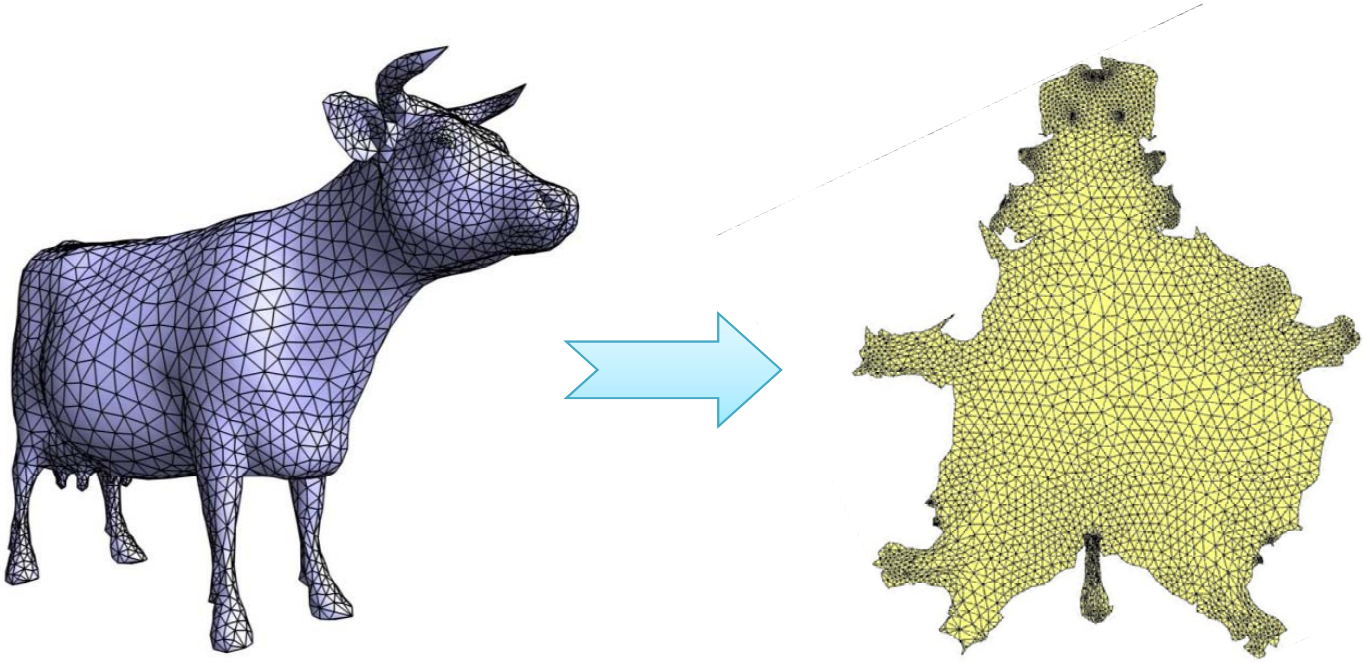
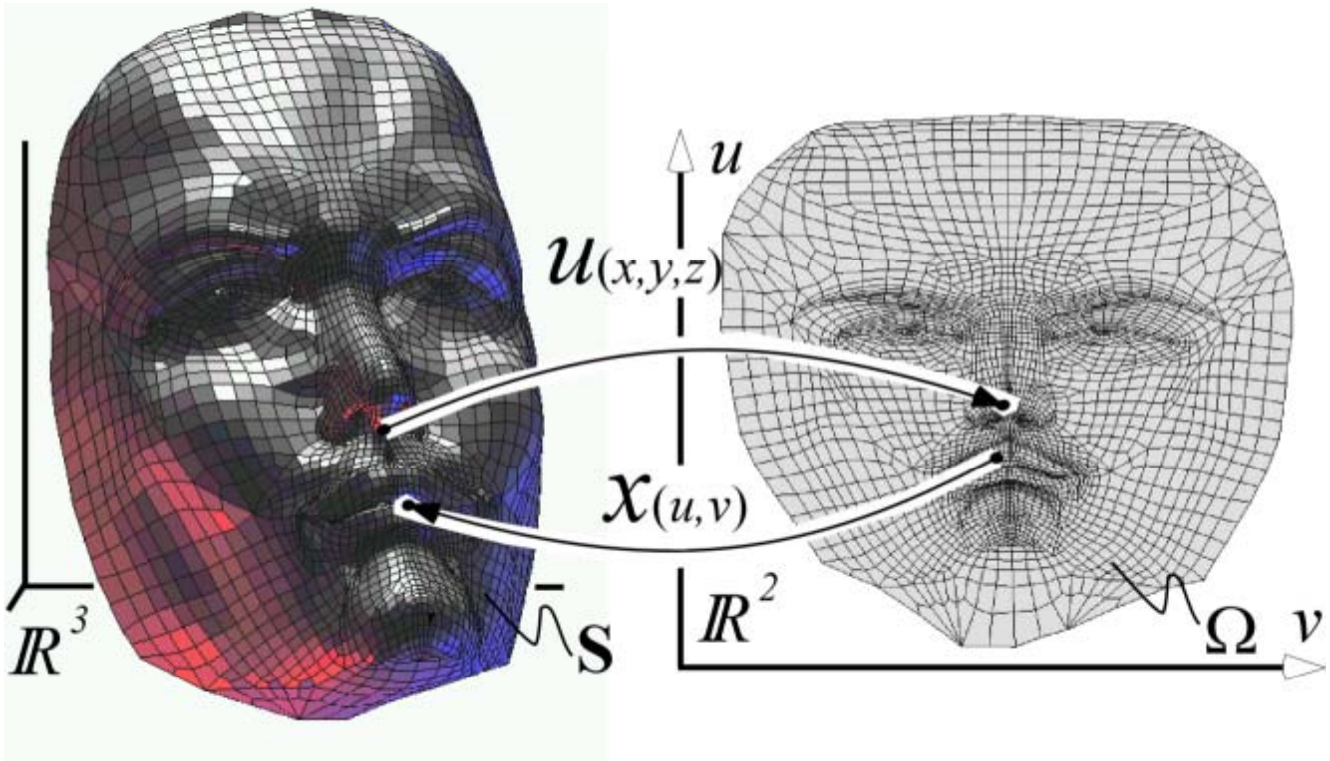
Parameterization I



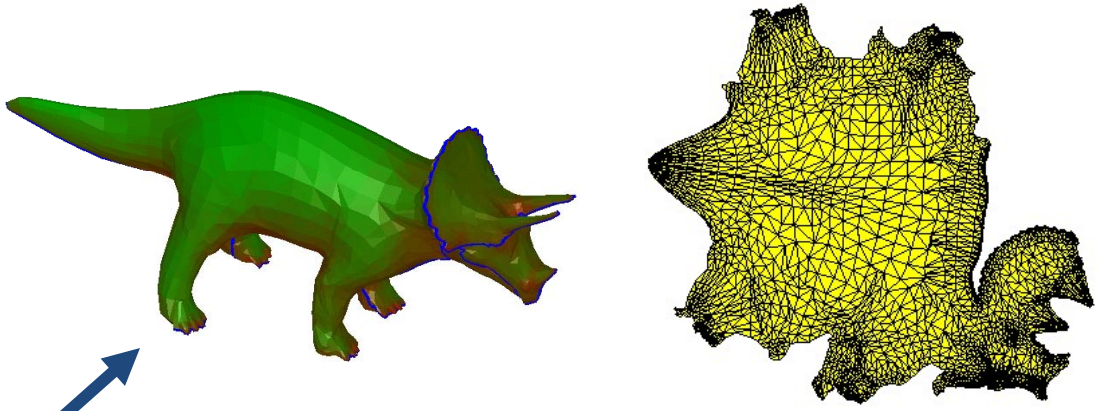
Problem Definition

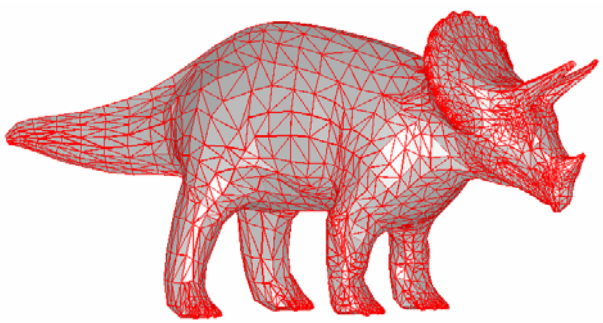
Given a surface (mesh) S in R^3 and a domain Ω
find a bijective F: $\Omega \leftrightarrow S$

