Parameterization I
Problem Definition

Given a surface (mesh) $S$ in $\mathbb{R}^3$ and a domain $\Omega$
find a bijective $F: \Omega \leftrightarrow S$
Typical Domains

disk = genus zero + boundary

sphere = closed genus zero
Cutting to a Disk

Creates artificial boundary
Applications

• Texture Mapping
Texture Mapping
Applications

- Normal Mapping

4M faces

8K faces

8K faces, normal-mapped

normal-map
Applications

- Normal Mapping
Applications

- Remeshing
Remeshing
Applications

• Shape interpolation
Applications

- Compression

Stanford Bunny
Desirable Properties

• Low distortion
• Bijective mapping
• Efficiently computable
Unfolding the World
Spherical Coordinates

\[ \theta \in [0, 2\pi), \phi \in [-\pi / 2, \pi / 2) \]

\[ x(\theta, \phi) = R \cos \theta \cos \phi \]

\[ y(\theta, \phi) = R \sin \theta \cos \phi \]

\[ z(\theta, \phi) = R \sin \phi \]
Definitions

• $f$ is **isometric** (length preserving), if the *length* of any curve on $S$ is preserved on $S^*$.  

• $f$ is **conformal** (angle preserving), if the *angle* of intersection of every pair of intersecting curves on $S$ is preserved on $S^*$.  

• $f$ is **equiareal** (area preserving) if the *area* of an area element on $S$ is preserved on $S^*$.  


Standard Map Projections

- **Orthographic**
  - preserves angles = **conformal**

- **Stereographic**
  - preserves angles = **conformal**

- **Mercator**
  - preserves area = **equiareal**

- **Lambert**
  - preserves area = **equiareal**
More Maps

- Mollweide-Projektion
- Mercator-Projektion
- Zylinderprojektion nach Miller
- Hammer-Aitoff-Projektion
- Peters-Projektion
- Längentreue Azimuthalprojektion
- Stereographische Projektion
- Behrman-Projektion
- Senkrechte Umgebungsprojektion
- Robinson-Projektion
- Hotine-Oblique Mercator-Projektion
- Sinusoidal-Projektion
- Gnomonische Projektion
- Flächenstreue Kegelprojektion
- Transverse Mercator-Projektion
- Cassini-Soldner-Projektion
Conformal Map

Similarity = Rotation + Scale
Preserves angles
Conformally flattened 2D mesh

Input
3D mesh

Output

similarity
Conformal Parameterization
Conformal

Minimal Stretch
Differential Geometry Revisited

• Parametric surface:

\[ \mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)) \]

• *Regular* if…
  - \( x_1(u, v), x_2(u, v), x_3(u, v) \) are smooth (differentiable)
  - tangent vectors are linearly independent

\[
\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u} \quad \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}
\]
Distortion Analysis

\[ \mathbf{X}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} \]

\[ J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} \]
Distortion Analysis

\[ d\mathbf{X} = J d\mathbf{U} \]

\[ |d\mathbf{X}|^2 = d\mathbf{U} J^T J d\mathbf{U} \]

\[ J^T J = \begin{pmatrix} x_u^T x_u & x_u^T x_v \\ x_v^T x_u & x_v^T x_v \end{pmatrix} = \mathbf{I} \quad \text{First fundamental form} \]
Fundamental Form Revisited

• Characterizes the surface locally

\[ I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \]

• Length element

\[ dx^2 = Edu^2 + 2Fdu\,d\nu + Gd\nu^2 \]

• Area element

\[ dA = \sqrt{EG - F^2} du\,d\nu \]
Mapping Surfaces

\[ f \] is allowable if \( x^* = f \circ x \) is regular
Isometric Maps

• **Theorem:** An allowable mapping $f$ from $S$ to $S^*$ is **isometric** (length-preserving), iff the first fundamental forms of $x$ and $x^* = f \cdot x$ are equal, i.e.

$$I = I^*$$

• Isometric surfaces have the same Gaussian curvature at corresponding pairs of points!
Conformal Maps

- **Theorem:** An allowable mapping $f$ from $S$ to $S^*$ is **conformal**, iff the first fundamental forms of $x$ and $x^* = f \cdot x$ are proportional, i.e., there exists a positive scalar function $n$, such that

\[ I = n(u, v)I^* \]

- A conformal map is always (locally) bijective.
Equiareal Maps

• Theorem: An allowable mapping $f$ from $S$ to $S^*$ is equiareal, iff the determinants of the first fundamental forms of $x$ and $x^* = f \cdot x$ are equal, i.e.,

$$det(I) = det(I^*)$$

• Area element:

$$dA = \sqrt{EG - F^2} \, du \, dv = \sqrt{det(I)} \, du \, dv$$
Relationships

• **Theorem**: Every isometric mapping is conformal and equiareal, and vice versa.

\[
\text{isometric} \iff \text{conformal} + \text{equiareal}
\]

• Isometric is ideal… but rare. In practice, we use:
  – conformal
  – equiareal
  – some balance between the two
Riemann Conformal Mapping Theorem

Any two simply connected compact planar regions can be mapped conformally onto each other.

