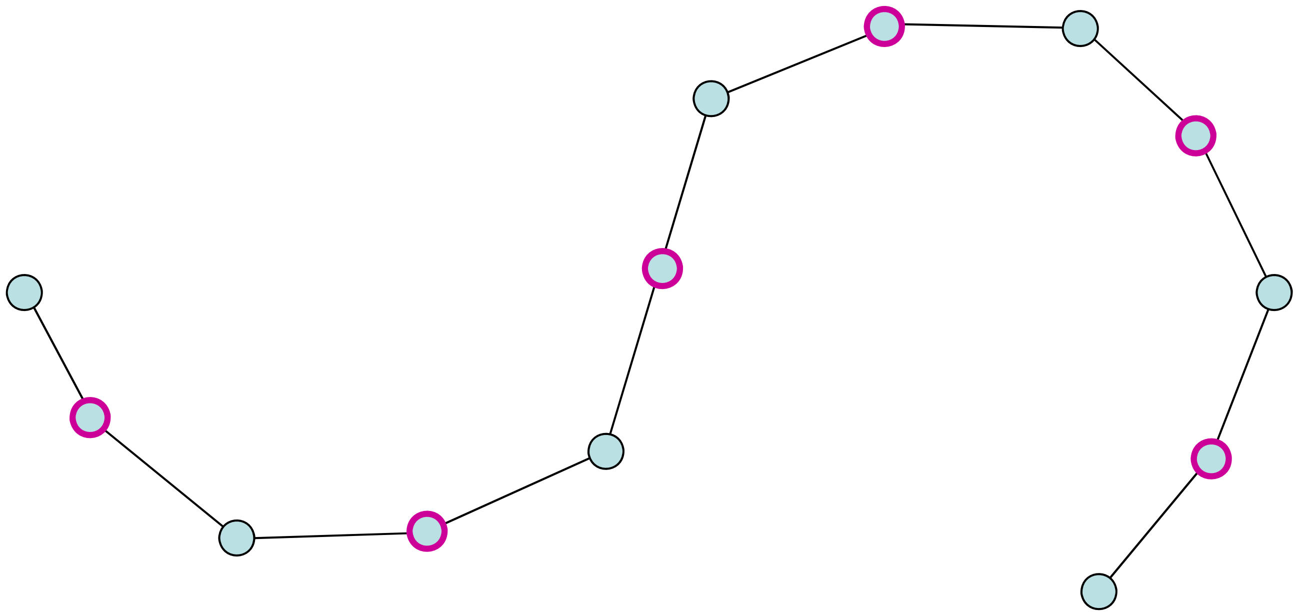
Interpolating



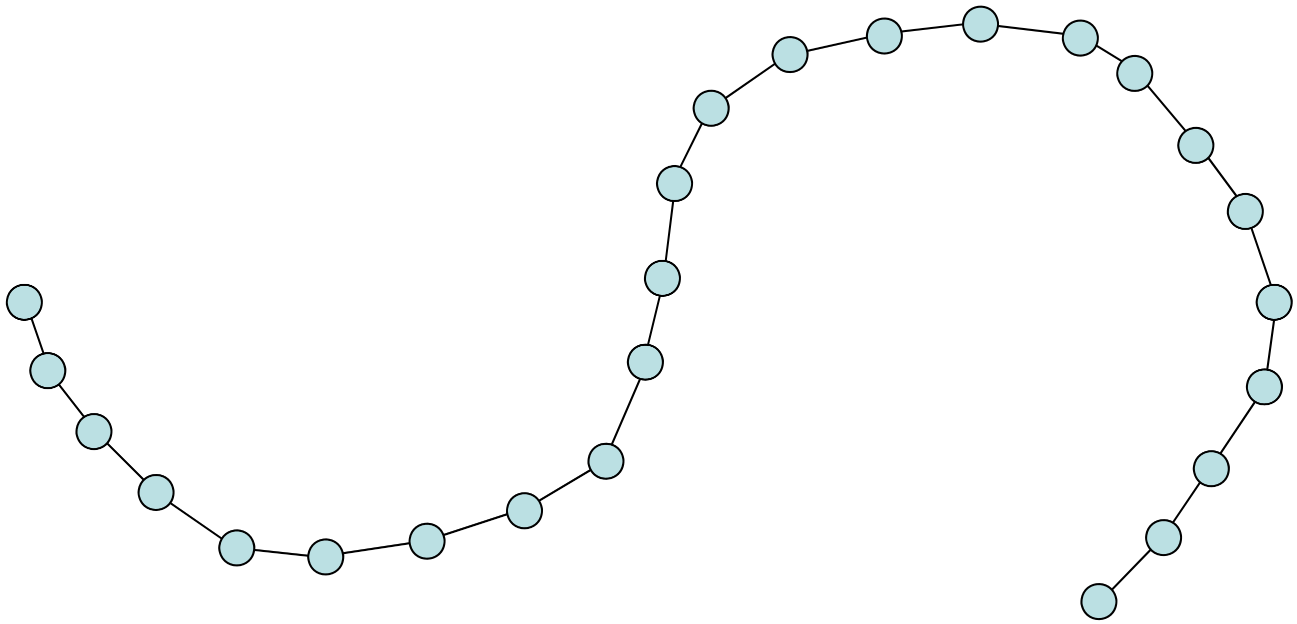
Interpolating



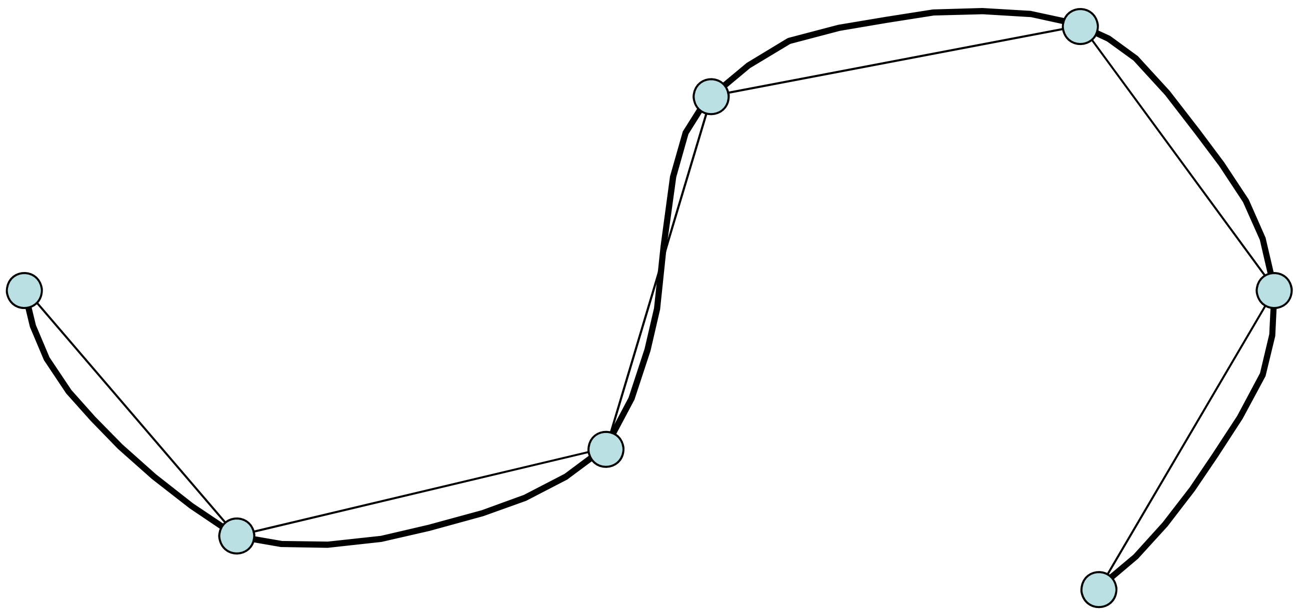
Interpolating



Interpolating



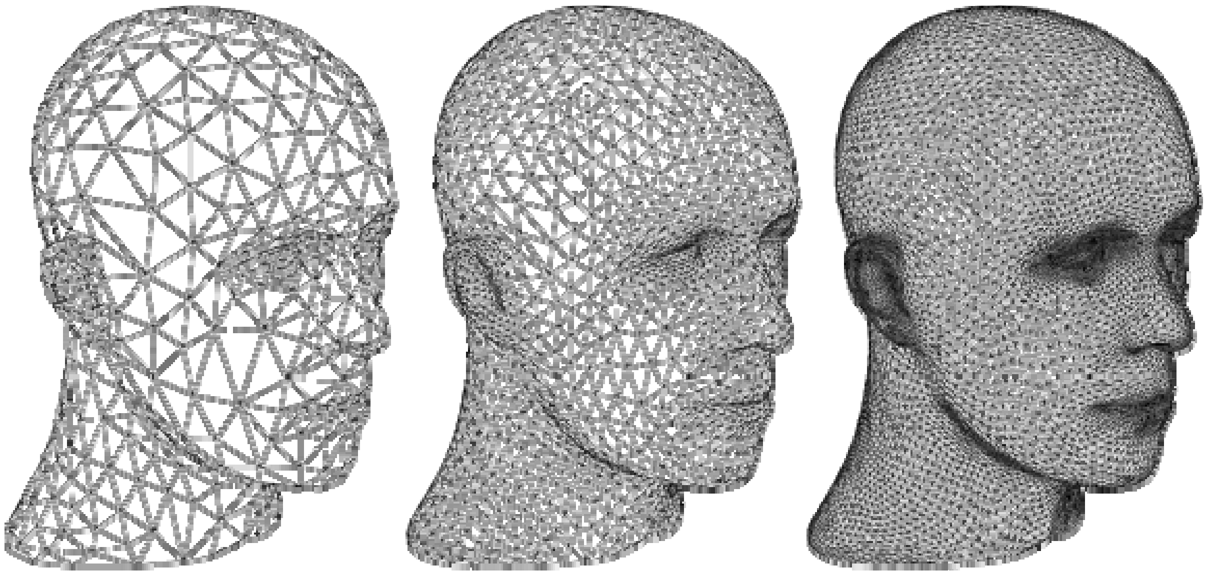
Interpolating



demo

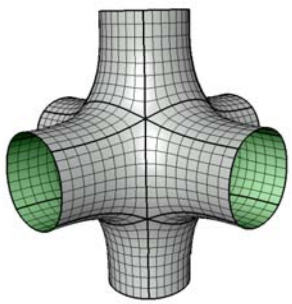
Subdivision Surfaces

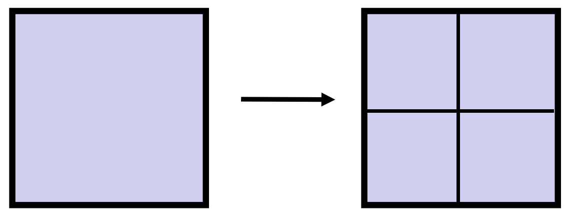
- No regular structure as for curves
  - Arbitrary number of edge-neighbors
  - Different subdivision rules for each valence