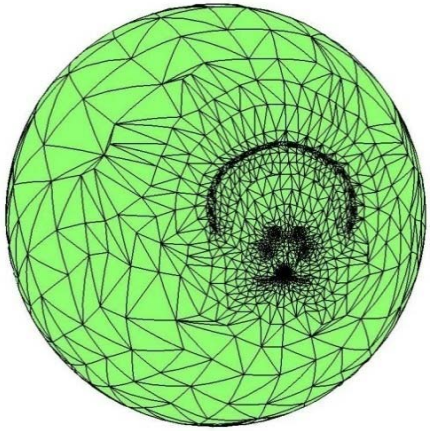


Typical Domains

disk = genus zero +
boundary

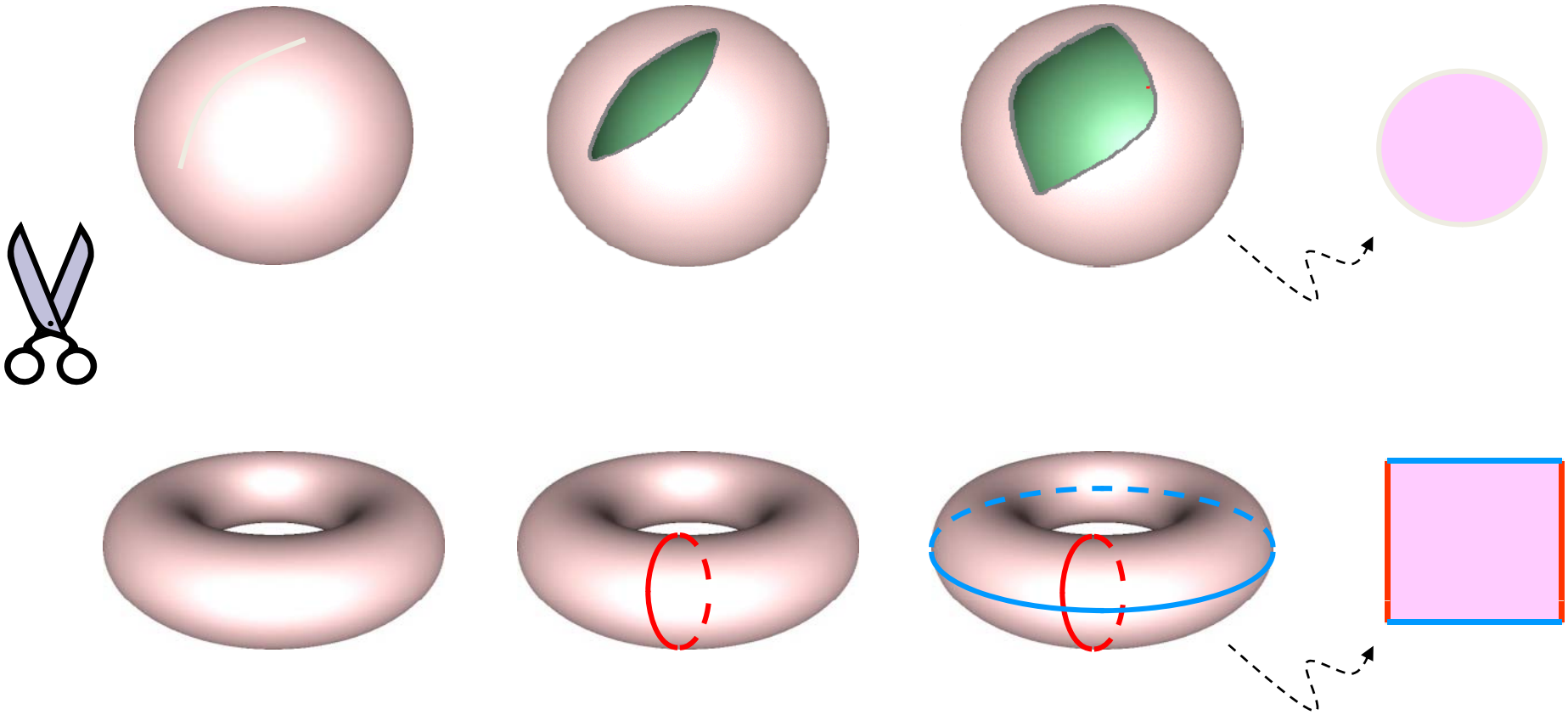






sphere = closed
genus zero

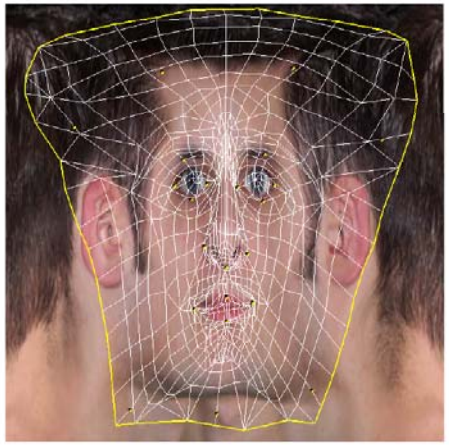
Cutting to a Disk

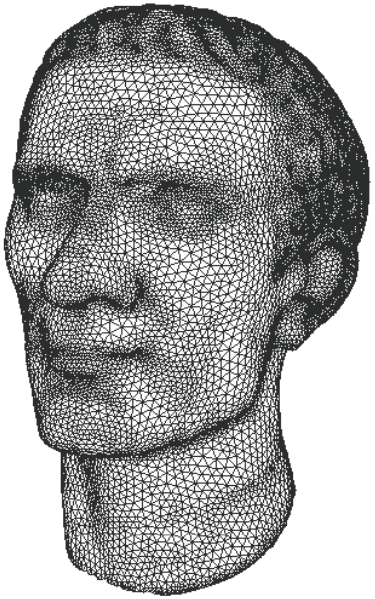


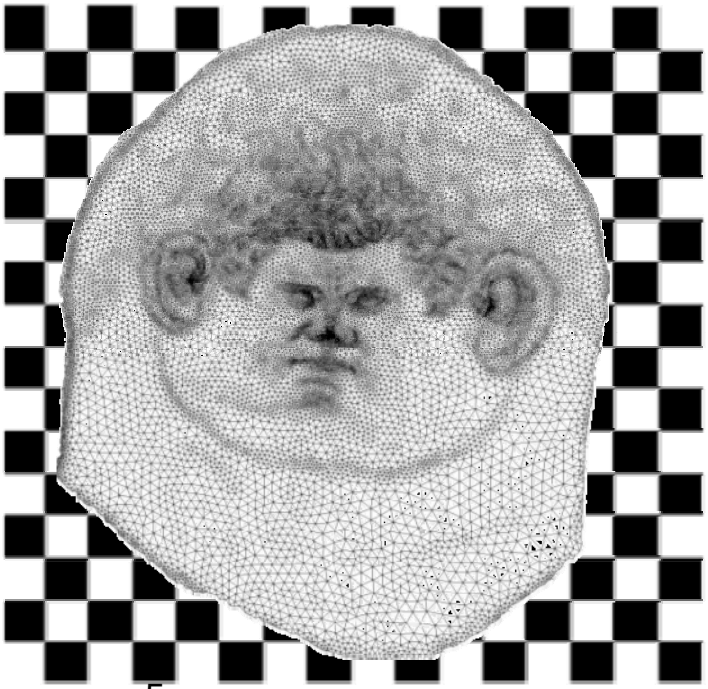
Creates artificial boundary

Applications

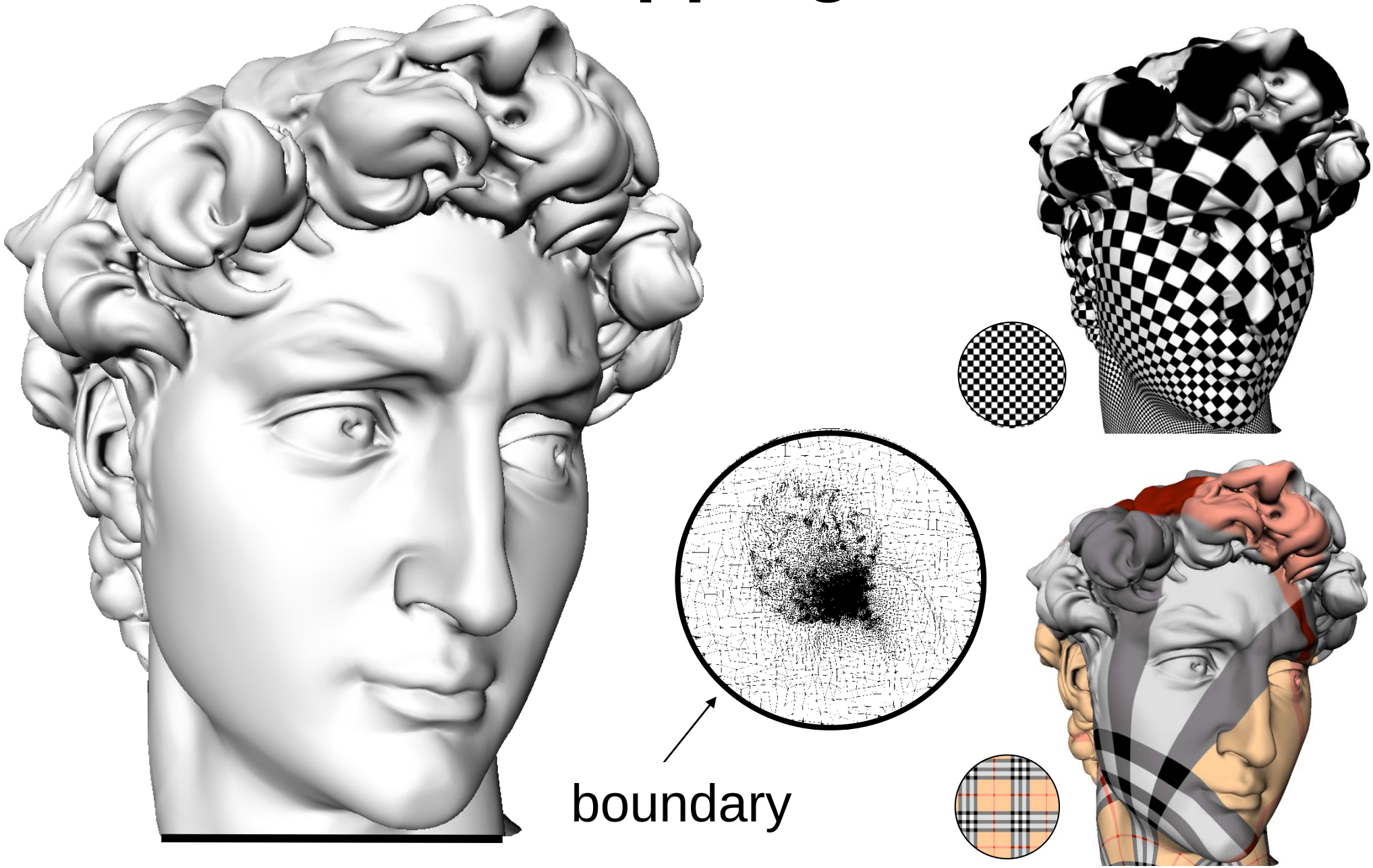
- Texture Mapping

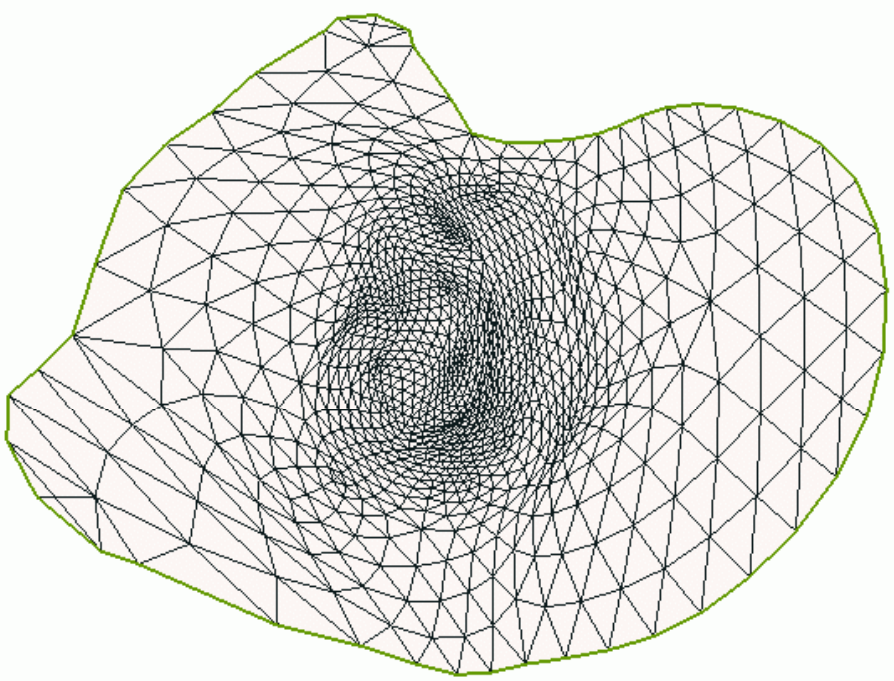


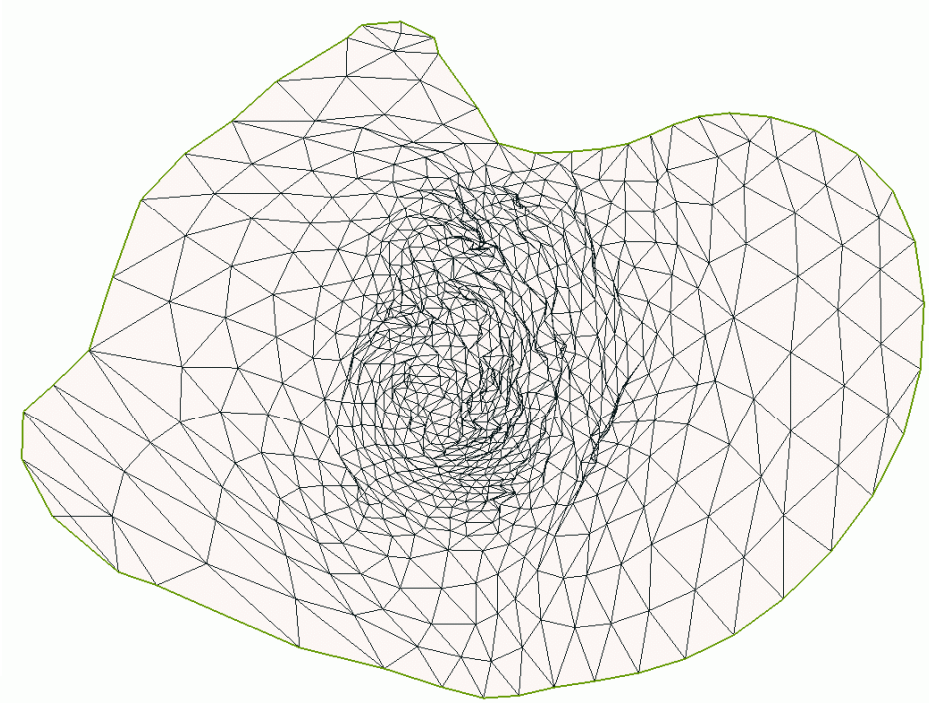


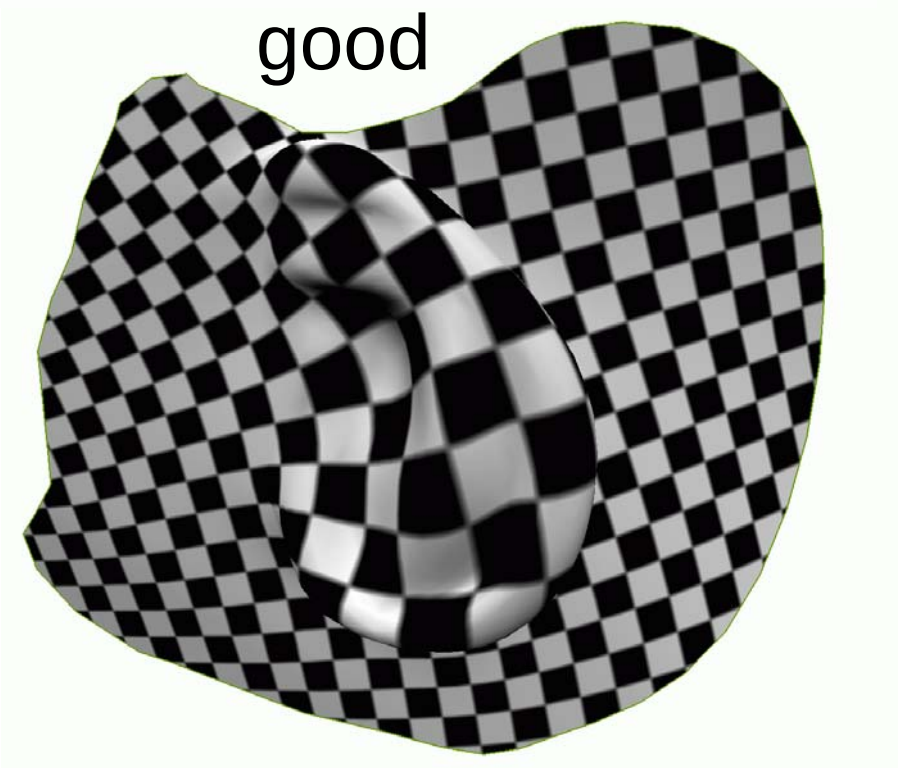


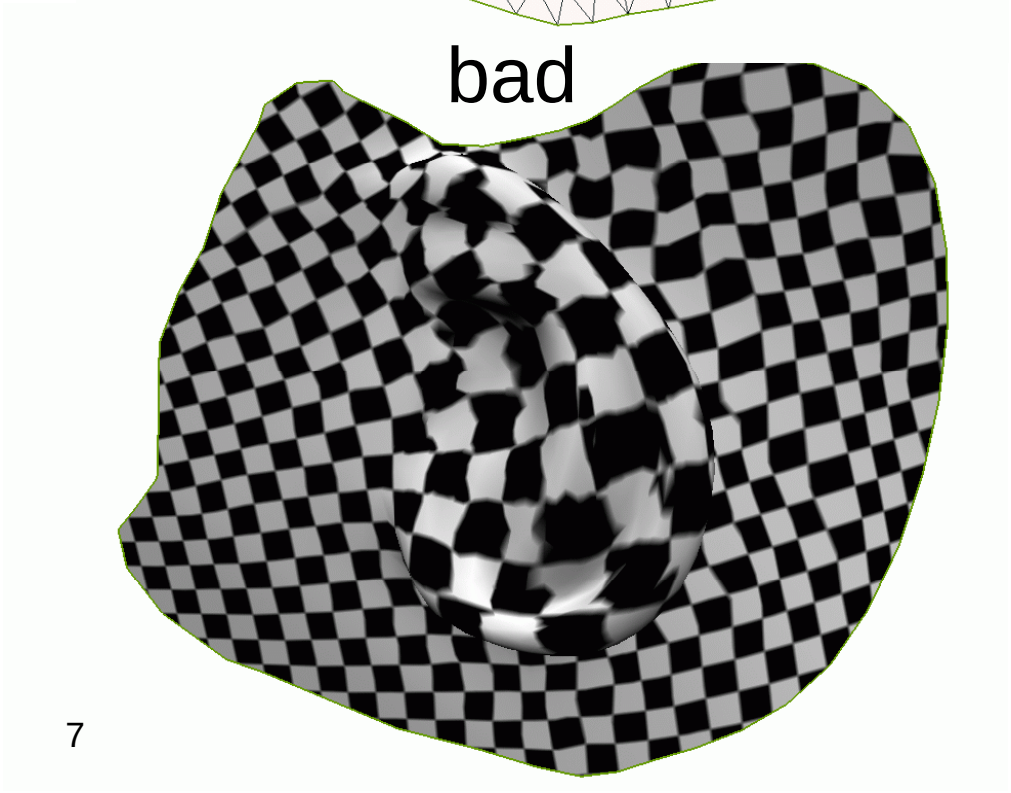
Texture Mapping





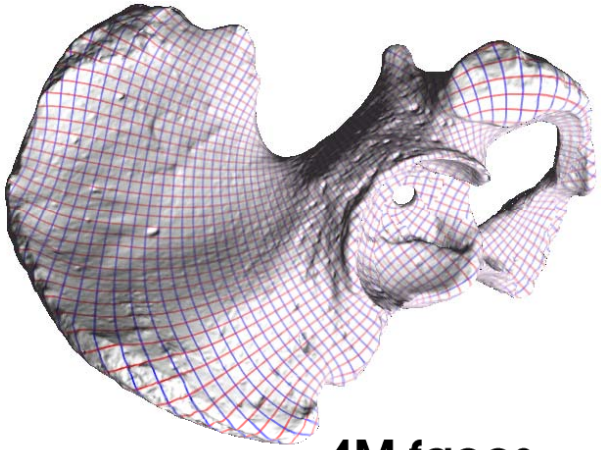



good

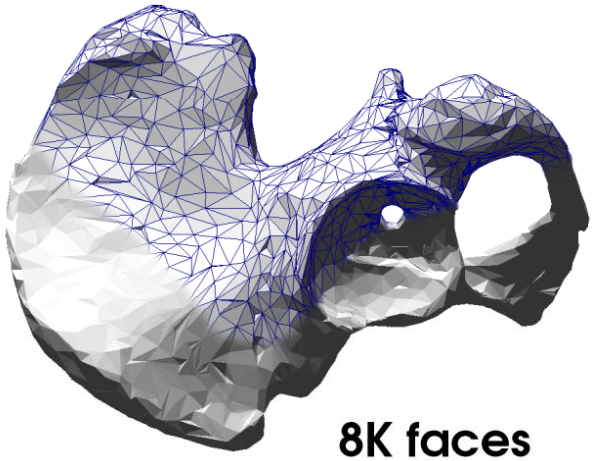

bad

Applications

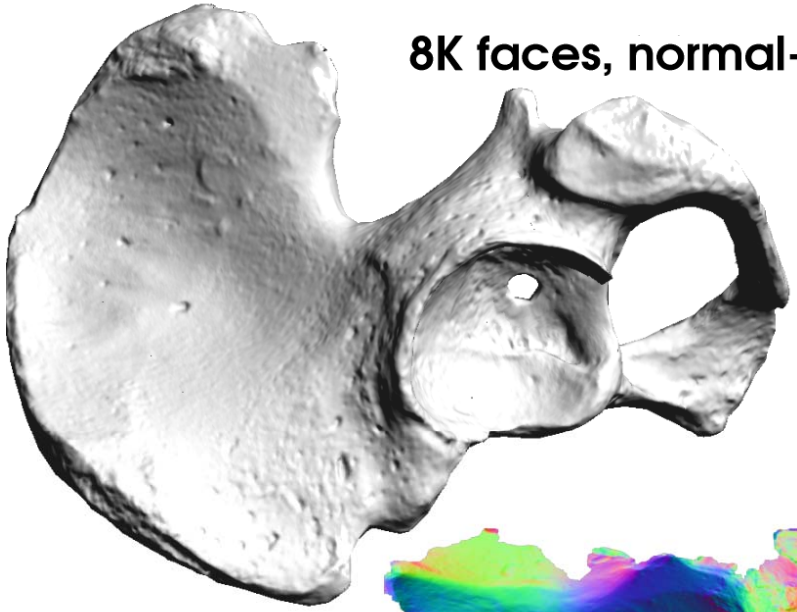
- Normal Mapping



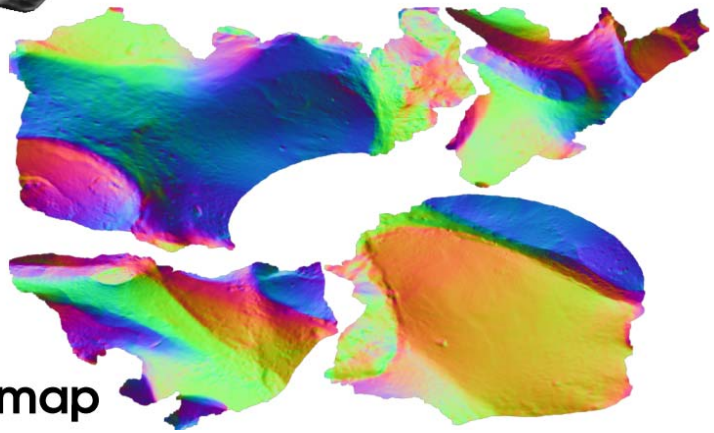