If \((x, y) \rightarrow (u, v)\) is a conformal mapping, then \(u(x, y)\) and \(v(x, y)\) satisfy the Cauchy-Riemann equations:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

thus both \(u\) and \(v\) are harmonic:

\[
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0 \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = 0
\]
Harmonic Maps

• $f$ is harmonic if satisfies (for each coordinate):
  \[ \Delta_s f = 0 \]

• isometric $\iff$ conformal $\iff$ harmonic

• Minimizes the Dirichlet energy
  \[ E_D(f) = \frac{1}{2} \int_S \| \nabla_s f \|^2 \]

  given boundary conditions
Harmonic Maps

• Easier to compute than conformal, but does not preserve angles. May not be bijective.
Harmonic Maps

• **Theorem** [Rado-Kneser-Choquet]:
  If $f: S \to R^2$ is harmonic and maps the boundary $\partial S$ homeomorphically onto the boundary $\partial S^*$ of some **convex** region $S^* \subset R^2$, then $f$ is **bijective**.
Discrete Harmonic Maps

- Piecewise linear map for triangulated, disk-like surface onto planar polygon

\[
\sum_{(i,j) \in E} w_{ij} (v_i - v_j) = 0
\]
2D Barycentric Drawings

- Fix 2D boundary to a convex polygon.
- Define drawing as a solution of
  \[
  \begin{align*}
  Wx &= b_x \\
  Wy &= b_y \\
  w_{ij} &= \begin{cases} 
  > 0 & (i, j) \in E \\
  -\sum_{j \neq i} w_{ij} & (i, i), i \notin B \\
  1 & (i, i), i \in B \\
  0 & \text{otherwise}
  \end{cases}
  \end{align*}
  \]

  \( W \) is symmetric: \( w_{ij} = w_{ji} \)

- Weights \( w_{ij} \) control triangle shapes
Why it Works

• **Theorem** *(Maxwell-Tutte)*

  If $G = \langle V,E \rangle$ is a 3-connected planar graph (triangular mesh) then any **barycentric** drawing is a valid embedding.
Example

\[ w_{ij} = 1 \]

\[
W = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & -5 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & -5 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -5 & 0
\end{pmatrix}
\]

\[
b_x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \end{pmatrix}, \quad b_y = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}
\]
Spring System

- Represent as configuration of springs on mesh edges

\[ E(v) = \frac{1}{2} \sum_{(i,j) \in E} w_{ij} \| v_i - v_j \|^2 \]

- Minimum of \( E(v) \) reached when gradients = 0

\[ \frac{\partial E(v)}{\partial v_i} = \sum_{(i,j) \in E} w_{ij} (v_i - v_j) = 0 \]
Uniform Weights

\[ w_{ij} = 1 \]

- No shape information – “equilateral” triangles
- Fastest to compute and solve
- Not 2D reproducible
Harmonic Weights

\[ w_{ij} = \frac{\cot(\alpha_{ij}) + \cot(\beta_{ij})}{2} \]

- Weights can be negative – not always valid
- Weights depend only on angles - close to conformal
- 2D reproducible
Mean-Value Weights

\[ w_{ij} = \frac{\tan(\gamma_{ij}/2) + \tan(\delta_{ij}/2)}{2 \| v_i - v_j \|} \]

- Result visually similar to harmonic
- No negative weights – always valid
- 2D reproducible
General Method

Select normalized weights
\[ \lambda_{ij} = \frac{w_{ij}}{\sum_{k \in N_i} w_{ik}} \]

so that
\[ \sum_{j \in N_1(i)} \lambda_{ij} = 1 \]

Re-express Laplace equation
\[ \sum_{j \in N_1(i)} \omega_{ij} (f(v_j) - f(v_i)) = 0 \]

as weighted average constraints
\[ f(v_i) = \sum_{j \in N_1(i)} \lambda_{ij} (f(v_j)) \]

Then, if \( \omega_{ij} \) are positive, so are are \( \lambda_{ij} \).
Example

uniform

harmonic

mean-value
Fixing the Boundary

- Simple convex shape (triangle, square, circle)
- Distribute points on boundary
  - Use chord length parameterization
- Fixed boundary can create high distortion
Non-Convex Boundary

- Convex boundary creates significant distortion.

- "Free" boundary is better.