Subdivision Rules

- How the connectivity changes





- How the geometry changes
  - Old points
  - New points

Subdivision Zoo

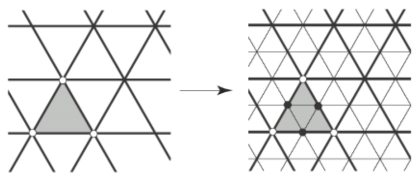
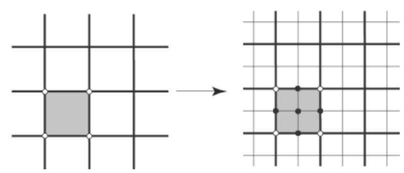
- Classification of subdivision schemes

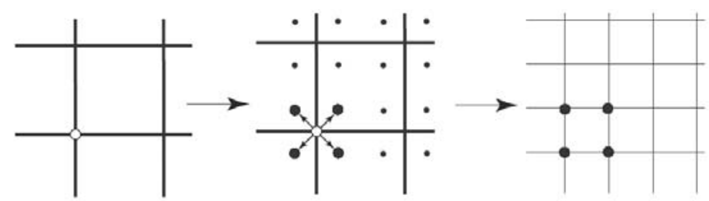
Primal	Faces are split into sub-faces
Dual	Vertices are split into multiple vertices

Approximating	Control points are not interpolated
Interpolating	Control points are interpolated

Subdivision Zoo

- Classification of subdivision schemes

Primal (face split)		
	 *Triangular meshes*	 *Quad Meshes*
Approximating	Loop(C^2)	Catmull-Clark(C^2)
Interpolating	Mod. Butterfly (C^1)	Kobbelt (C^1)



Dual (vertex split)
Doo-Sabin, Midedge(C^1)
Biquartic (C^2)

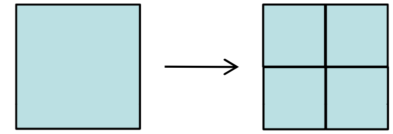
- Many more...

Subdivision Zoo

- Classification of subdivision schemes

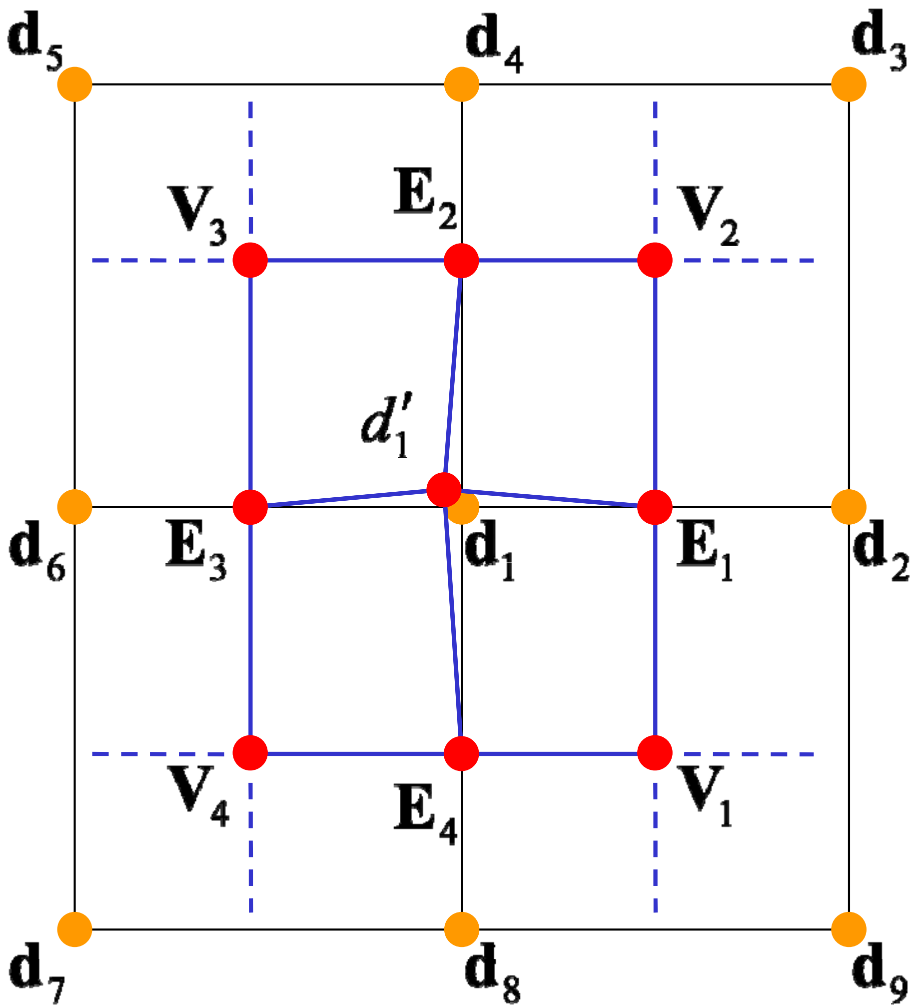
<table>
<tr><td rowspan="2"></td><td colspan="2">Primal</td><td rowspan="2">Dual</td></tr>
<tr><td>Triangles</td><td>Rectangles</td></tr>
<tr><td>Approximating</td><td>Loop</td><td>Catmull-Clark</td><td rowspan="2">Doo-Sabin
Midedge</td></tr>
<tr><td>Interpolating</td><td>Butterfly</td><td>Kobbelt</td></tr>
</table>

Catmull-Clark Subdivision



- Generalization of *bi-cubic B-Splines*
- Primal, approximation subdivision scheme
- Applied to *polygonal* meshes
- Generates G^2 *continuous* limit surfaces:
  - C^1 for the set of finite extraordinary points
    - Vertices with valence $\neq 4$
  - C^2 continuous everywhere else

Catmull-Clark Subdivision



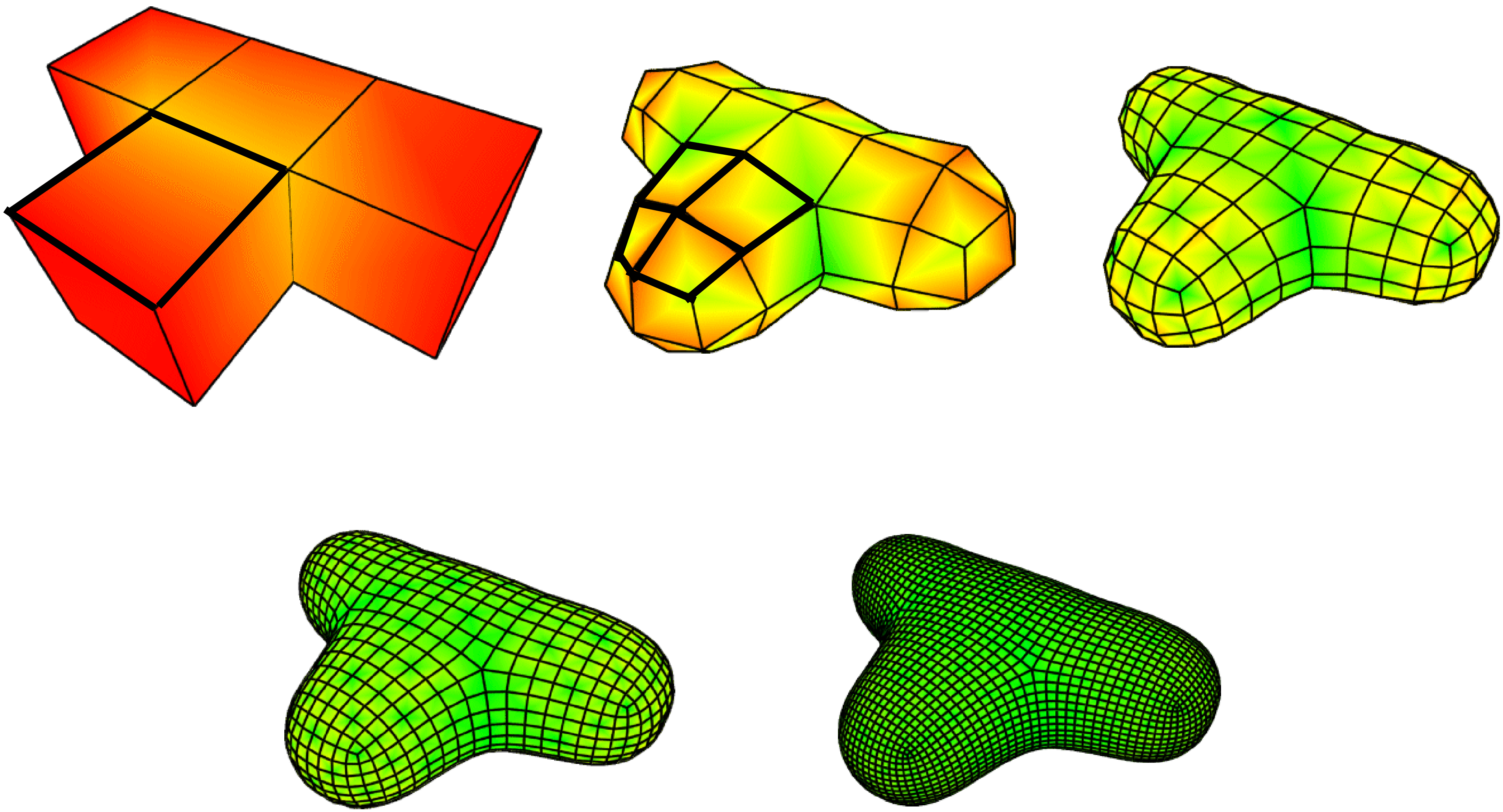
$$\mathbf{V}_2 = \frac{1}{n} \times \sum_{j=1}^{n} \mathbf{d}_j$$