4M faces



8K faces



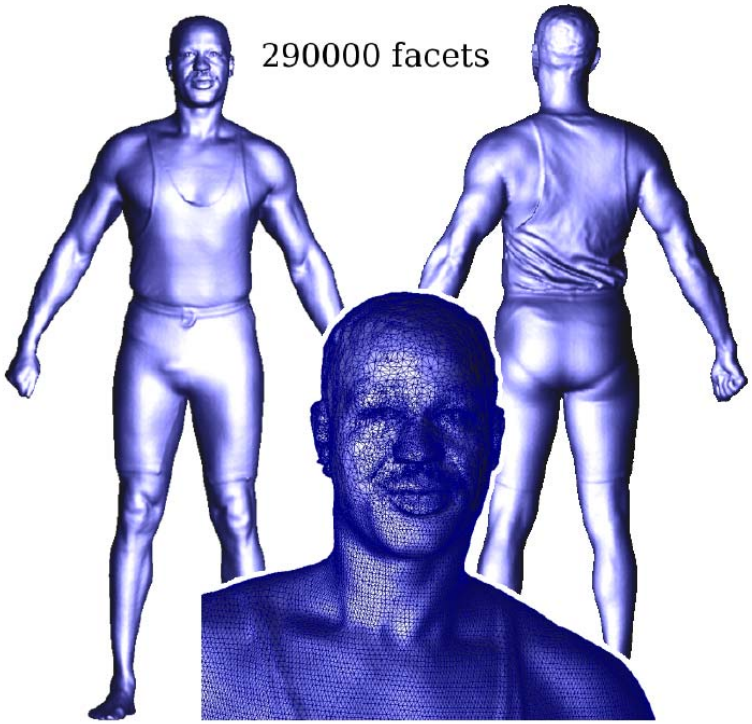
8K faces, normal-mapped

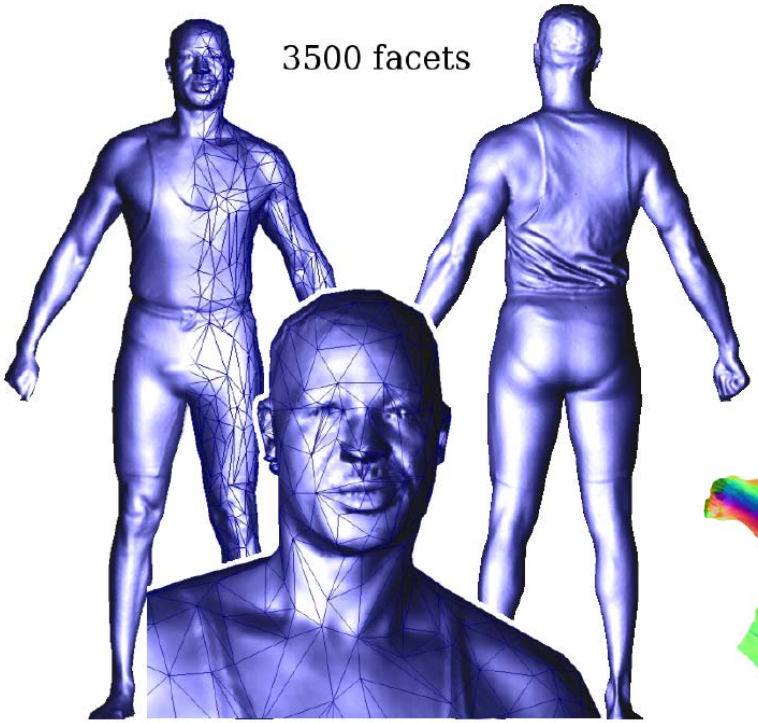


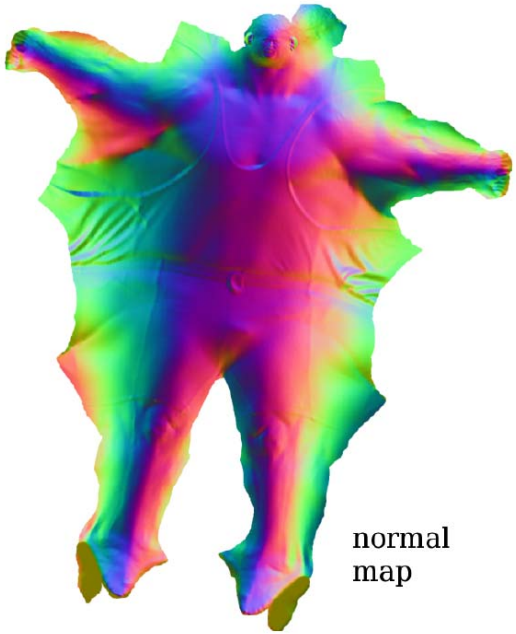
normal-map

Applications

- Normal Mapping

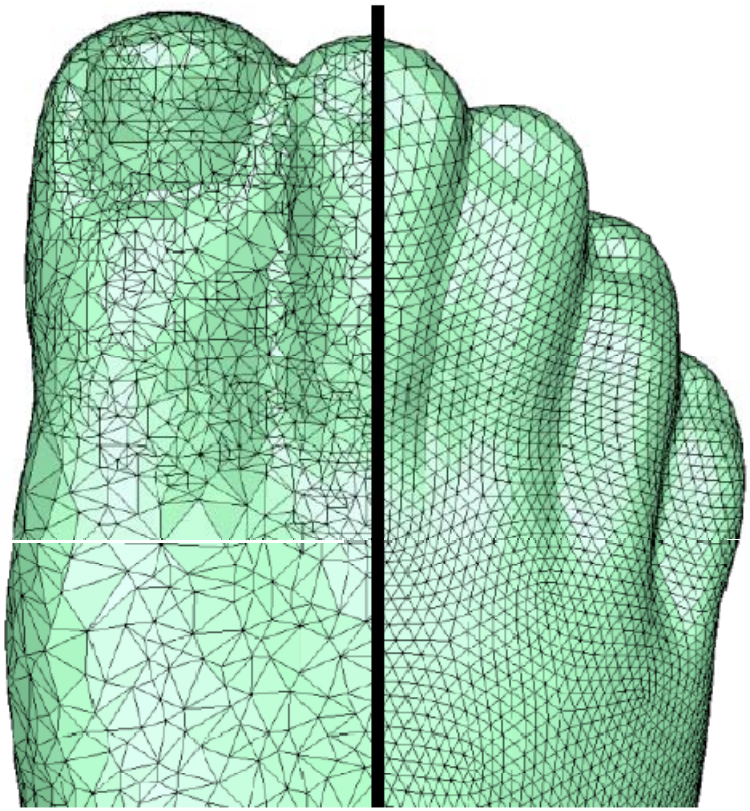


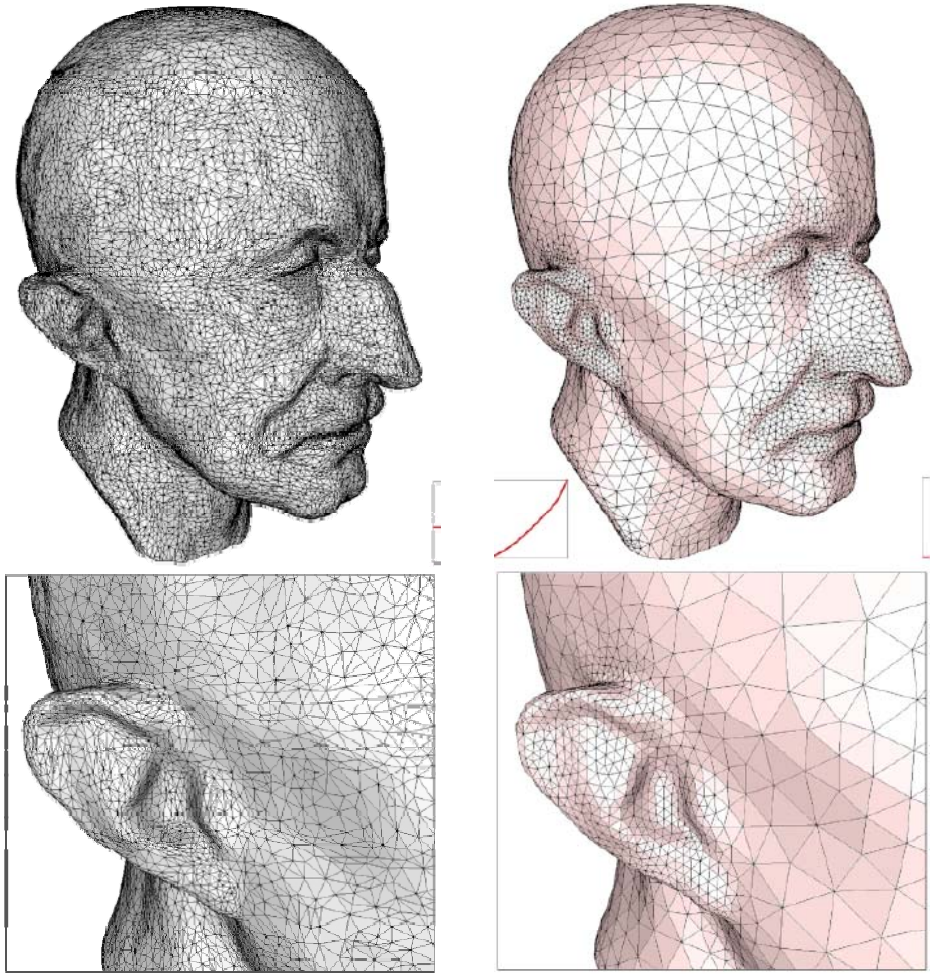




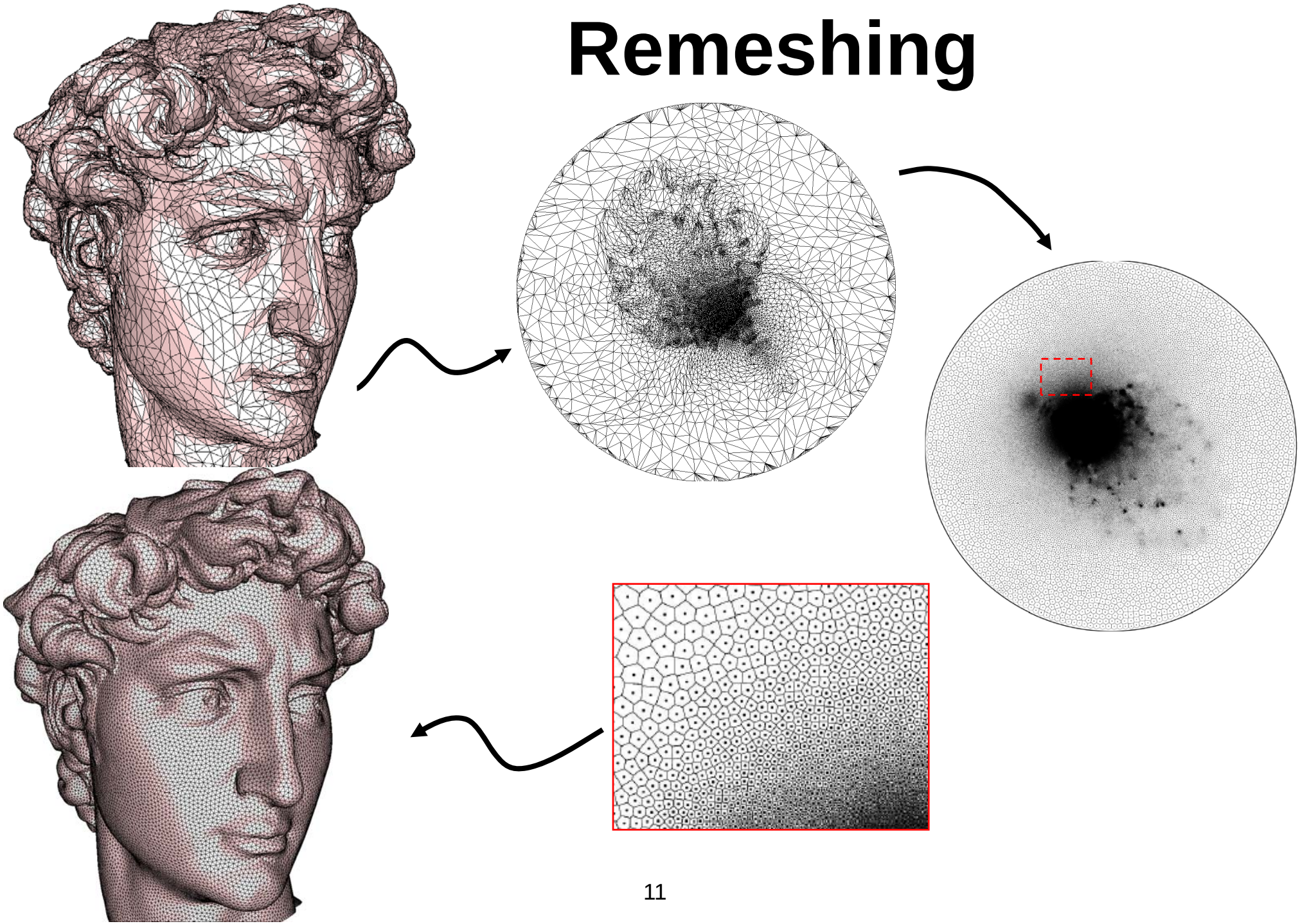
Applications

- Remeshing



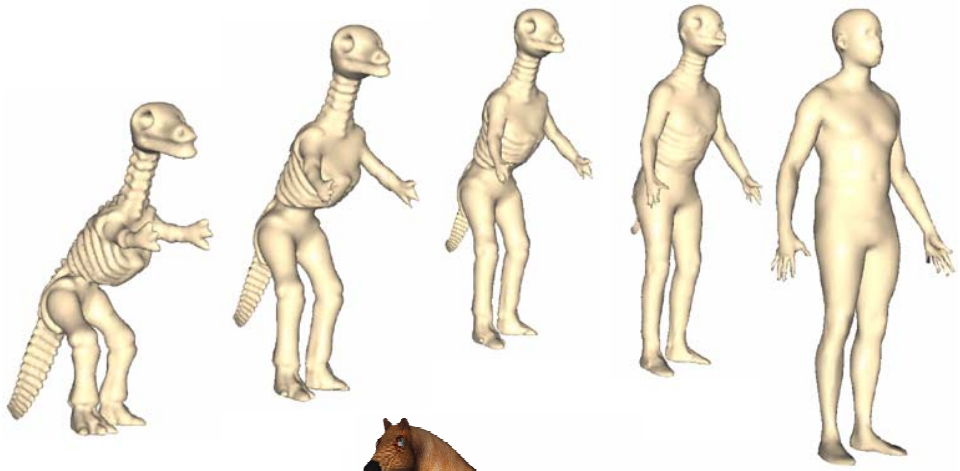


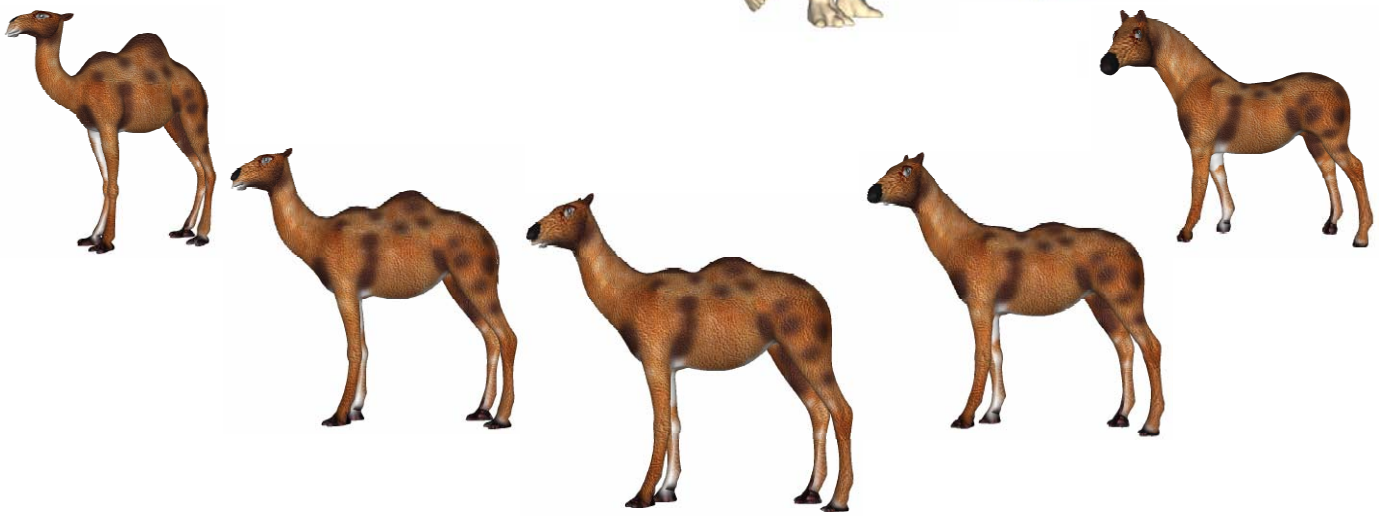
Remeshing



Applications

- Shape interpolation

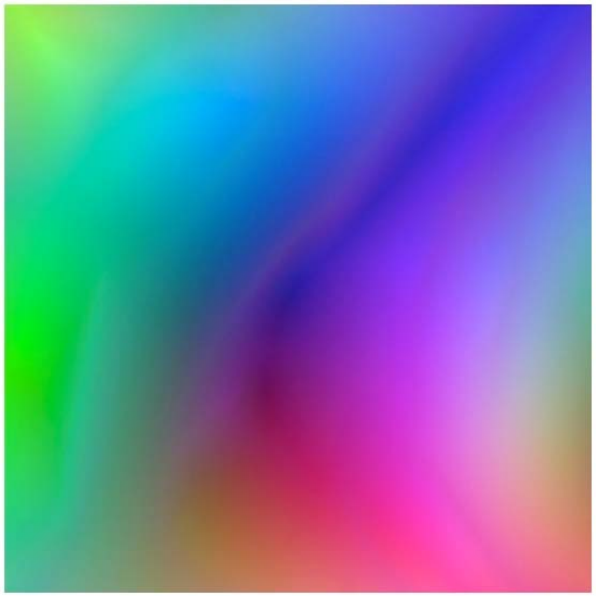




Movie

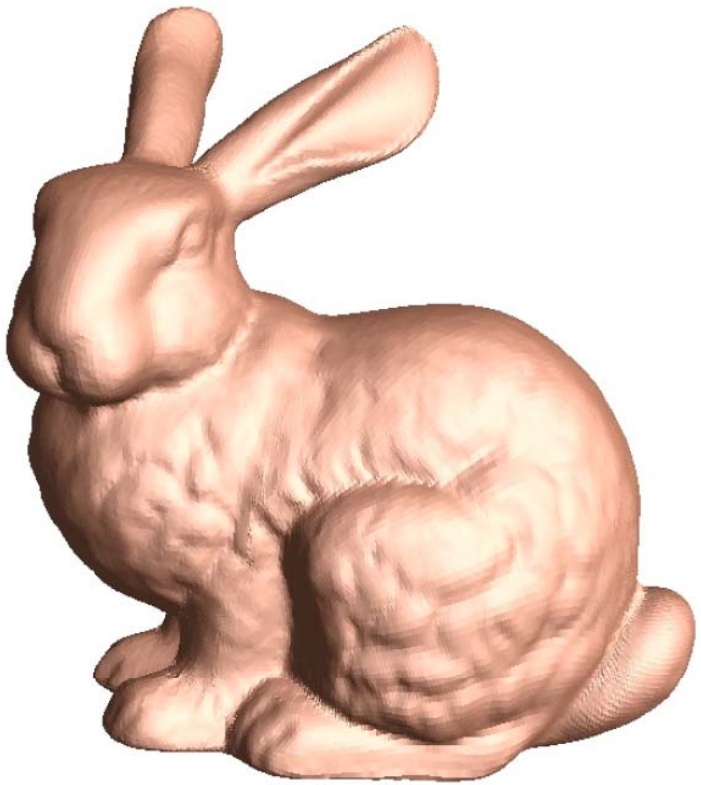
Applications

- Compression



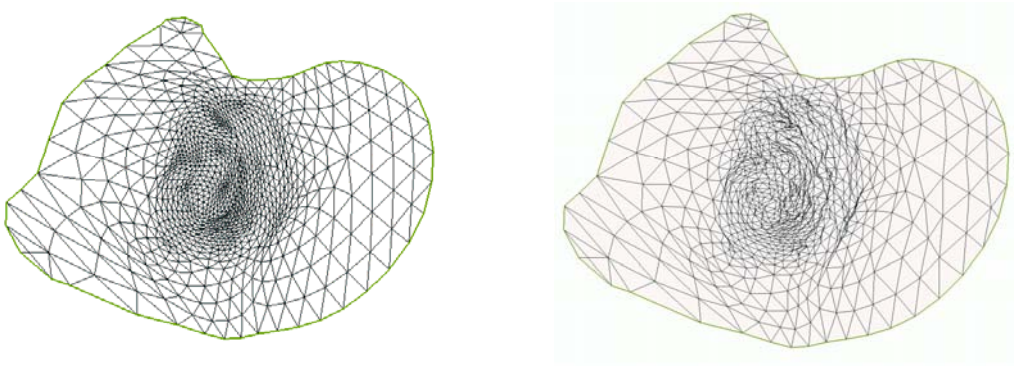
Stanford Bunny

=



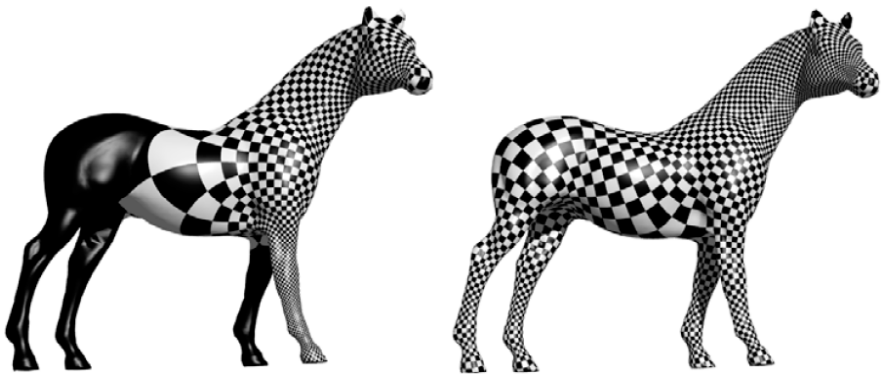
Desirable Properties

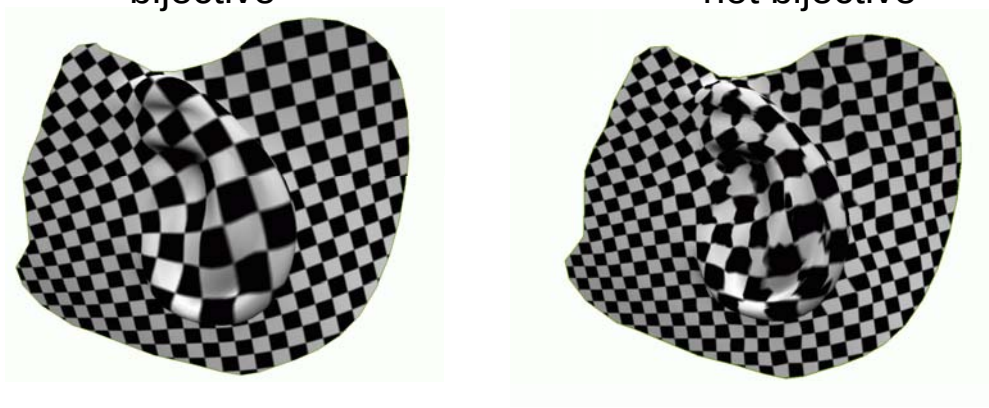
- Low distortion