$$\mathbf{E}_i = \frac{1}{4}\left(\mathbf{d}_1 + \mathbf{d}_{2i} + \mathbf{V}_i + \mathbf{V}_{i+1}\right)$$

$$\mathbf{d}_1' = \frac{(n-3)}{n}\mathbf{d}_1 + \frac{2}{n}\mathbf{R} + \frac{1}{n}\mathbf{S}$$

$$\mathbf{R} = \frac{1}{m}\sum_{i=1}^{m}\mathbf{E}_i \quad \mathbf{S} = \frac{1}{m}\sum_{i=1}^{m}\mathbf{V}_i$$

Catmull-Clark Subdivision

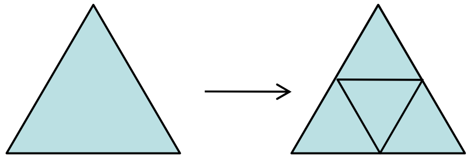


Classic Subdivision Operators

- Classification of subdivision schemes

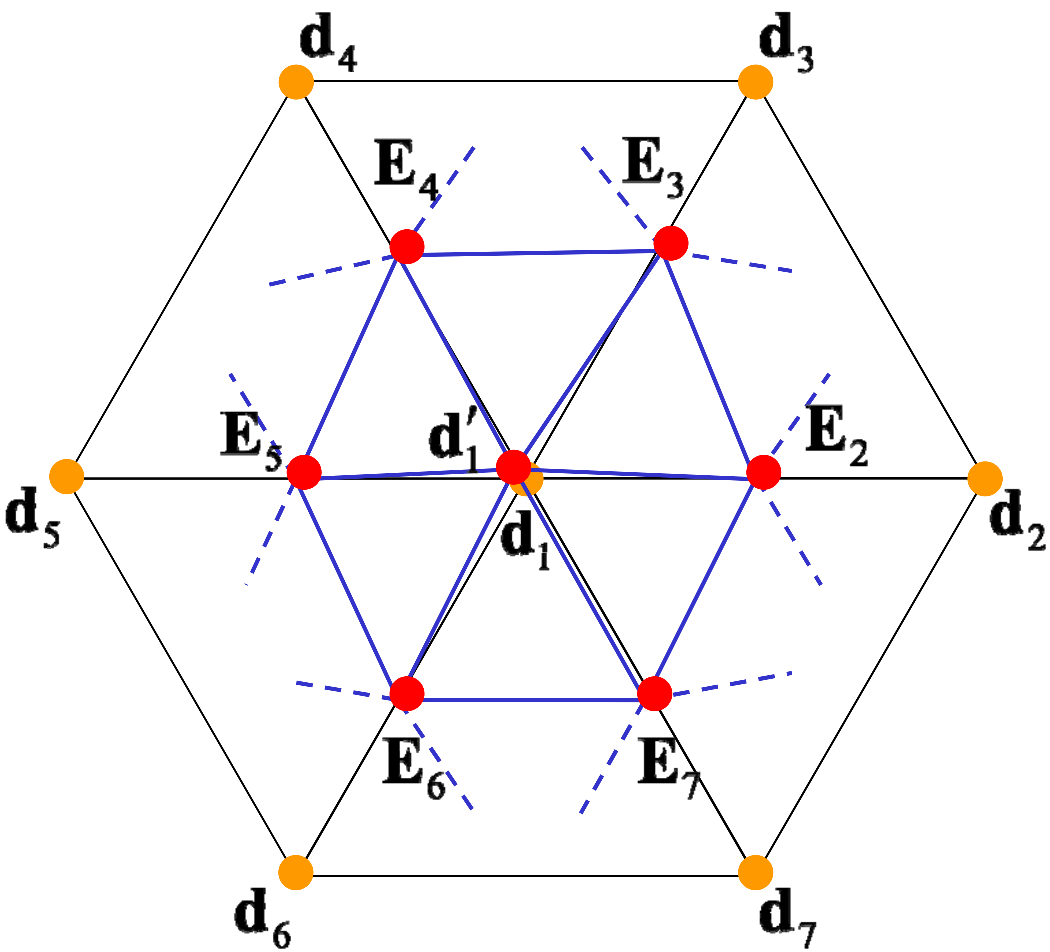
<table>
<tr><td rowspan="2"></td><td colspan="2">Primal</td><td rowspan="2">Dual</td></tr>
<tr><td>Triangles</td><td>Rectangles</td></tr>
<tr><td>Approximating</td><td>Loop</td><td>Catmull-Clark</td><td rowspan="2">Doo-Sabin
Midedge</td></tr>
<tr><td>Interpolating</td><td>Butterfly</td><td>Kobbelt</td></tr>
</table>

Loop Subdivision



- Generalization of *box splines*
- Primal, approximating subdivision scheme
- Applied to *triangle* meshes
- Generates G^2 *continuous* limit surfaces:
  - C^1 for the set of finite extraordinary points
    - Vertices with valence $\neq 6$
  - C^2 continuous everywhere else

Loop Subdivision

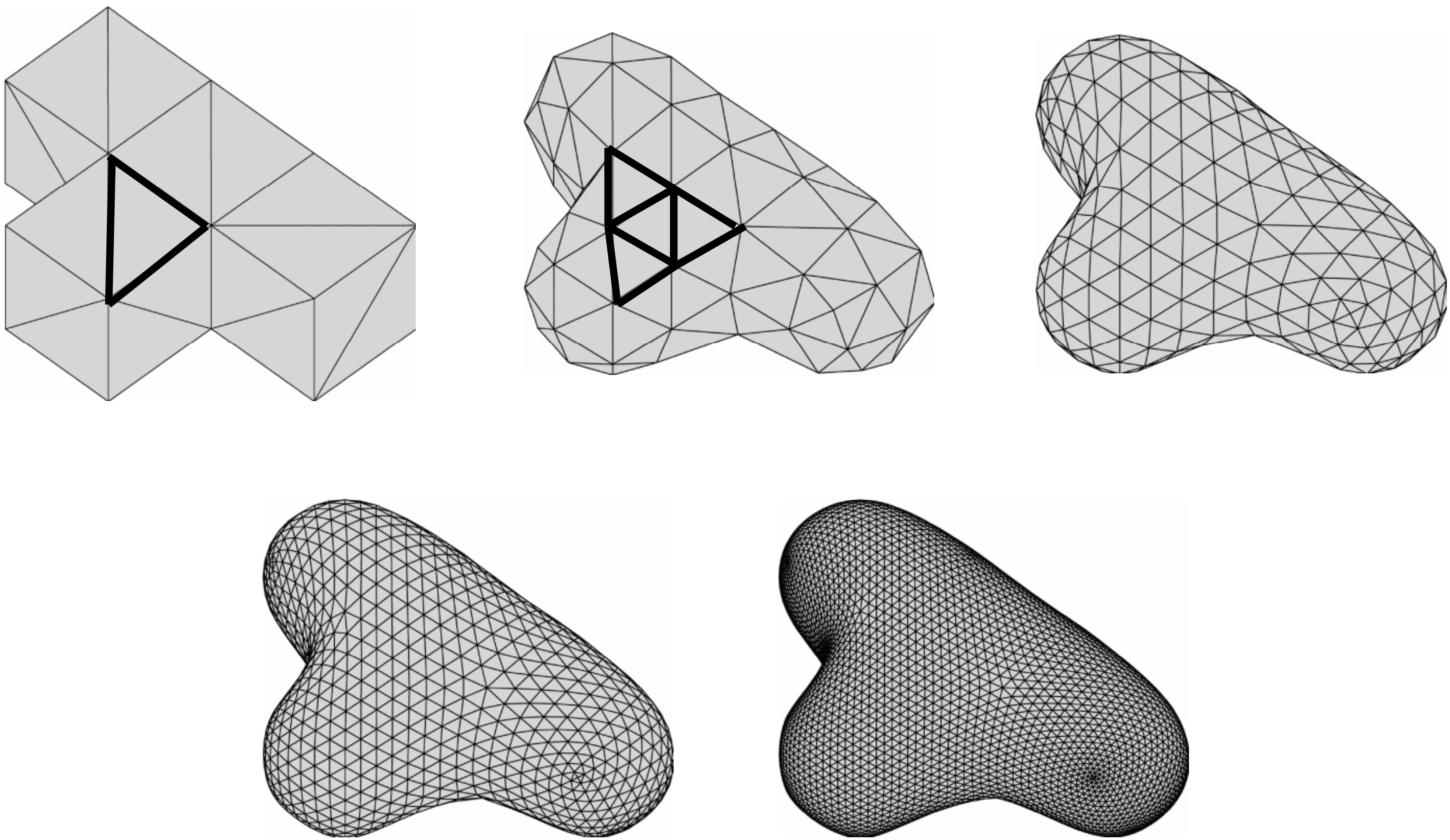


$$E_i = \frac{3}{8}\left(d_1 + d_i\right) + \frac{1}{8}\left(d_{i-1} + d_{i+1}\right)$$

$$\mathbf{d}_1' = \alpha_n \mathbf{d}_1 + \frac{\left(1 - \alpha_n\right)}{n} \sum_{j=2}^{n+1} \mathbf{d}_j$$

$$\alpha_n = \frac{3}{8} + \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{n}\right)^2$$

Loop Subdivision

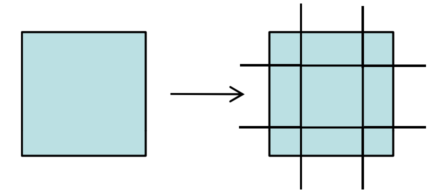


Subdivision Zoo

- Classification of subdivision schemes

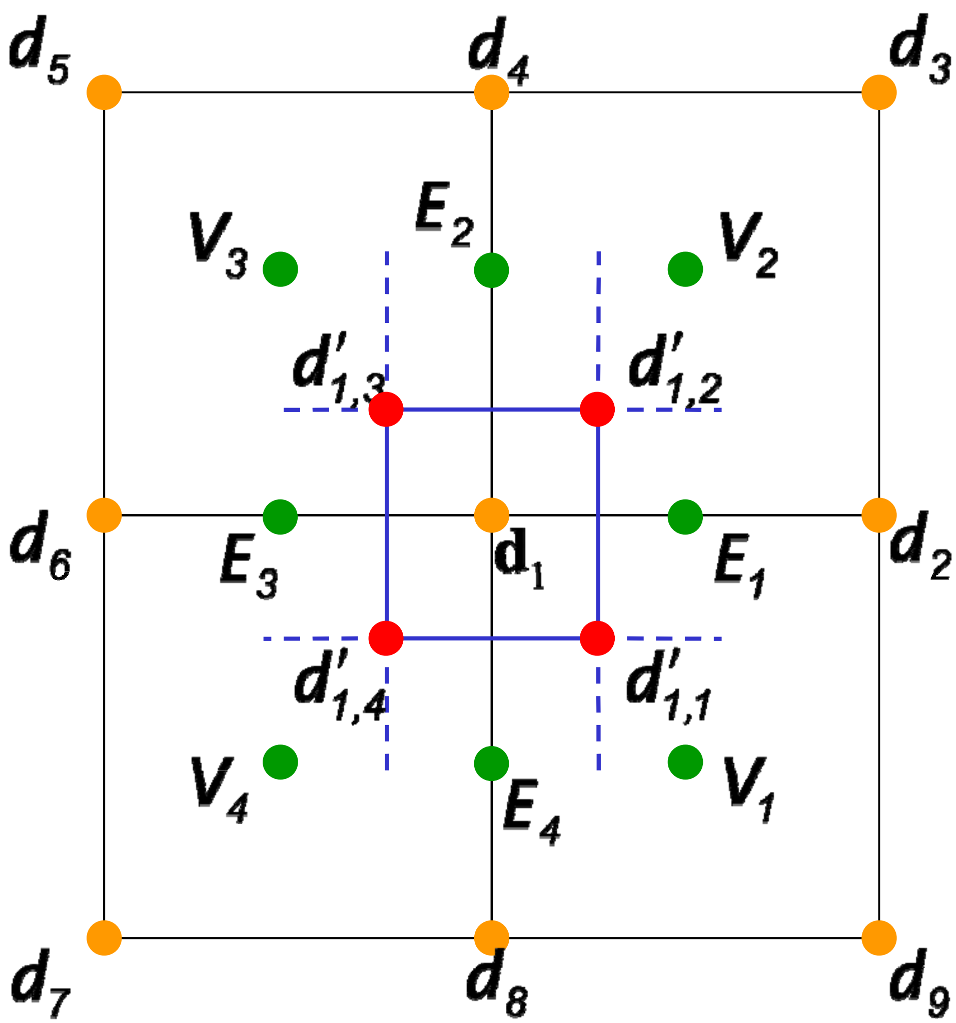
	Primal		Dual
	Triangles	**Rectangles**	
Approximating	Loop	Catmull-Clark	**Doo-Sabin** **Midedge**
Interpolating	Butterfly	Kobbelt	

Doo-Sabin Subdivision



- Generalization of *bi-quadratic B-Splines*
- Dual, approximating subdivision scheme
- Applied to *polygonal* meshes
- Generates G^1 *continuous* limit surfaces:
  - C^0 for the set of finite extraordinary points
    - Center of irregular polygons after 1 subdivision step
  - C^1 continuous everywhere else