- Bijective mapping
- Efficiently computable

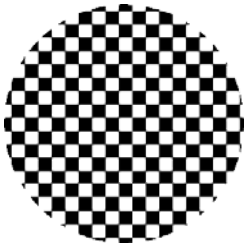


bijective

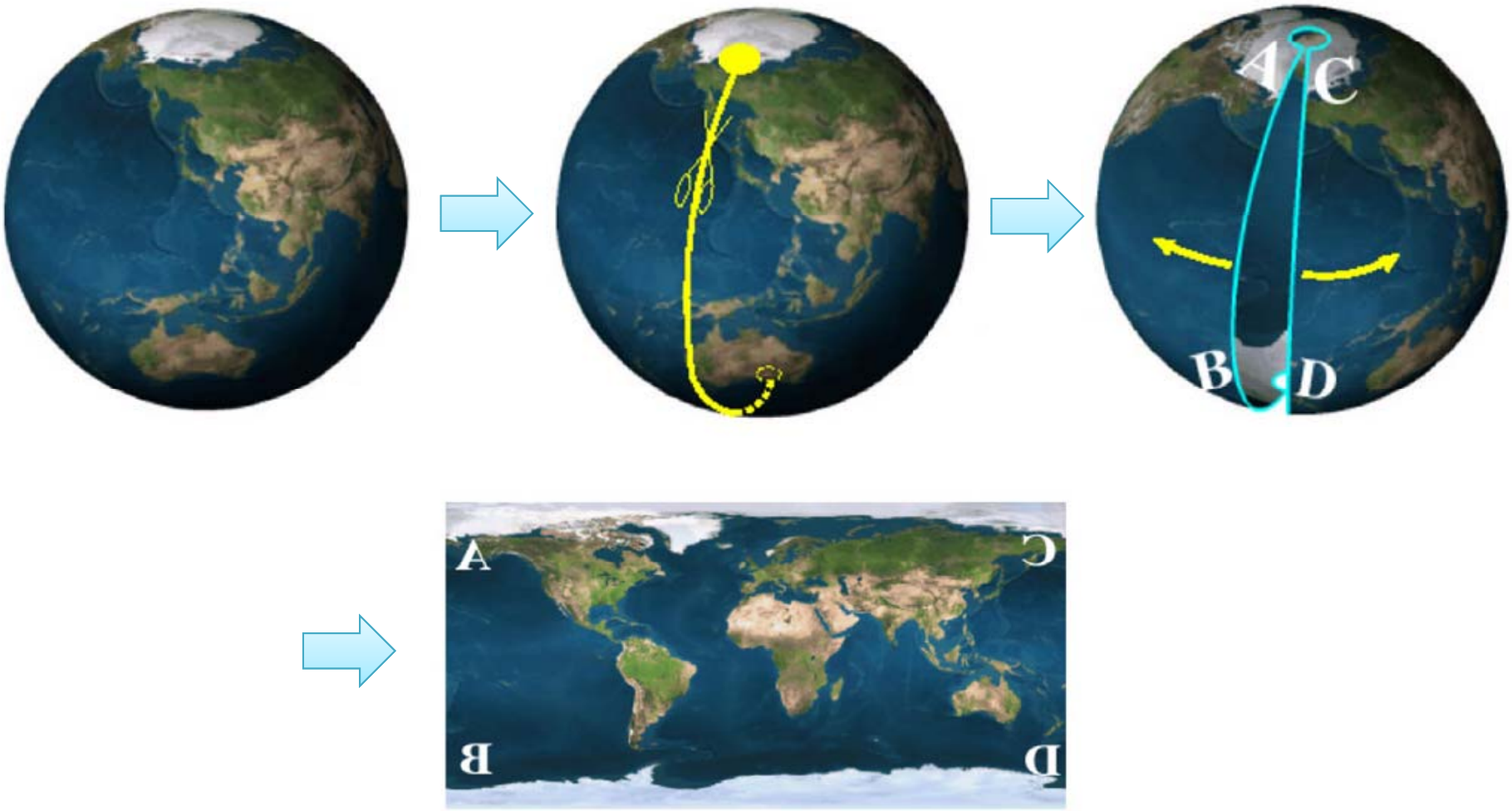
not bijective







Unfolding the World



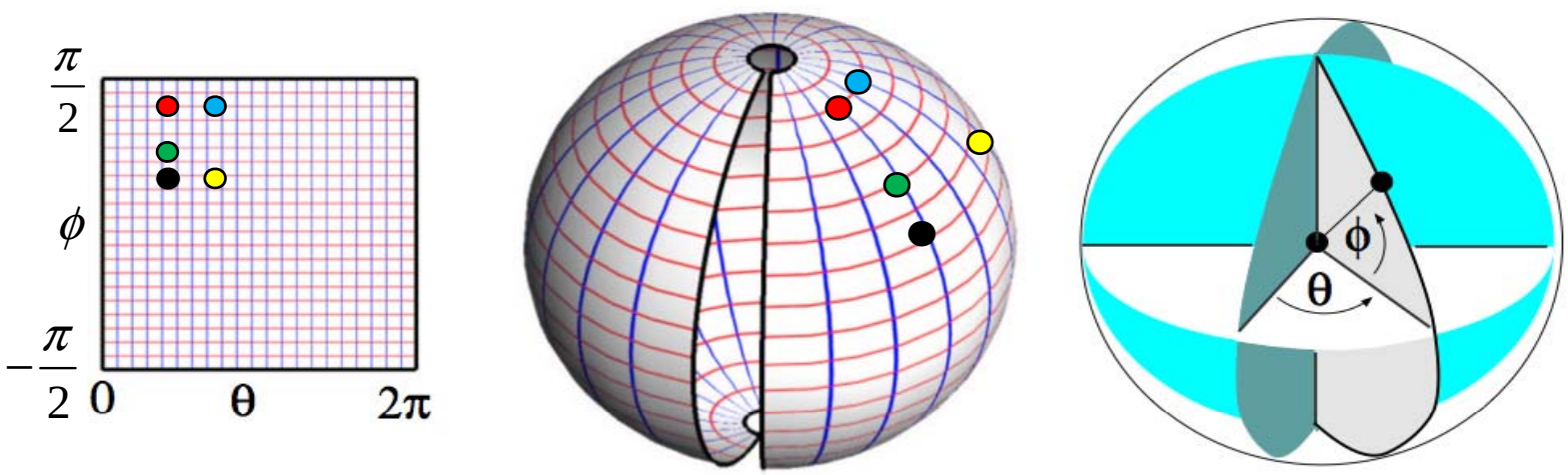
Spherical Coordinates

$$\theta \in [0, 2\pi), \phi \in [-\pi/2, \pi/2)$$

$$x(\theta, \phi) = R\cos\theta\cos\phi$$

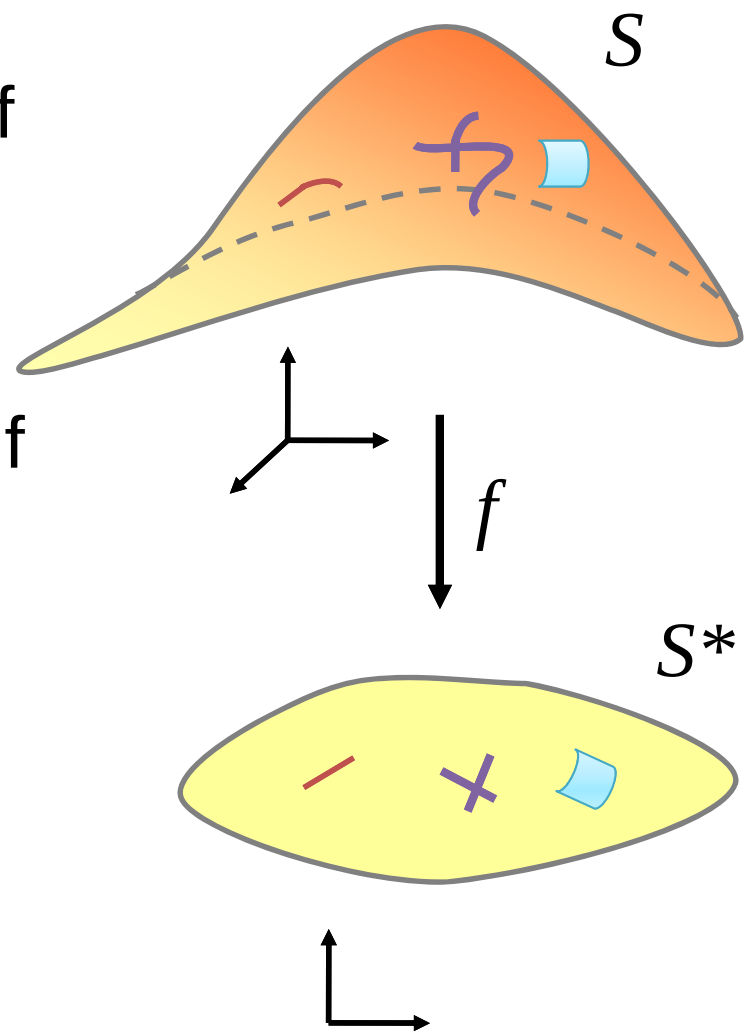
$$y(\theta, \phi) = R\sin\theta\cos\phi$$

$$z(\theta, \phi) = R\sin\phi$$



Definitions

- f is ***isometric*** (length preserving), if the *length* of any curve on S is preserved on S^*.
- f is ***conformal*** (angle preserving), if the *angle* of intersection of every pair of intersecting curves on S is preserved on S^*.
- f is ***equiareal*** (area preserving) if the *area* of an area element on S is preserved on S^*.