Doo-Sabin Subdivision

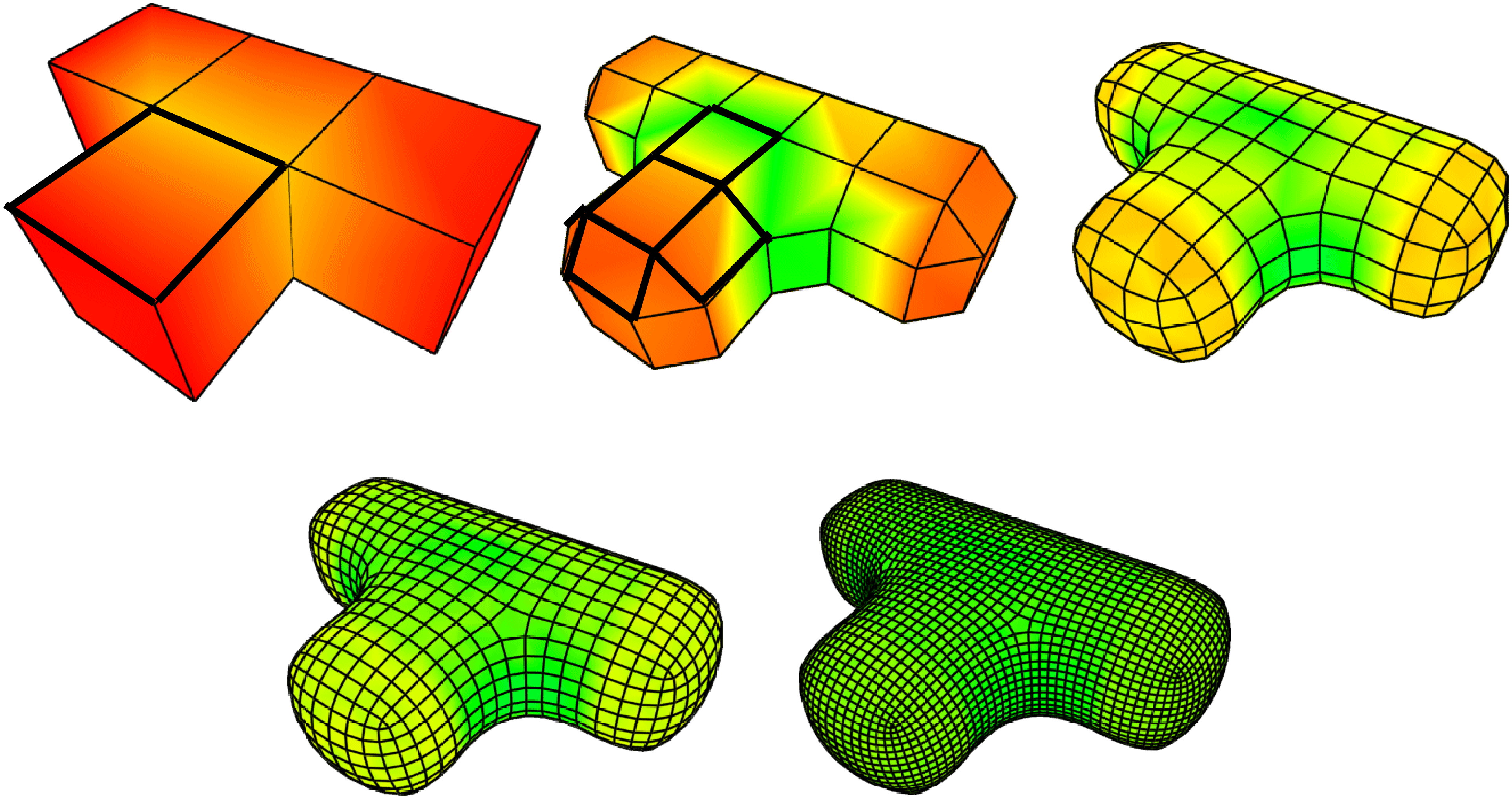


$$V_2 = \frac{1}{n} \times \sum_{j=1}^{n} d_j$$

$$E_i = \frac{1}{2}\left(d_1 + d_{2i}\right)$$

$$d'_{1,j} = \frac{1}{4}\left(d_1 + E_j + E_{j-1} + V_j\right)$$

Doo-Sabin Subdivision



Classic Subdivision Operators

- Classification of subdivision schemes

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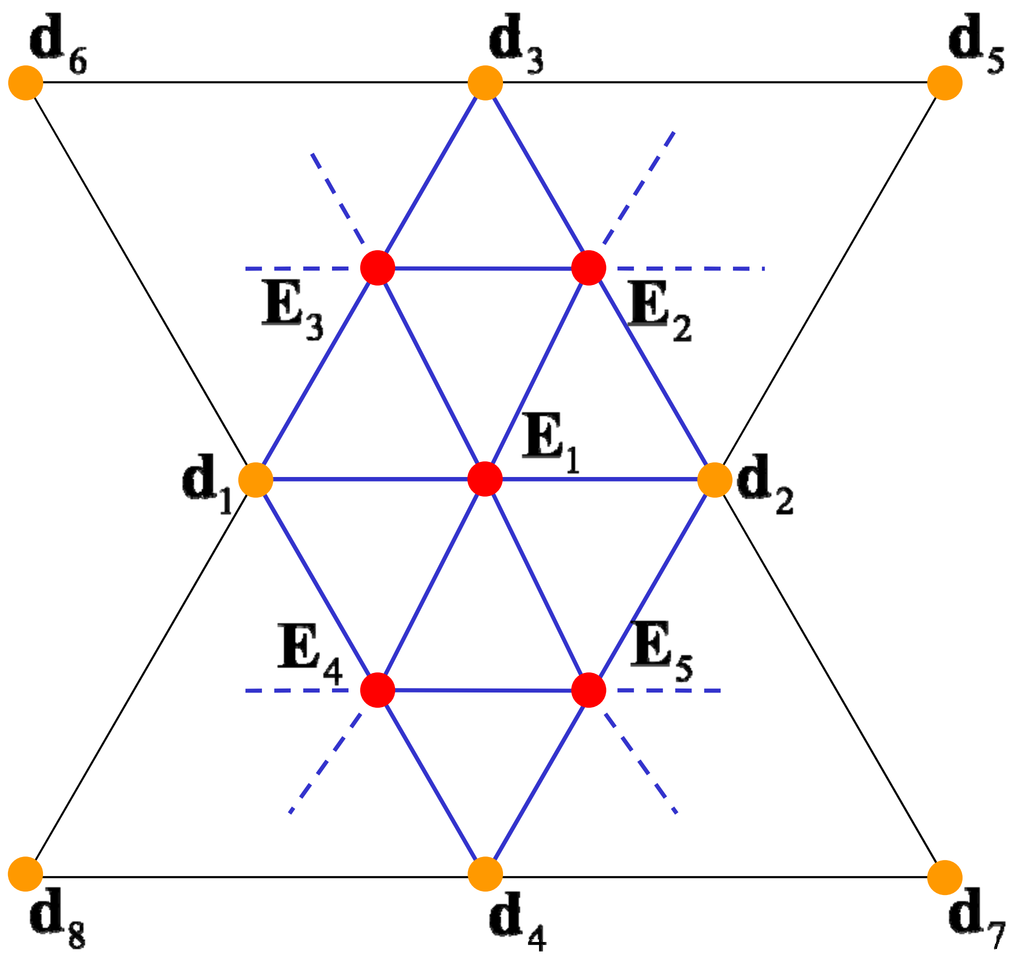
</table>

Butterfly Subdivision

- Primal, interpolating scheme
- Applied to *triangle* meshes
- Generates G^1 *continuous* limit surfaces:
  - C^o for the set of finite extraordinary points
    - Vertices of valence = 3 or > 7
  - C^1 continuous everywhere else

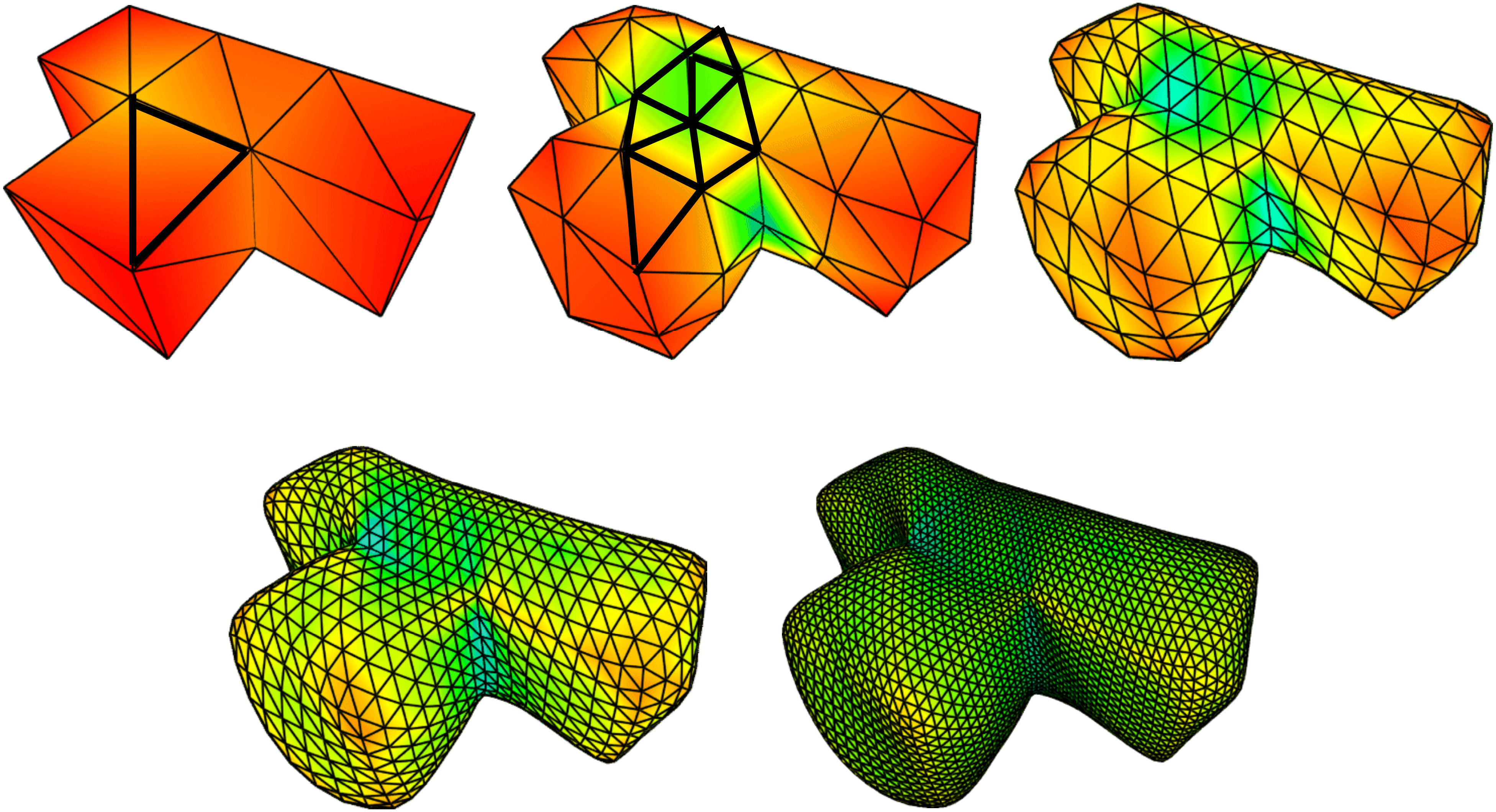
Butterfly Subdivision



$$\boxed{\begin{aligned}\mathbf{E}_1 = \frac{1}{2}\left(\mathbf{d}_1+\mathbf{d}_2\right)+\omega\left(\mathbf{d}_3+\mathbf{d}_4\right)- \\ -\frac{\omega}{2}\left(\mathbf{d}_5+\mathbf{d}_6+\mathbf{d}_7+\mathbf{d}_8\right)\end{aligned}}$$

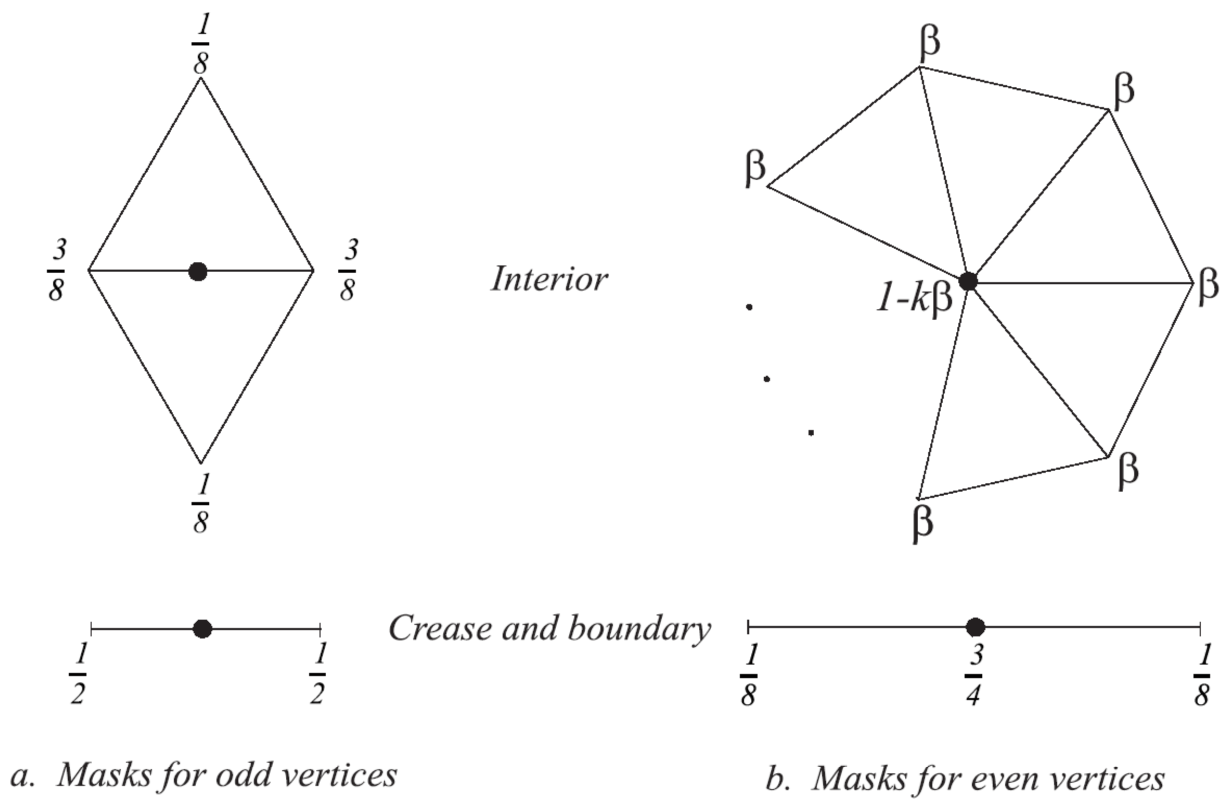
$$\boxed{\mathbf{d}_i' = \mathbf{d}_i}$$

Butterfly Subdivision

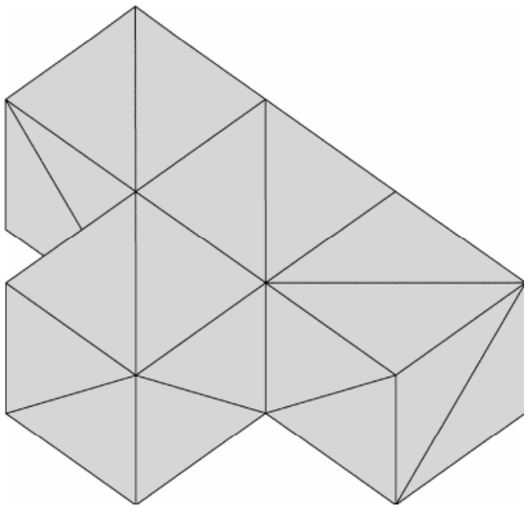


Remark

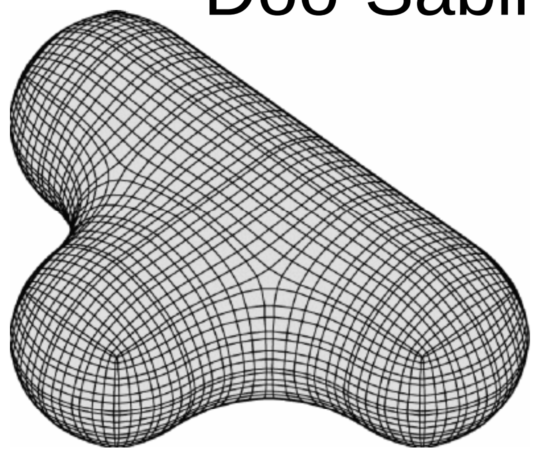
- Different masks apply on the boundary
- Example: Loop



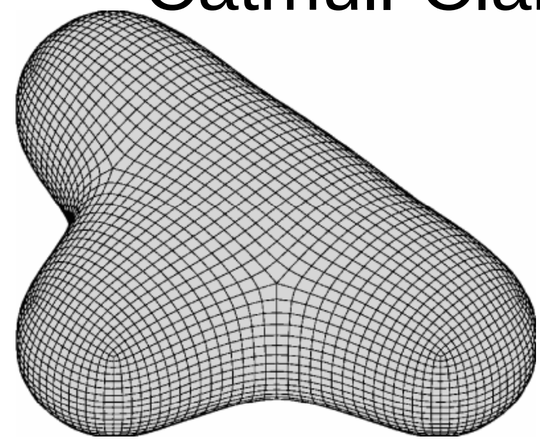
Comparison



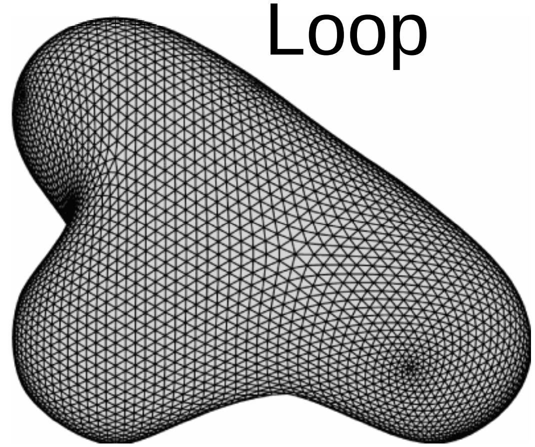
Doo-Sabin



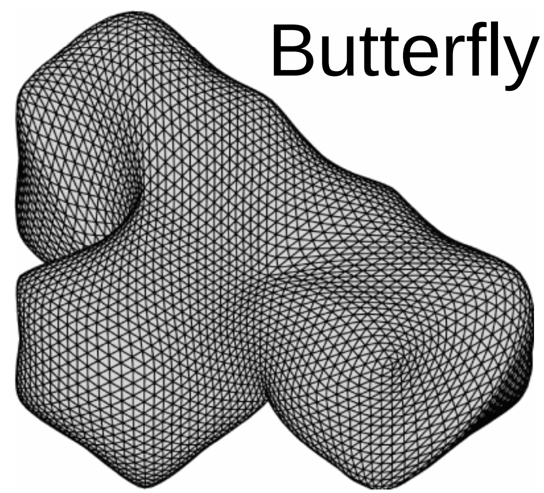
Catmull-Clark



Loop

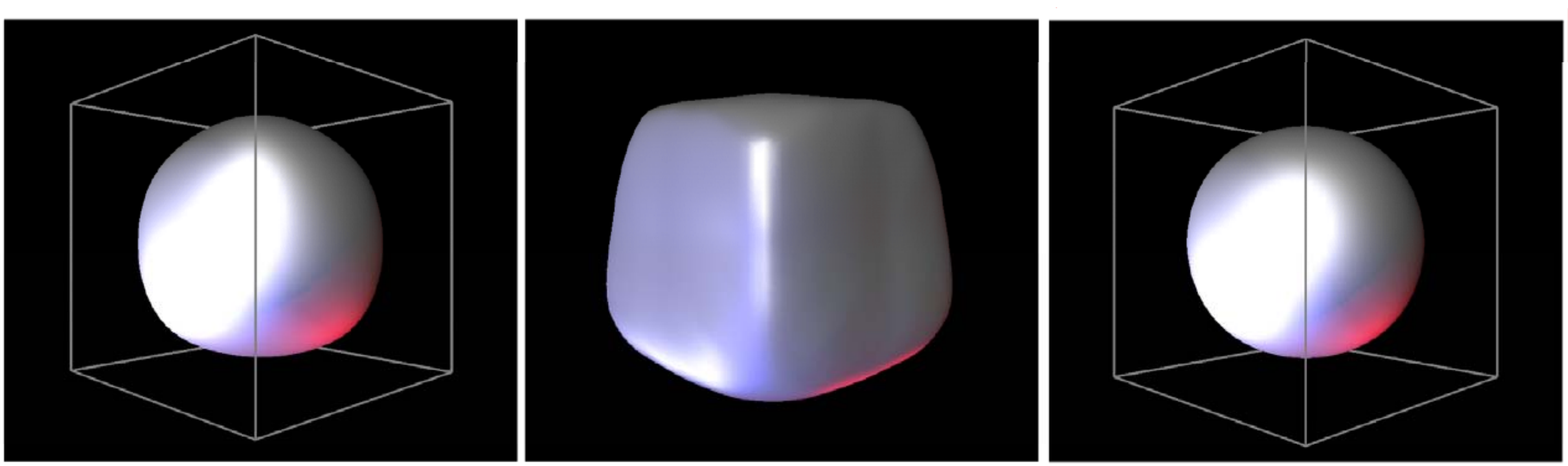


Butterfly



Comparison

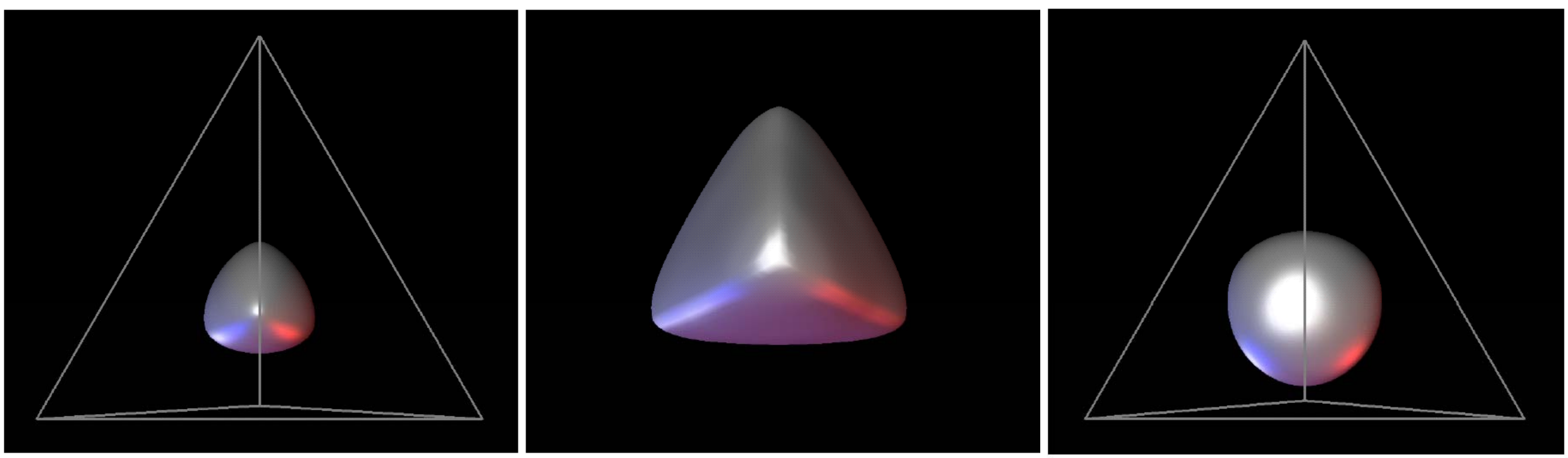
- Subdividing a cube
  - Loop result is assymetric, because cube was triangulated first
  - Both Loop and Catmull-Clark are better then Butterfly (C^2 vs. C^1)
  - Interpolation vs. smoothness