Standard Map Projections

orthographic

stereographic

Mercator

Lambert

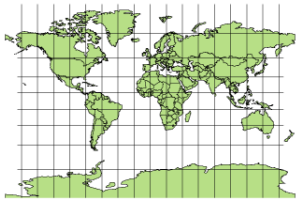
preserves angles = **conformal**

preserves area = **equiareal**

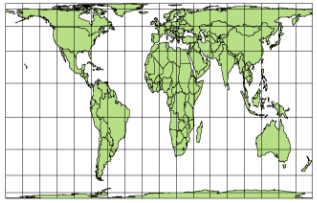
More Maps

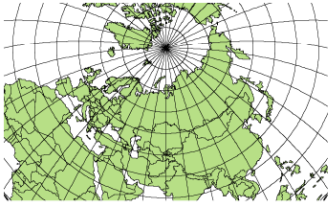

Mollweide-Projektion

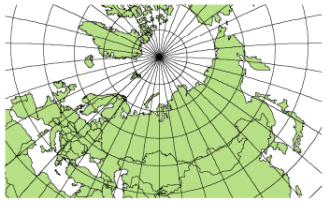

Mercator-Projektion

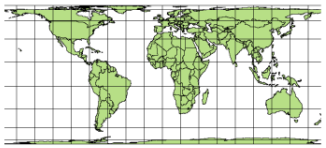

Zylinderprojektion nach Miller

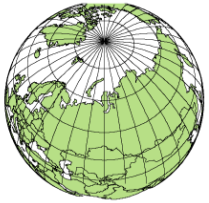

Hammer-Aitoff-Projektion

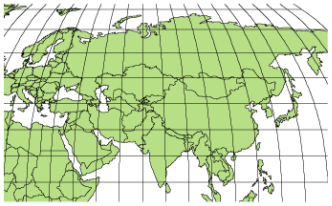

Peters-Projektion

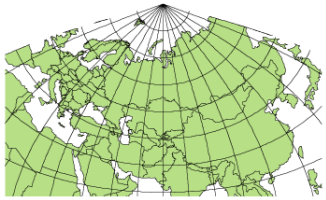

Längentreue Azimuthalprojektion

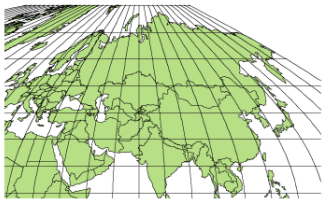

Stereographische Projektion

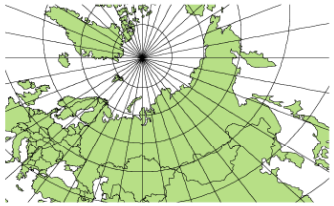

Behrmann-Projektion


Senkrechte Umgebungsperspektive

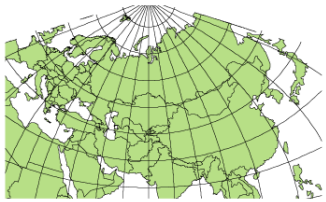

Robinson-Projektion

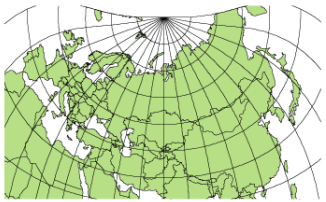

Hotine Oblique Mercator-Projektion


Sinusoidale Projektion

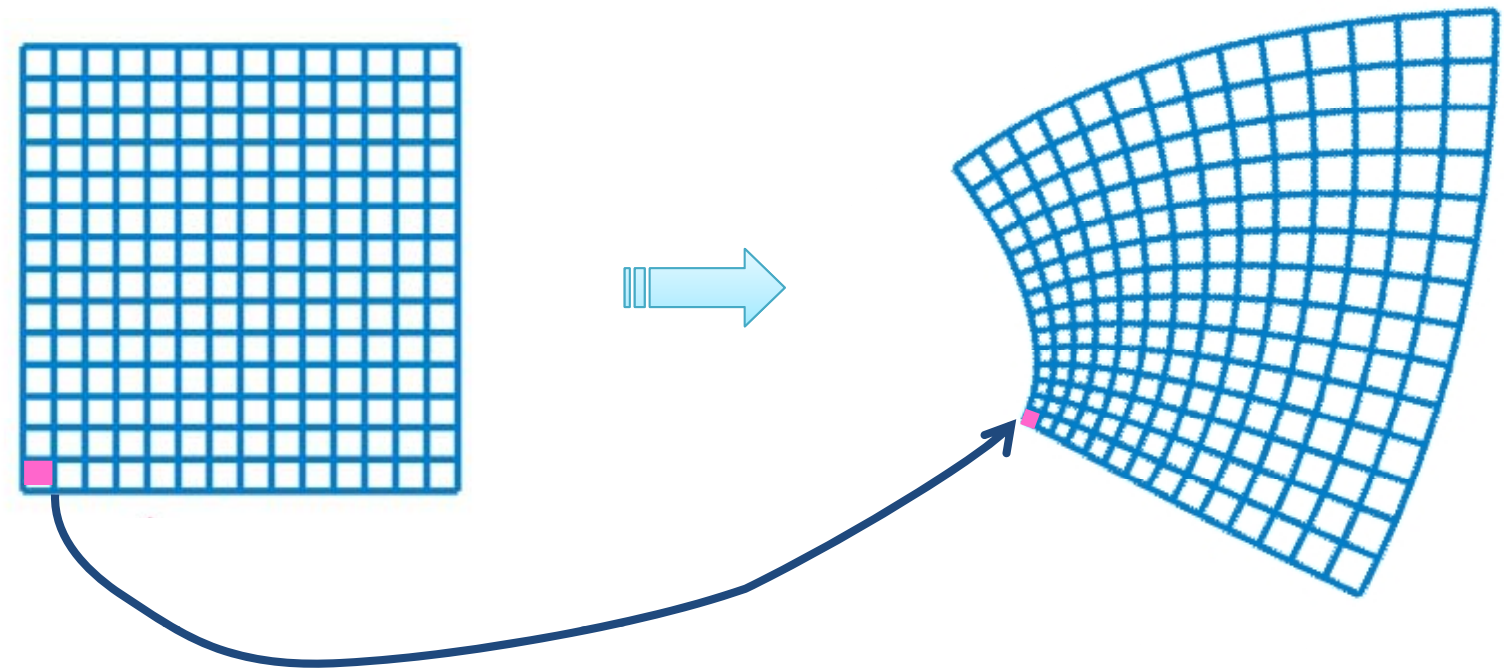

Gnomonische Projektion


Flächentreue Kegelprojektion


Transverse Mercator-Projektion

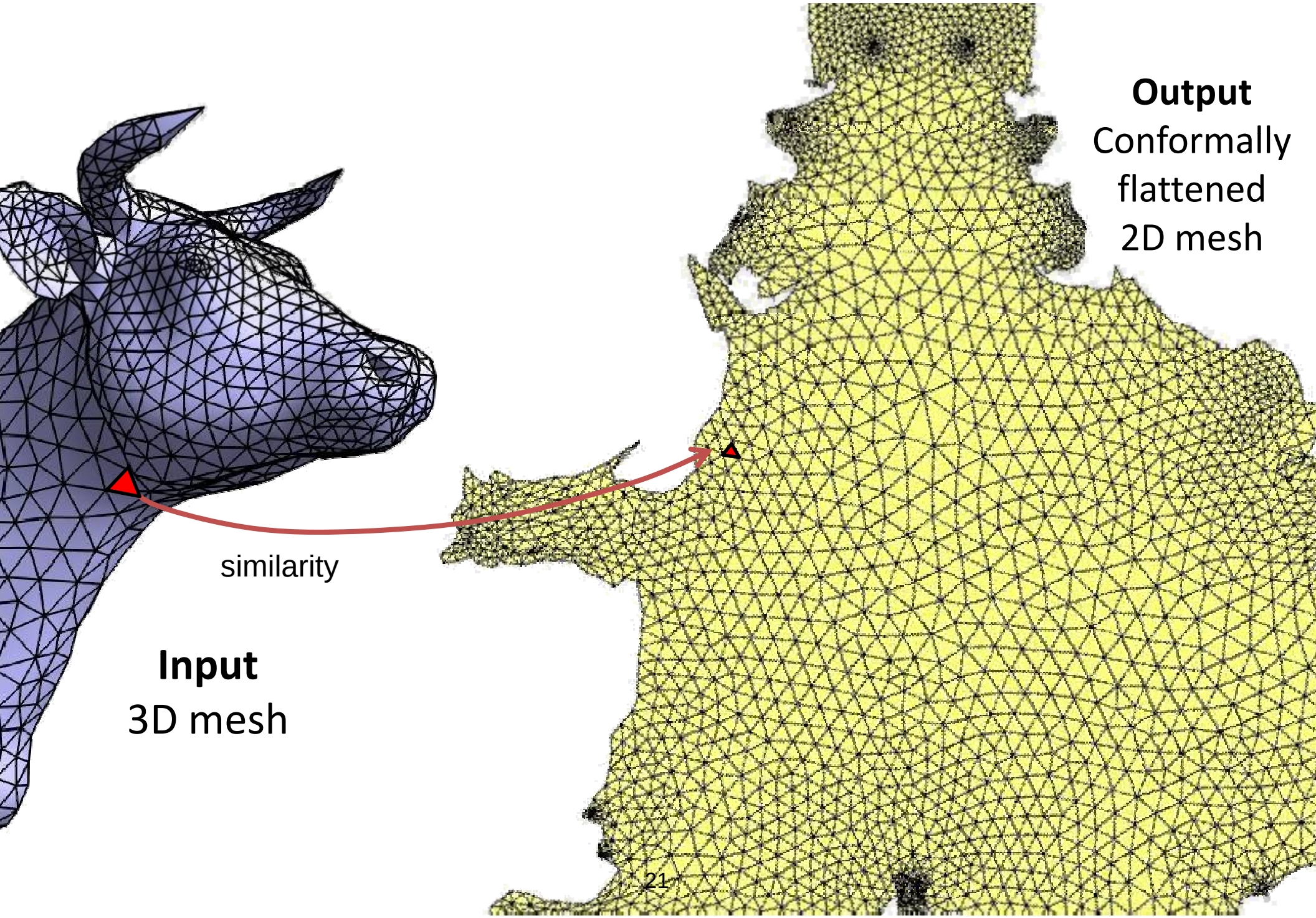

Cassini-Soldner-Projektion

Conformal Map

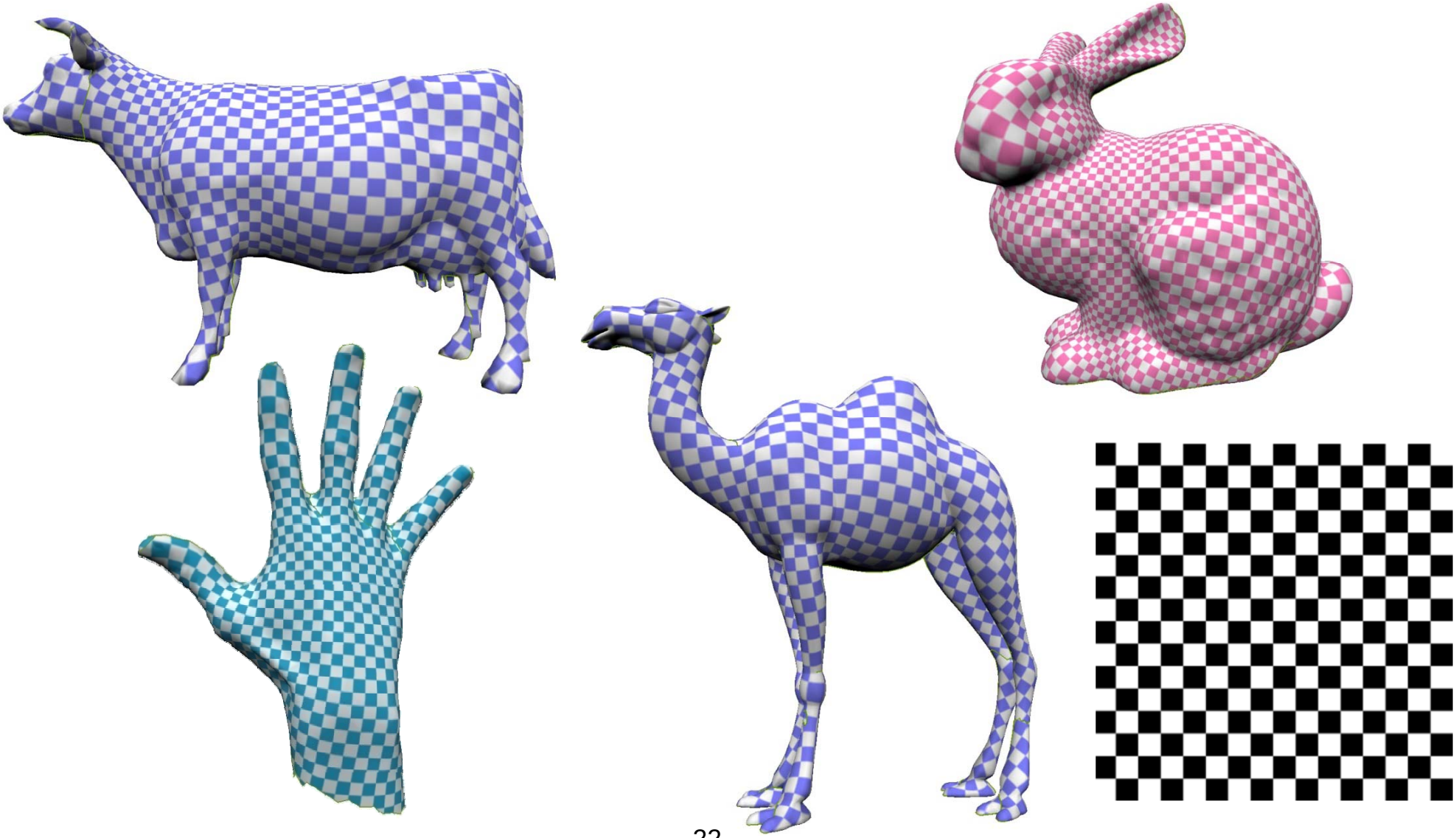


Similarity = Rotation + Scale

Preserves angles

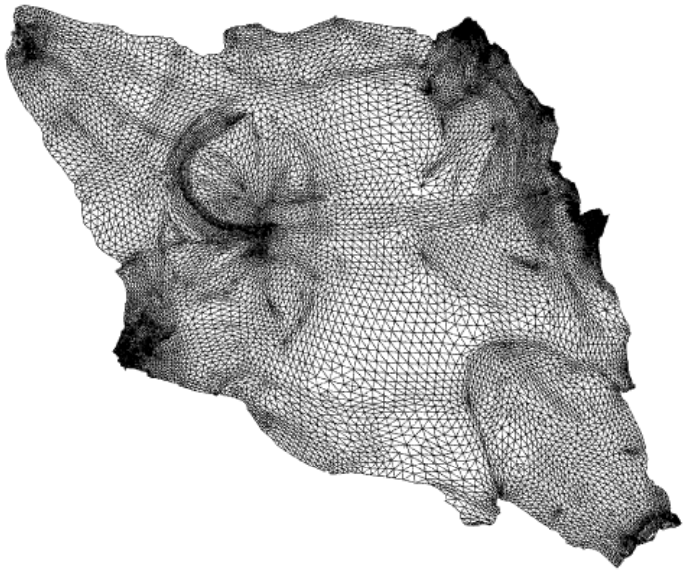

Output
Conformally
flattened
2D mesh
similarity
Input
3D mesh

Conformal Parameterization

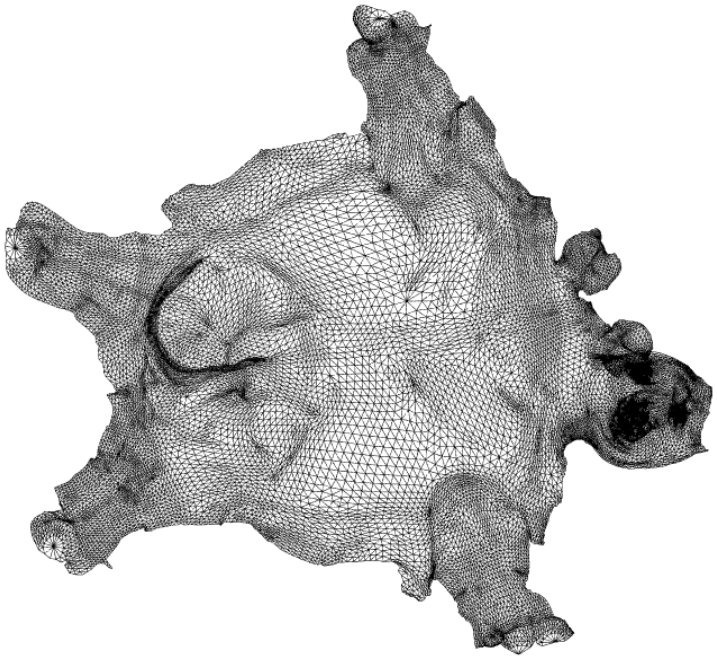


Conformal

Minimal Stretch



Conformal



Minimal Stretch

Differential Geometry Revisited

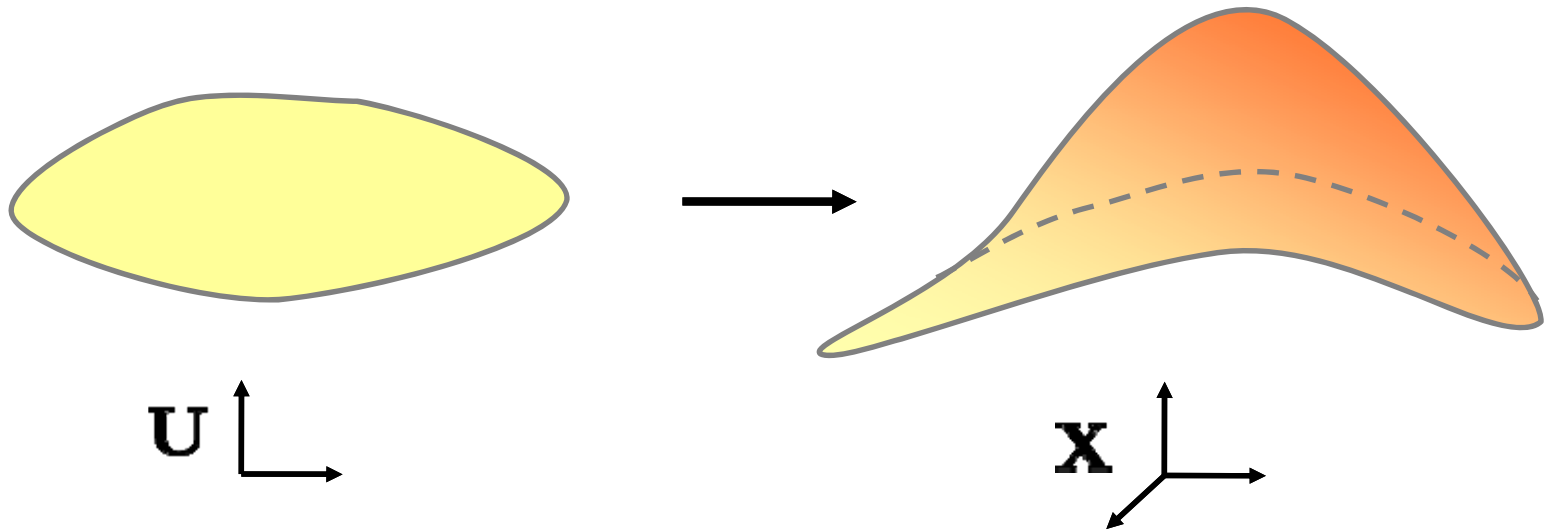
- Parametric surface:

$$\mathbf{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$$

- ***Regular*** if…
  - $x_1(u,v), x_2(u,v), x_3(u,v)$ are smooth (differentiable)
  - tangent vectors are linearly independent