Loop *Butterfly* *Catmull-Clark*

Comparison

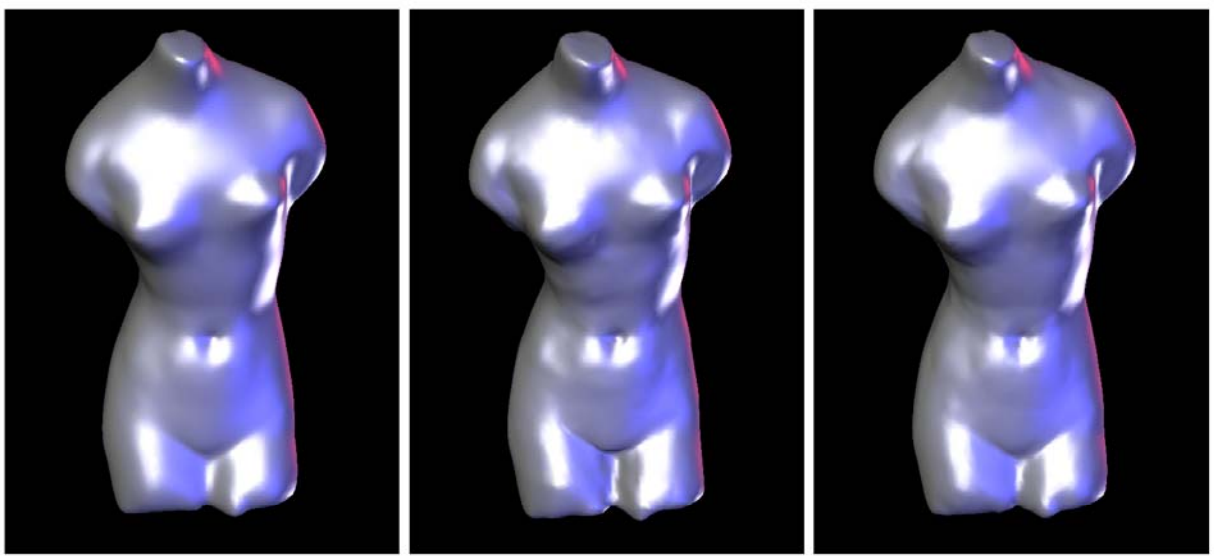
- Subdividing a tetrahedron
  - Same insights
  - Severe shrinking for approximating schemes



Loop *Butterfly* *Catmull-Clark*

Comparison

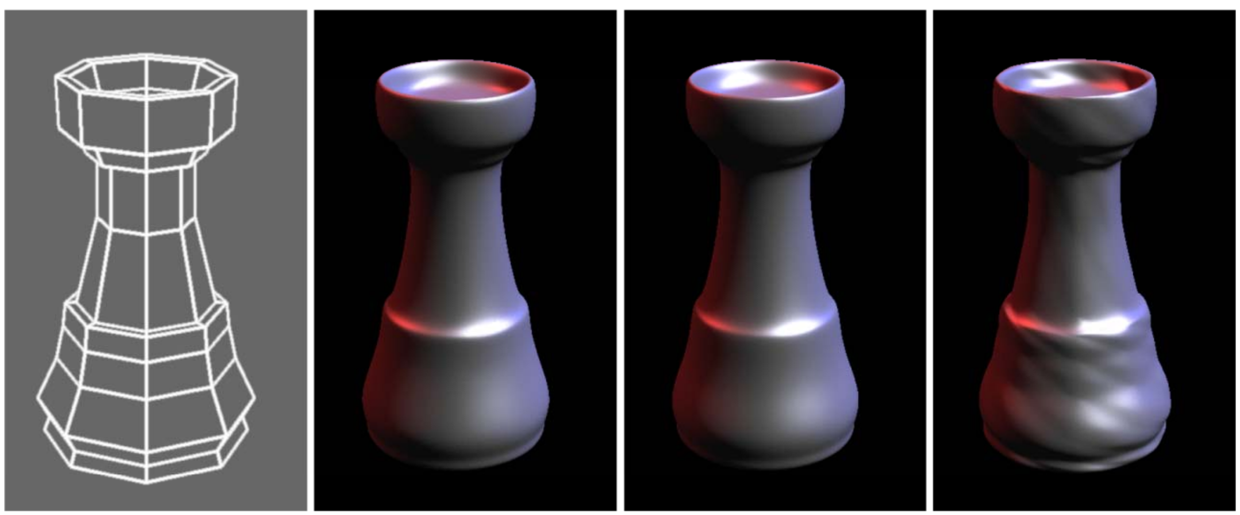
- Spot the difference?
- For smooth meshes with uniform triangle size, different schemes provide very similar results
- Beware of interpolating schemes for control polygons with sharp features



Loop *Butterfly* *Catmull-Clark*

So Who Wins?

- Loop and Catmull-Clark best when interpolation is not required
- Loop best for triangular meshes
- Catmull-Clark best for quad meshes
  - Don’t triangulate and then use Catmull-Clark



Initial mesh *Loop* *Catmull-Clark* *Catmull-Clark,after triangulation*

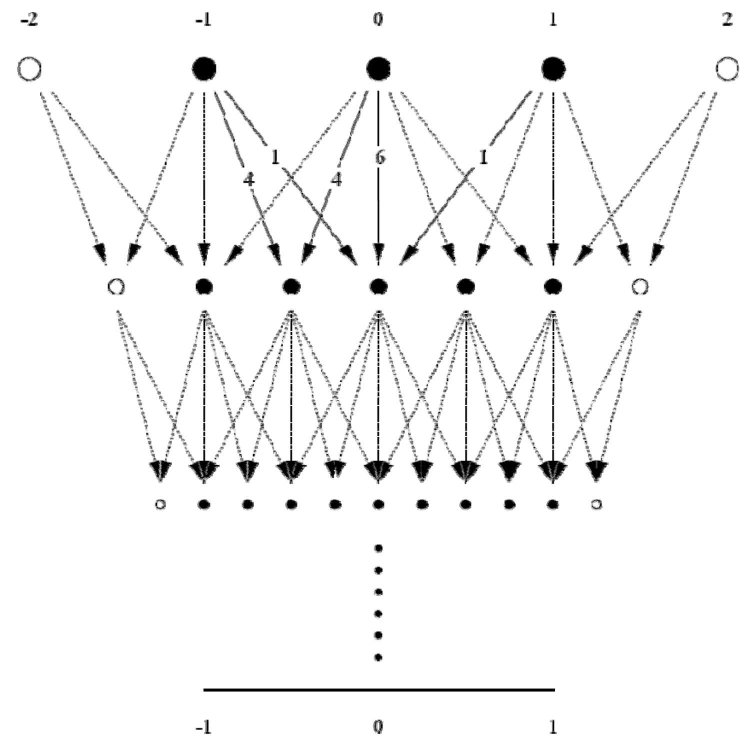
Analysis of Subdivision

- Invariant neighborhoods
  - How many control-points affect a small neighborhood around a point ?
- Subdivision scheme can be analyzed by looking at a *local* subdivision matrix

Local Subdivision Matrix

- Example: Cubic B-Splines

$$\begin{pmatrix} \mathbf{p}_{-2}^{j+1} \\ \mathbf{p}_{-1}^{j+1} \\ \mathbf{p}_{0}^{j+1} \\ \mathbf{p}_{1}^{j+1} \\ \mathbf{p}_{2}^{j+1} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p}_{-2}^{j} \\ \mathbf{p}_{-1}^{j} \\ \mathbf{p}_{0}^{j} \\ \mathbf{p}_{1}^{j} \\ \mathbf{p}_{2}^{j} \end{pmatrix}$$



- Invariant neighborhood size: 5

Analysis of Subdivision

- Analysis via eigen-decomposition of matrix S
  - Compute the eigenvalues

$$\{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$$

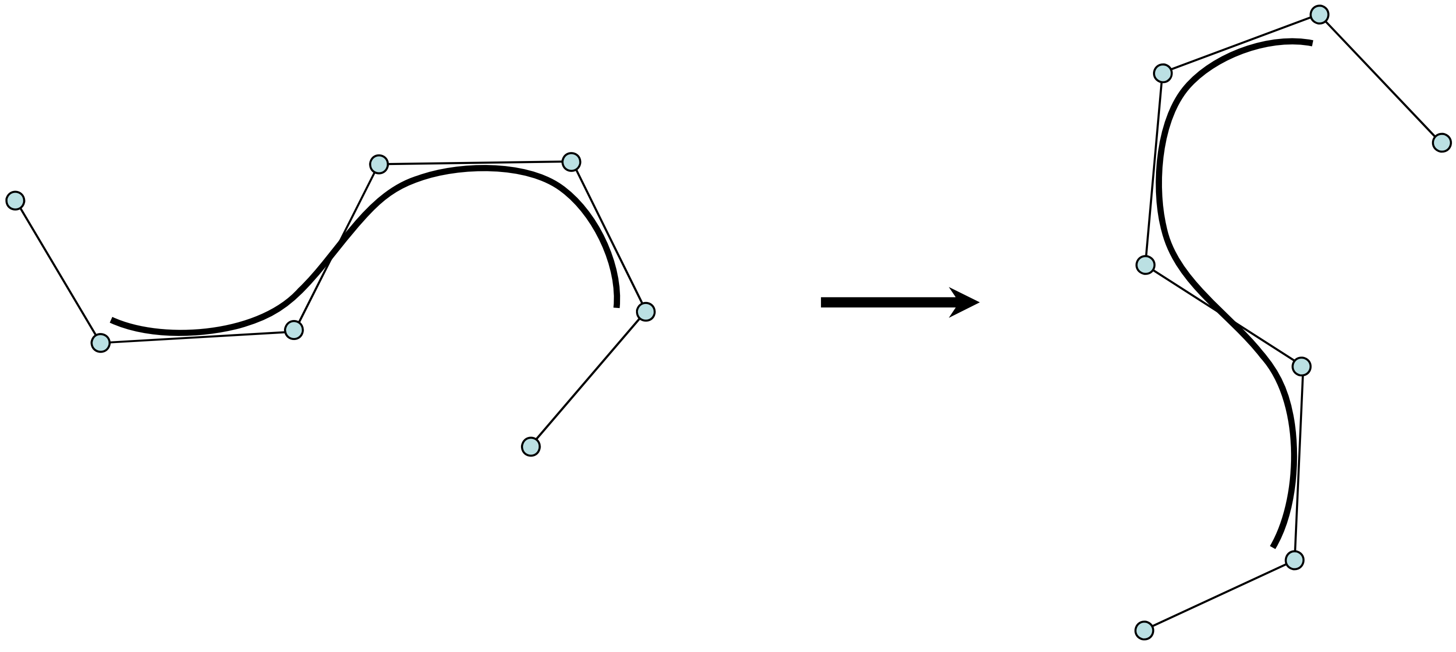
  - and eigenvectors

$$X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$$

  - Let $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ be real and X a complete set of eigenvectors