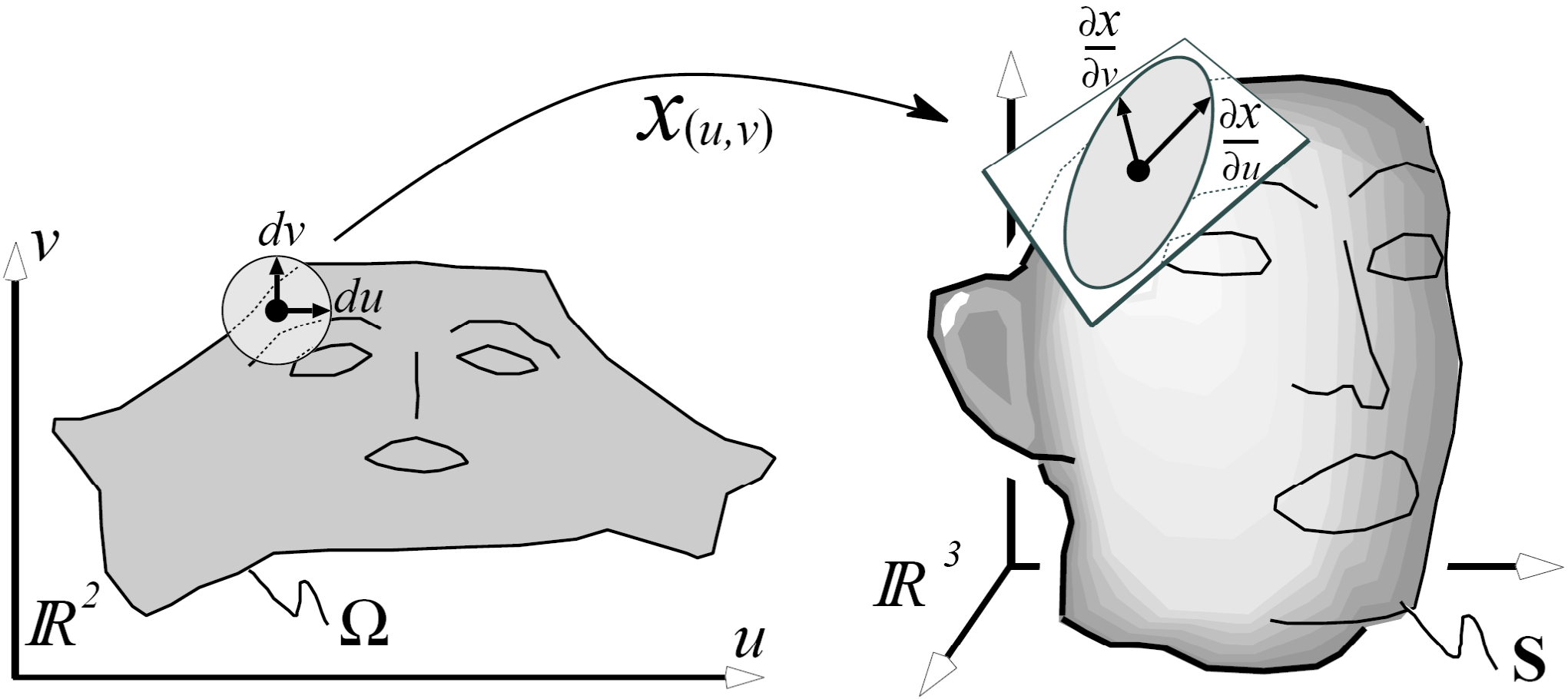
$$\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u} \qquad \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$$

Distortion Analysis

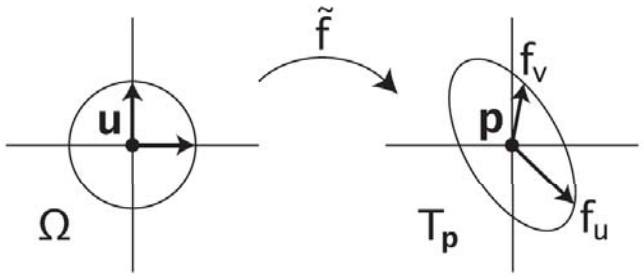


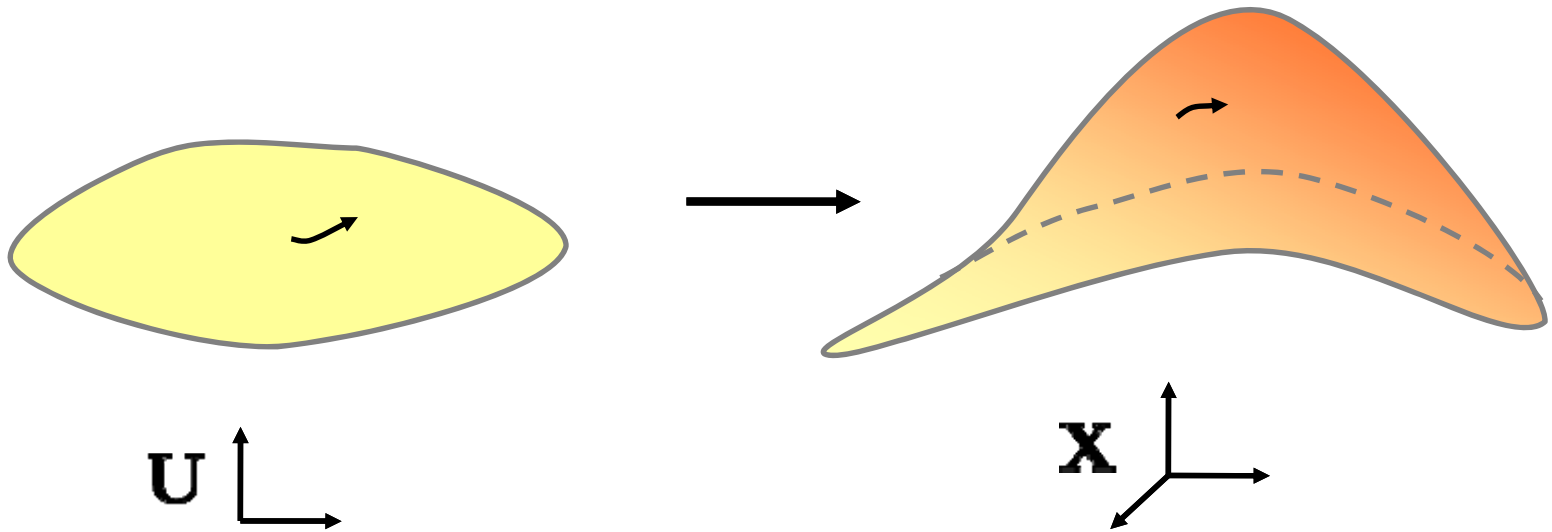
Jacobian

$$\mathbf{X}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \qquad J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$


$x_{(u,v)}$
$\frac{\partial x}{\partial v}$
$\frac{\partial x}{\partial u}$
v
dv
du
$\mathbb{R}^2$
Ω
u
$\mathbb{R}^3$
$\mathbf{S}$

Distortion Analysis





$$d\mathbf{X} = J d\mathbf{U}$$

$$|d\mathbf{X}||^2 = d\mathbf{U}\ J^T J\ d\mathbf{U} \qquad J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

$$J^T J = \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_v^T \mathbf{x}_u & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} = \mathbf{I}$$

First fundamental form

Fundamental Form Revisited

- Characterizes the surface locally

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

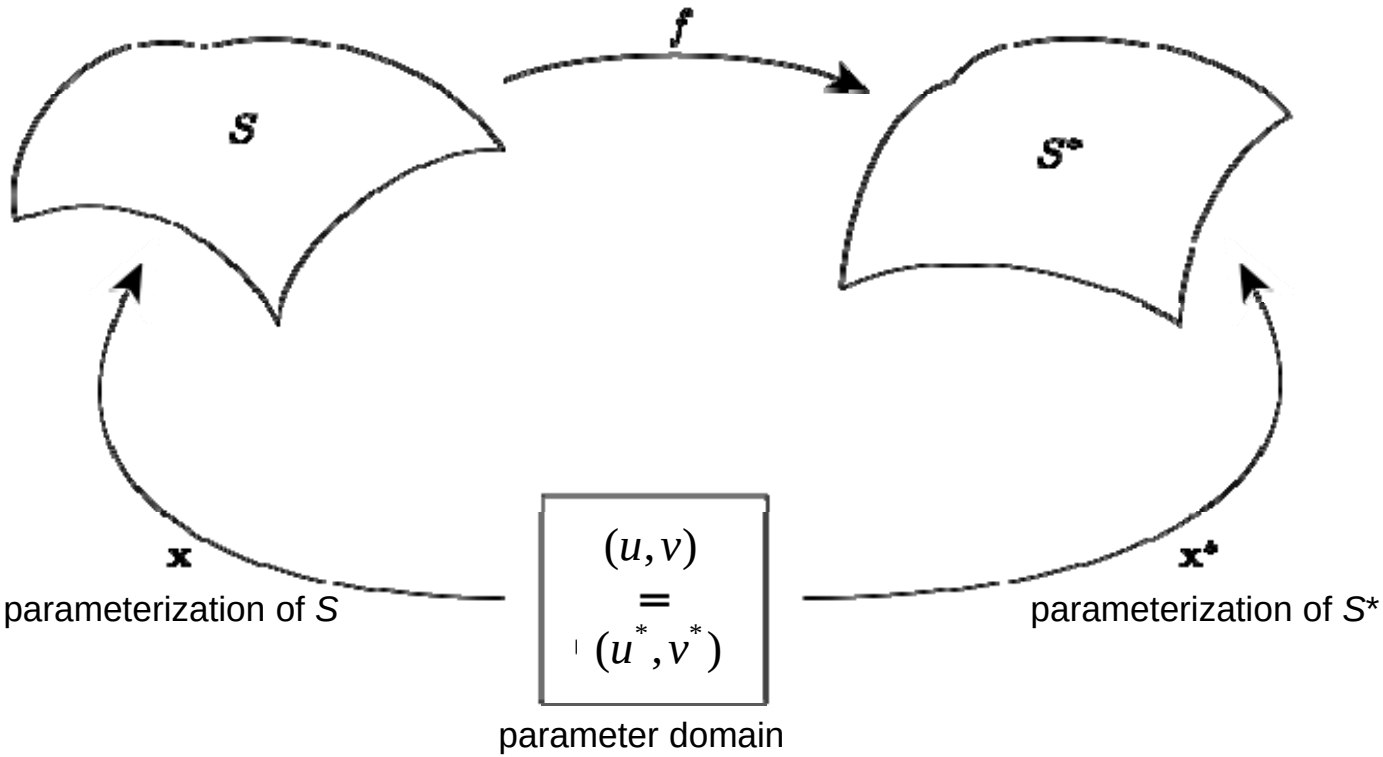
- Length element

$$d\mathbf{x}^2 = E du^2 + 2F du dv + G dv^2$$

- Area element

$$dA = \sqrt{EG - F^2}\, du dv$$

Mapping Surfaces



f is allowable if $\mathbf{x}^* = f \circ \mathbf{x}$ is regular

Isometric Maps

- **Theorem:** An allowable mapping f from S to S^* is ***isometric*** (length-preserving), iff the first fundamental forms of x and $x^* = f \cdot x$ are equal, i.e.

$$\boxed{\mathbf{I} = \mathbf{I}^*}$$

- Isometric surfaces have the same Gaussian curvature at corresponding pairs of points!

Conformal Maps

- **Theorem:** An allowable mapping f from S to S^* is ***conformal***, iff the first fundamental forms of x and $x^* = f \cdot x$ are proportional, i.e., there exists a positive scalar function n, such that

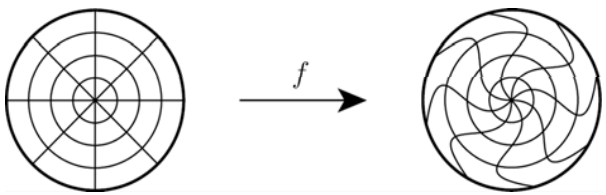
$$\boxed{\mathbf{I} = n(u, v)\mathbf{I}^*}$$

- A conformal map is always (locally) bijective.

Equiareal Maps

- **Theorem:** An allowable mapping f from S to S^* is ***equiareal***, iff the determinants of the first fundamental forms of x and $x^* = f \cdot x$ are equal, i.e.,

$$det(\mathbf{I}) = det(\mathbf{I}^*)$$



- Area element:

$$dA = \sqrt{EG - F^2}dudv = \sqrt{\det(\mathbf{I})}dudv$$

Relationships

- **Theorem:** Every isometric mapping is conformal and equiareal, and vice versa.

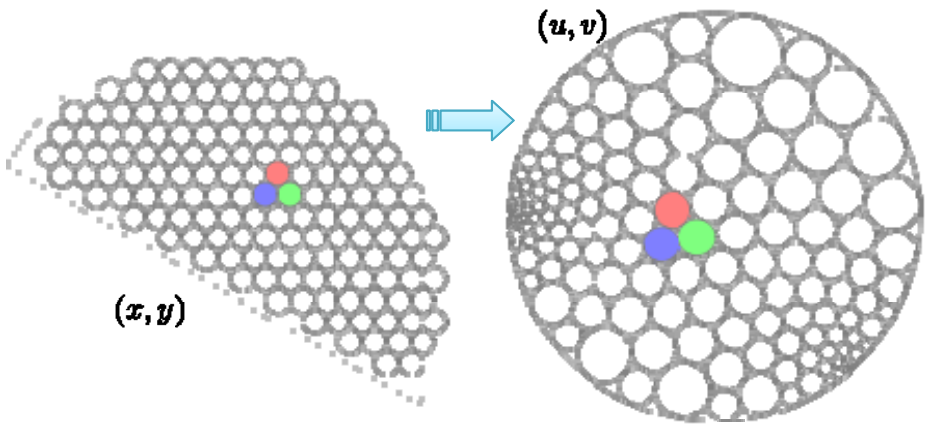
$$\text{isometric} \Longleftrightarrow \text{conformal} + \text{equiareal}$$

- Isometric is ideal… but rare. In practice, we use:
  - conformal
  - equiareal
  - some balance between the two

Riemann Conformal Mapping Theorem

Any two simply connected compact planar regions

can be mapped conformally onto each other.



$$J(u,v) = \begin{pmatrix} a(u,v) & b(u,v) \\ -b(u,v) & a(u,v) \end{pmatrix}$$

If $(x,y) \rightarrow (u,v)$ is a conformal mapping, then $u(x,y)$ and $v(x,y)$ satisfy

the Cauchy-Riemann equations: $$\frac{\partial u}{dx} = \frac{\partial v}{dy} \qquad \frac{\partial u}{dy} = -\frac{\partial v}{dx}$$

thus both u and v are harmonic: $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0 \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = 0$$

Harmonic Maps

- f is *harmonic* if satisfies (for each coordinate):

$$\Delta_S f = 0$$

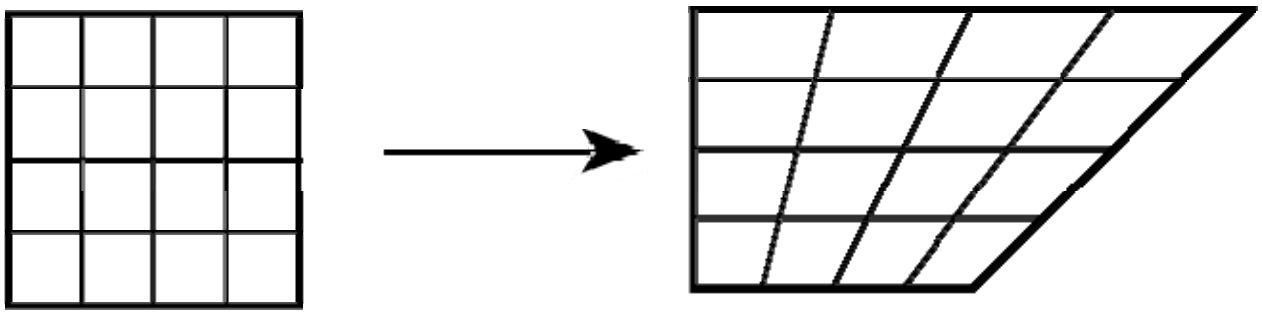
- isometric $\Longrightarrow$ conformal $\Longrightarrow$ harmonic
- Minimizes the Dirichlet energy

$$E_D(f) = \frac{1}{2}\int_S \|\nabla_S f\|^2$$

given boundary conditions

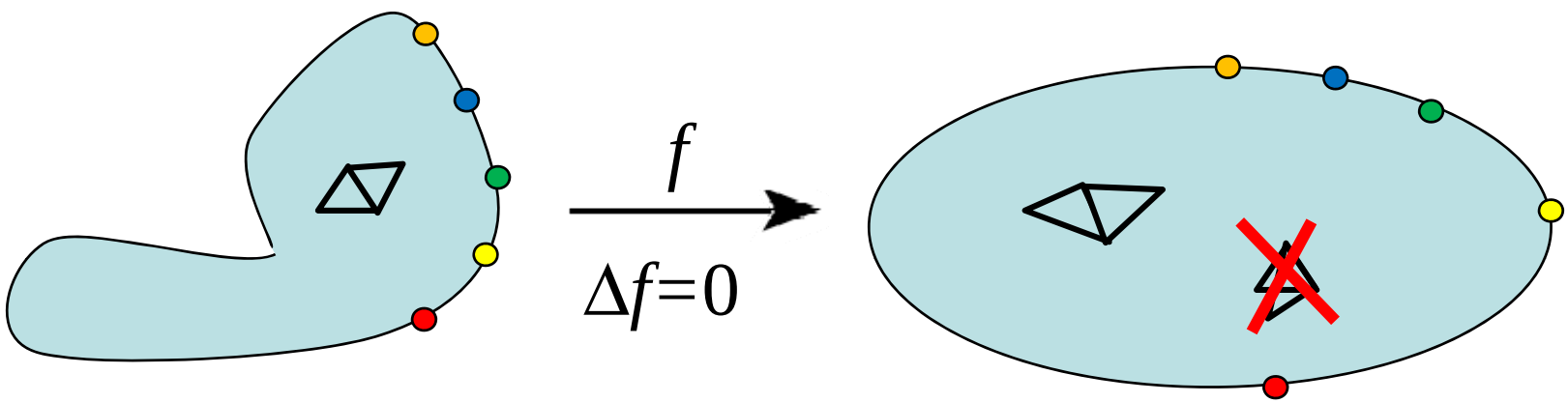
Harmonic Maps

- Easier to compute than conformal, but does not preserve angles. May not be bijective.



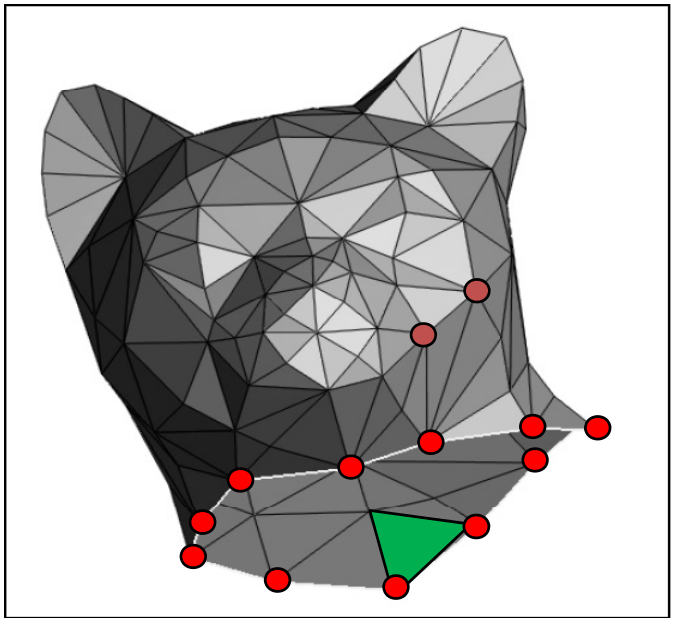
Harmonic Maps

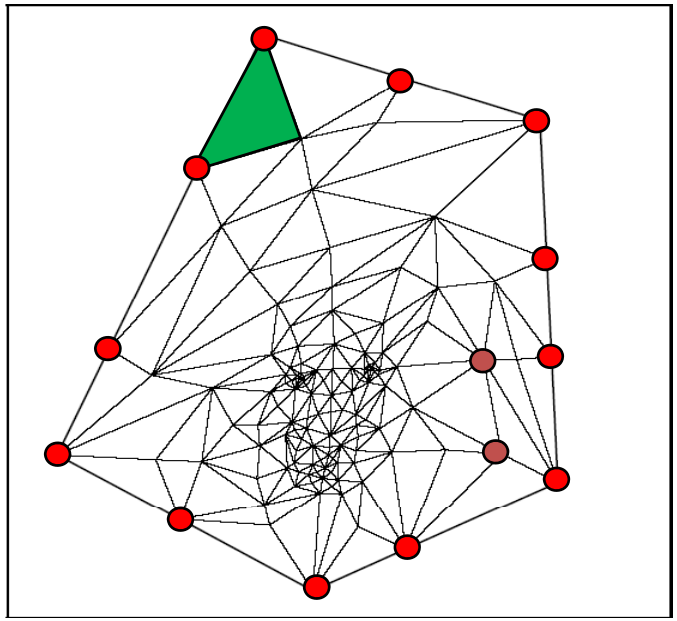
- **Theorem** [Rado-Kneser-Choquet]**:**
  If $f : S \to R^2$ is harmonic and maps the boundary ∂S homeomorphically onto the boundary ∂S^* of some **convex** region $S^* \subset R^2$, then f is **bijective**.



Discrete Harmonic Maps

- Piecewise linear map for triangulated, disk-like surface onto planar polygon





$$\text{Laplace equation: } \sum_{(i,j)\in E} w_{ij}(v_i - v_j) = 0$$

2D Barycentric Drawings

- Fix 2D boundary to **convex polygon**.
- Define drawing as a solution of

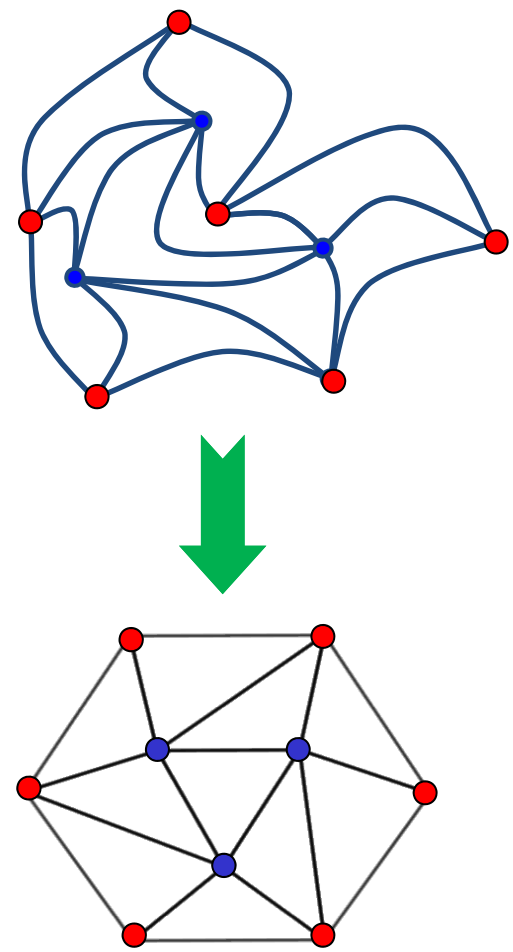
$$Wx = b_x \qquad Wy = b_y$$

$$w_{ij} = \begin{cases} > 0 & (i,j) \in E \\ -\sum_{j \neq i} w_{ij} & (i,i), i \notin B \\ 1 & (i,i), i \in B \\ 0 & otherwise \end{cases}$$

$$W \text{ is } symmetric: \; w_{ij} = w_{ji}$$

- Weights w_{ij} control triangle shapes

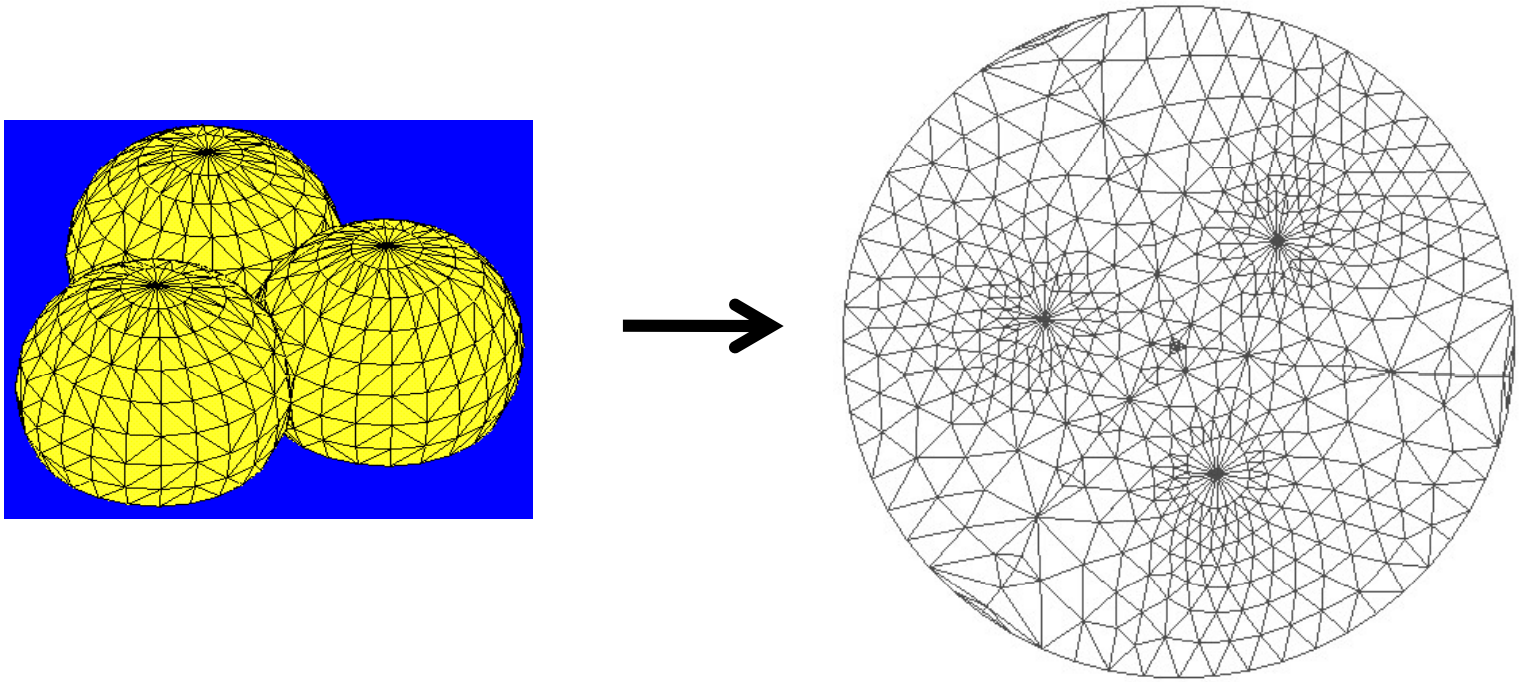
B = Boundary vertices



Why it Works

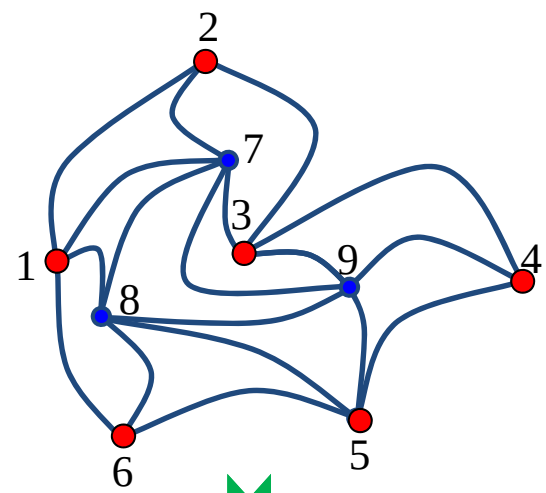
- **Theorem** (Maxwell-Tutte)

  If $G = \langle V,E \rangle$ is a 3-connected planar graph (triangular mesh) then any **barycentric** drawing is a valid embedding.



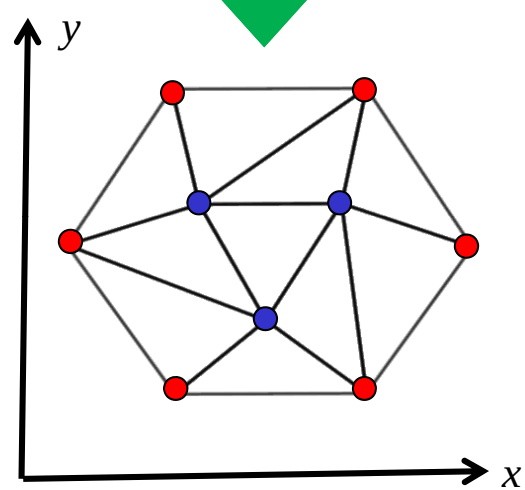
Example

$$\boxed{w_{ij} = 1}$$



Laplacian Matrix

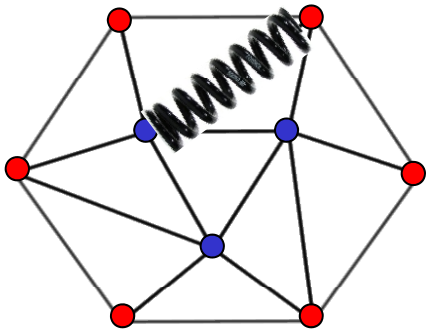
$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -5 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & -5 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -5 \end{pmatrix} \quad b_x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad b_y = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



Spring System

- Represent as configuration of springs on mesh edges

$$E(v) = \frac{1}{2} \sum_{(i,j) \in E} w_{ij} \, || \, v_i - v_j \, ||^2$$



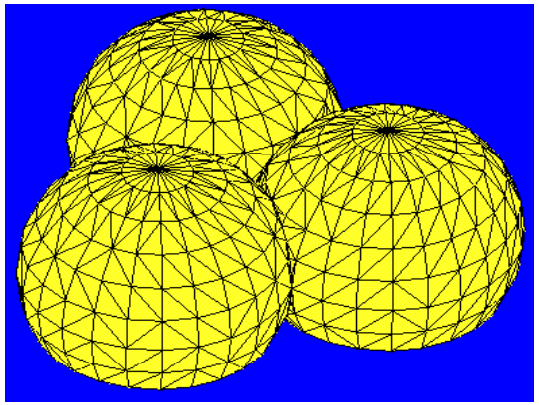
- Minimum of $E(v)$ reached when gradients = 0

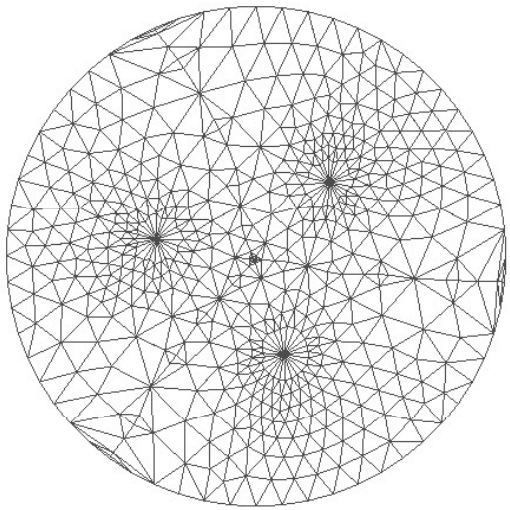
$$\frac{\partial E(v)}{\partial v_i} = \sum_{(i,j) \in E} w_{ij} (v_i - v_j) = 0$$

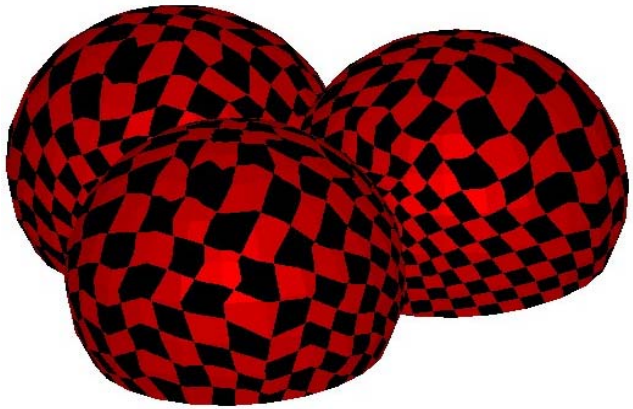
Uniform Weights

$$\boxed{w_{ij} = 1}$$

- No shape information – “equilateral” triangles
- Fastest to compute and solve
- Not 2D reproducible