Analysis of Subdivision

- Invariance under affine transformations
  - transform(subdivide(P)) = subdivide(transform(P))



Analysis of Subdivision

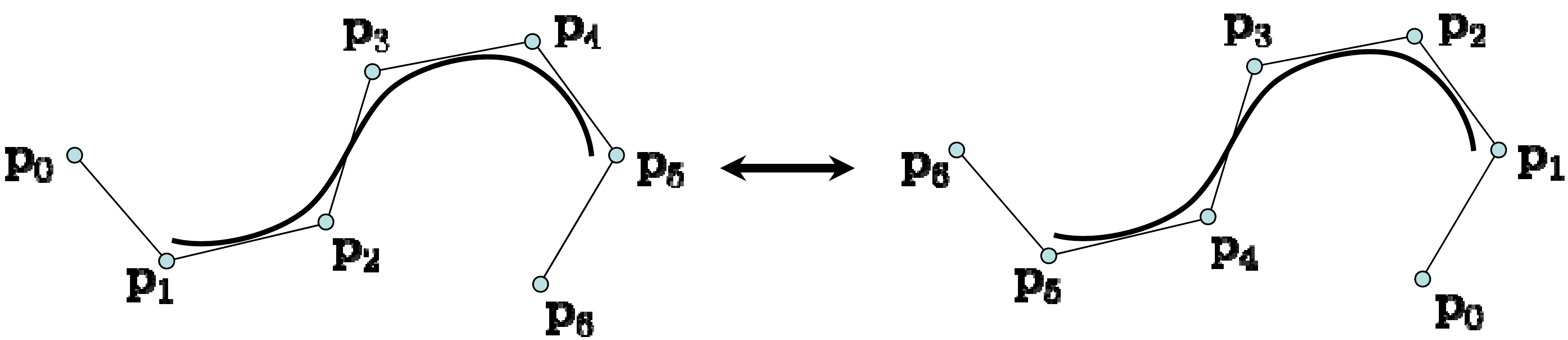
- Invariance under affine transformations
  - transform(subdivide(P)) = subdivide(transform(P))

- Rules have to be affine combinations
  - Even and odd weights each sum to 1

$$\sum_j S_{2i,j} = \sum_j S_{2i+1,j} = 1$$

Analysis of Subdivision

- Invariance under reversion of point ordering
- Subdivision rules (matrix rows) have to be symmetric



Analysis of Subdivision

Conclusion: 1 has to be eigenvector of S with eigenvalue $\lambda_0=1$

Limit Behavior - Position

- Any vector is linear combination of eigenvectors:

$$\mathbf{p} = \sum_{i=0}^{n-1} a_i \mathbf{x}_i \qquad a_i = \tilde{\mathbf{x}}_i^T \mathbf{p}$$

rows of X^{-1}

- Apply subdivision matrix:

$$S\mathbf{p}^0 = S\sum_{i=0}^{n-1} a_i \mathbf{x}_i = \sum_{i=0}^{n-1} a_i S\mathbf{x}_i = \sum_{i=0}^{n-1} a_i \lambda_i \mathbf{x}_i$$

Limit Behavior - Position

- For convergence we need $1 = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1}$
- Limit vector:

$$\mathbf{p}^{\infty} = \lim_{j \to \infty} S^j \mathbf{p}^0 = \lim_{j \to \infty} \sum_{i=0}^{n-1} a_i \lambda_i^j \mathbf{x}_i = a_0 \cdot \mathbf{1}$$

$$\mathbf{p}_i^{\infty} = a_0 = \tilde{\mathbf{x}}_0^T \mathbf{p}^j$$ independent of *j* !

Limit Behavior - Tangent

- Set origin at a_0:

$$\mathbf{p}^j = \sum_{i=1}^{n-1} a_i \lambda_i^j \mathbf{x}_i$$

- Divide by λ_1^j

$$\frac{1}{\lambda_1^j}\mathbf{p}^j = a_1\mathbf{x}_1 + \sum_{i=2}^{n-1} a_i \left(\frac{\lambda_i}{\lambda_1}\right)^j \mathbf{x}_i$$

- Limit tangent given by:

$$\mathbf{t}_i^\infty = a_1 = \tilde{\mathbf{x}}_1^T \mathbf{p}^j$$

Limit Behavior - Tangent

- Curves:
  - All eigenvalues of S except $\lambda_0=1$ should be less than λ_1 to ensure existence of a tangent, i.e.

$$1 = \lambda_0 > \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_{n-1}$$

- Surfaces:
  - Tangents determined by λ_1 and λ_2

$$1 = \lambda_0 > \lambda_1 = \lambda_2 > \lambda_3 \geq \cdots \geq \lambda_{n-1}$$

Example: Cubic Splines

- Subdivision matrix & rules

$$S = \frac{1}{8}\begin{pmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{pmatrix} \qquad \begin{aligned} \mathbf{p}_{2i}^{j+1} &= \frac{1}{8}\mathbf{p}_{i-1}^{j} + \frac{6}{8}\mathbf{p}_{i}^{j} + \frac{1}{8}\mathbf{p}_{i+1}^{j} \\ \mathbf{p}_{2i+1}^{j+1} &= \frac{1}{2}\mathbf{p}_{i}^{j} + \frac{1}{2}\mathbf{p}_{i+1}^{j} \end{aligned}$$

- Eigenvalues

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4,) = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

Example: Cubic Splines

- Eigenvectors

$$X = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 0 & -\frac{1}{11} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad X^{-1} = \begin{pmatrix} 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 \\ 0 & -1 & 0 & 1 & 0 \\ & & \dots & & \\ & & \dots & & \\ & & \dots & & \end{pmatrix}$$

- Limit position and tangent

$$\mathbf{p}_i^\infty = \tilde{\mathbf{x}}_0^T \mathbf{p}^j = \frac{1}{6}\left(\mathbf{p}_{i-1}^j + 4\mathbf{p}_i^j + \mathbf{p}_{i+1}^j\right)$$

$$\mathbf{t}_i^\infty = \tilde{\mathbf{x}}_1^T \mathbf{p}^j = \mathbf{p}_{i+1}^j - \mathbf{p}_i^j$$

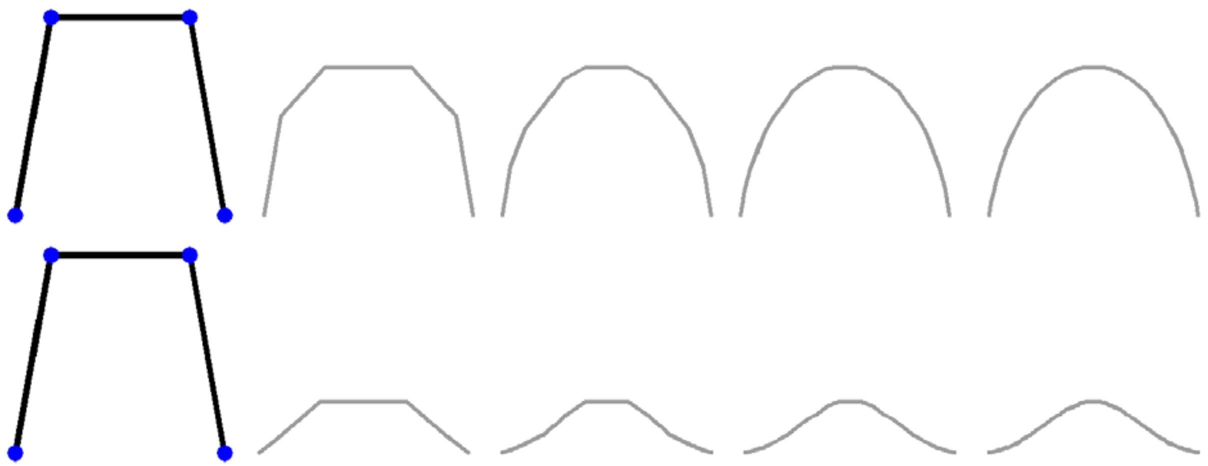
Properties of Subdivision

- Flexible modeling
  - Handle surfaces of arbitrary topology
  - Provably smooth limit surfaces
  - Intuitive control point interaction
- Scalability
  - Level-of-detail rendering
  - Adaptive approximation
- Usability
  - Compact representation
  - Simple and efficient code

Beyond Subdivision Surfaces

- Non-linear subdivision [Schaefer et al. 2008]

  Idea: replace arithmetic mean with other function

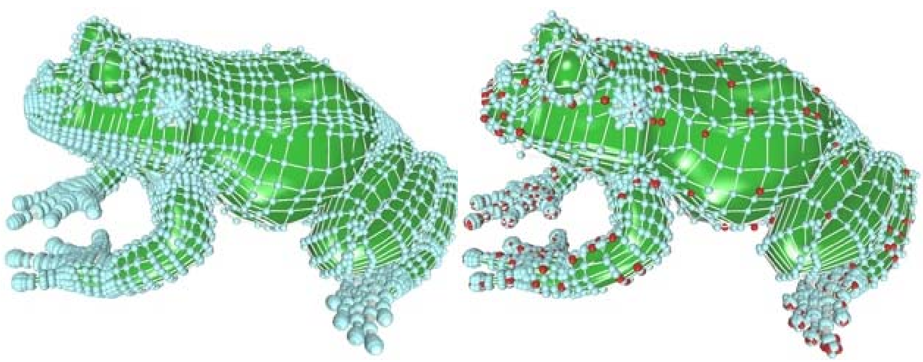


de Casteljau with $\frac{a+b}{2}$

de Casteljau with $\sqrt{ab}$

Beyond Subdivision Surfaces

- T-Splines [Sederberg et al. 2003]
  – Allows control points to be *T-junctions*
  – Can use less control points
  – Can model different topologies with single surface



NURBS T-Splines

NURBS T-Splines

Beyond Subdivision Surfaces

- How do you subdivide a teapot?