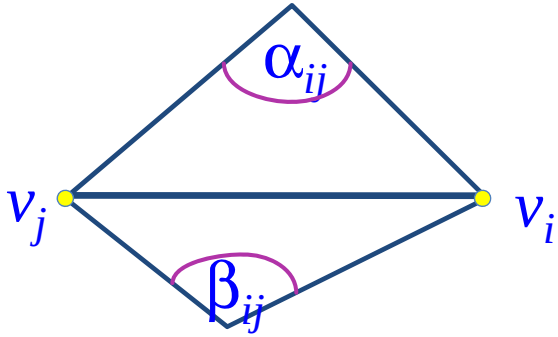




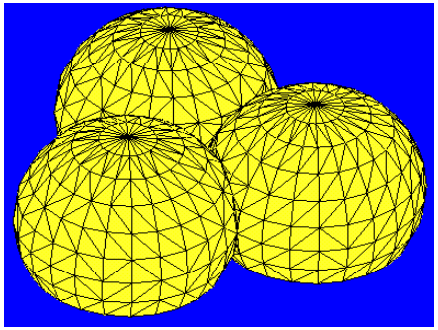


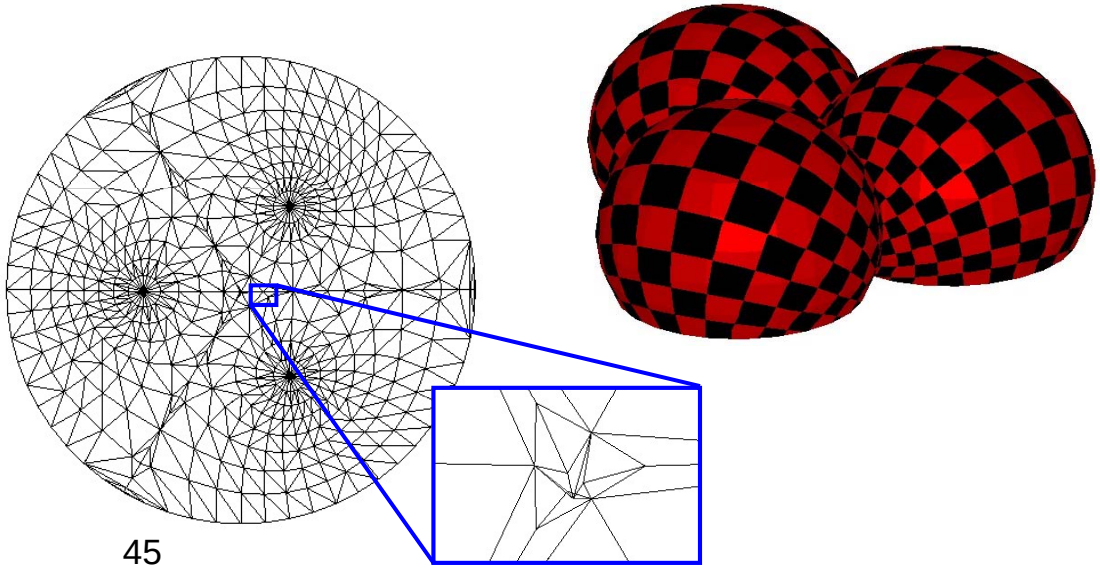
Harmonic Weights

$$w_{ij} = \frac{\cot(\alpha_{ij}) + \cot(\beta_{ij})}{2}$$



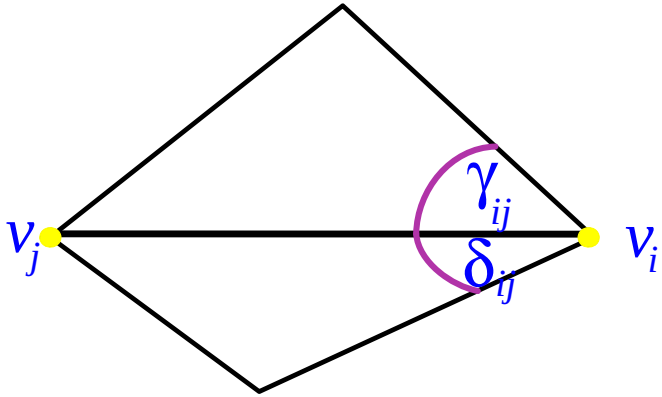
- Weights can be negative – not always valid
- Weights depend only on angles - close to conformal
- 2D reproducible



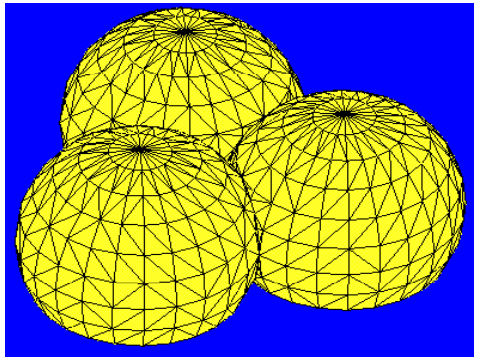


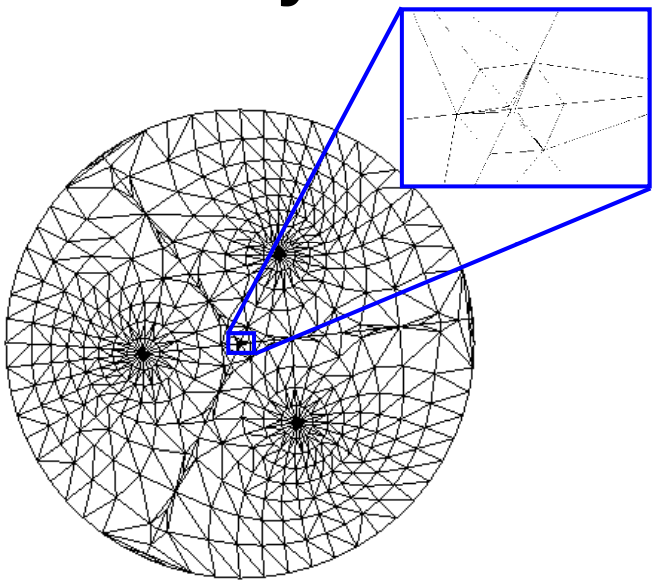
Mean-Value Weights

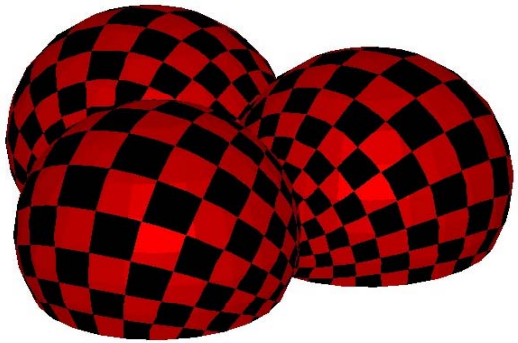
$$w_{ij} = \frac{\tan(\gamma_{ij}/2) + \tan(\delta_{ij}/2)}{2\,||\,v_i - v_j\,||}$$



- Result visually similar to harmonic
- No negative weights – **always** valid
- 2D reproducible







General Method

Select normalized weights $\lambda_{ij} = w_{ij} \Big/ \sum_{k \in N_i} w_{ik}$

so that $\sum_{j \in N_1(i)} \lambda_{ij} = 1$

Re-express Laplace equation $\sum_{j \in N_1(i)} \omega_{ij}(f(v_j) - f(v_i)) = 0$

as weighted average constraints $f(v_i) = \sum_{j \in N_1(i)} \lambda_{ij}(f(v_j))$

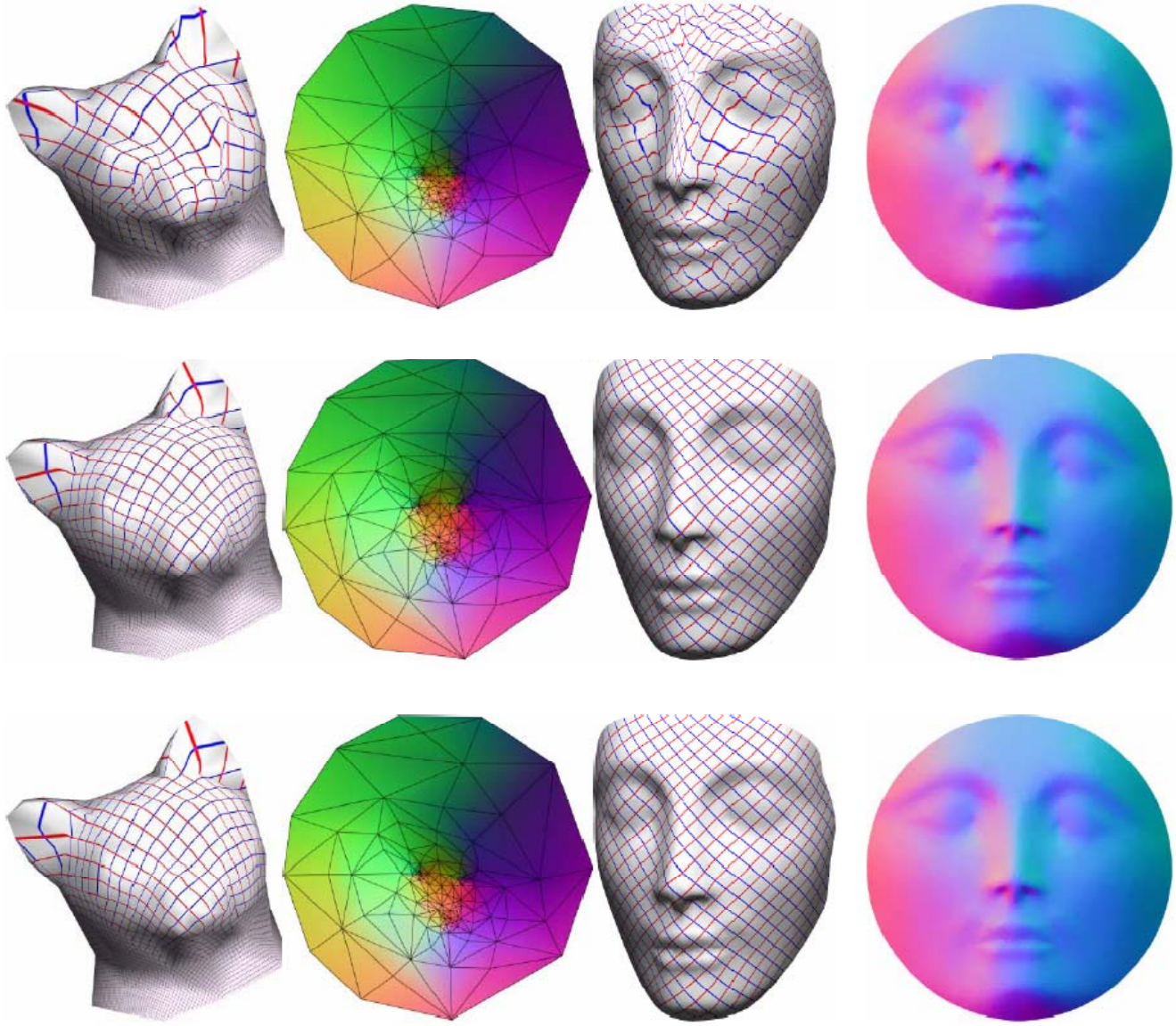
Then, if ω_{ij} are positive, so are are λ_{ij} .

Example

uniform

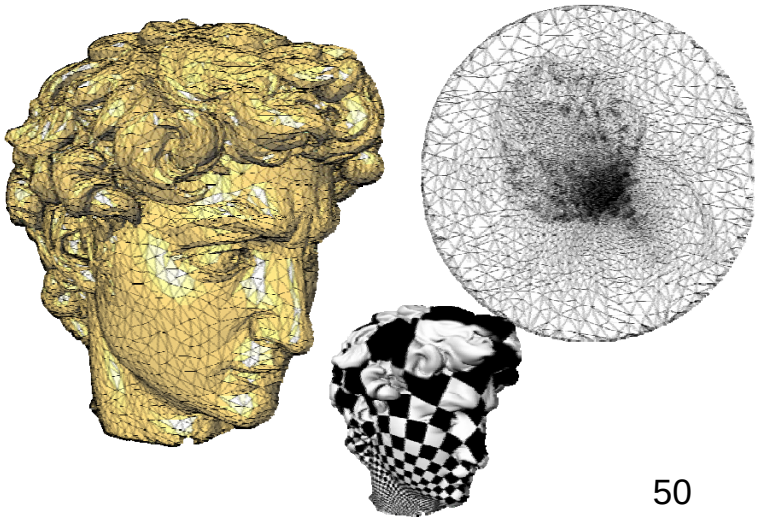
harmonic

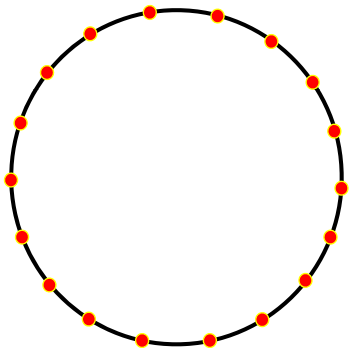
mean-value



Fixing the Boundary

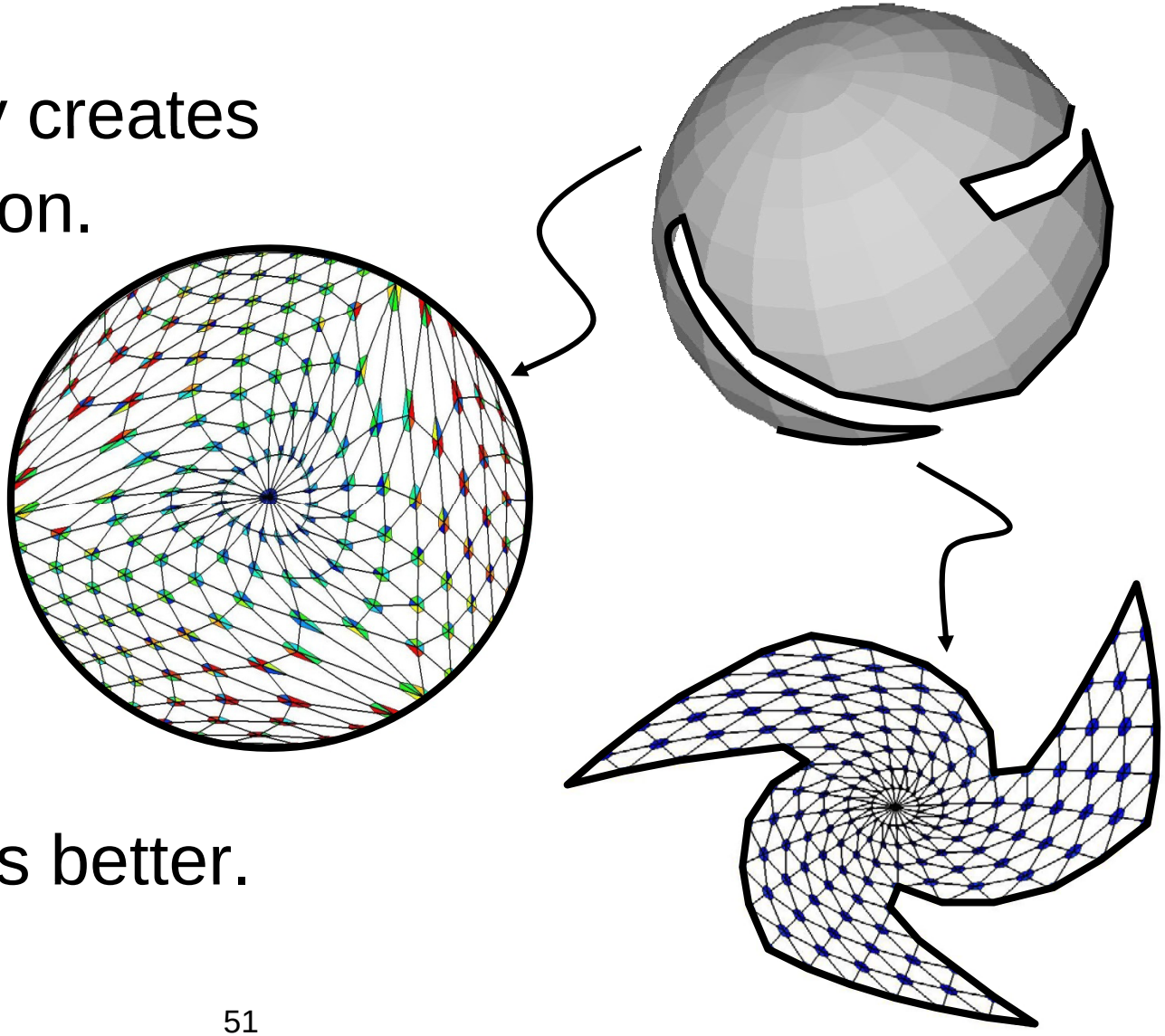
- Simple convex shape (triangle, square, circle)
- Distribute points on boundary
  - Use chord length parameterization
- Fixed boundary can create high distortion





Non-Convex Boundary

- Convex boundary creates significant distortion.



- “Free” boundary is better.