Discrete Exterior Calculus

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\[ \text{grad} f \equiv \nabla f \equiv \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \]

\[ \nabla \equiv \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \]
Vector Calculus in $\mathbb{R}^3$

\[ \text{div } \vec{v} \equiv \nabla \cdot \vec{v} \equiv \sum_i \frac{\partial v_i}{\partial x_i} \]

\[ \text{curl } \vec{v} \equiv \nabla \times \vec{v} \equiv \ldots \]

\[ \Delta f \equiv \nabla \cdot \nabla f \equiv \sum_i \frac{\partial^2 f}{\partial x_i^2} \]
Famous Theorems (in $R^2$)

\[ \int_{\Omega} \text{div} \, \vec{v} \, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{n} \, dl \]  
“Divergence Theorem”

\[ \int_{\Omega} \text{curl} \, \vec{v} \, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{t} \, dl \]  
“Green’s Theorem”
**Famous Theorems (in $R^2$)**

\[ \int_{\Omega} \text{div} \: \vec{v} \: dA = \int_{\partial\Omega} \vec{v} \cdot \vec{n} \: dl \]

“Divergence Theorem”

\[ \int_{\Omega} \text{curl} \: \vec{v} \: dA = \int_{\partial\Omega} \vec{v} \cdot \vec{t} \: dl \]

Scalars in $R^2$  

“Green’s Theorem”
Famous Theorems (in $R^2$)

“Divergence Theorem”

“Green’s Theorem”
Some Questions

- Why are these theorems similar?
- How do we generalize?
- How do these work on surfaces?
Some Questions

- Why are these theorems similar?
- How do we generalize?
- How do these work on surfaces?
- What's up with the cross product?
<math>
Introducing... Exterior Calculus

Extension of vector calculus to surfaces (and manifolds).
Everything must be intrinsic!
Basic Assumption

Everything must be **intrinsic**!

Vector fields are **tangent**!
For each point $p$ on a surface:

$k$ vectors in the tangent space at $p$ → **Differential $k$-form** → $\mathbb{R}$
For each point $p$ on a surface:

- $k$ vectors in the tangent space at $p$
- **Differential $k$-form**
- $k$-linear
- Alternating
For each point $p$ on a surface:

$k$ vectors in the tangent space at $p$ → \textit{Differential $k$-form} → $\mathbb{R}$

$k$-linear
Alternating

[Challenge: In $n$ dimensions, $p$-forms are zero for $p > n$.]
For each point $p$ on a surface:

$k$ vectors in the tangent space at $p$ → Differential $k$-form → $f : \Sigma \rightarrow \mathbb{R}$

The output is an $o$-form.
Expressing Vector Fields

Vector field
\[ \vec{v} : \Sigma \rightarrow T\Sigma \subset \mathbb{R}^3 \]

1-form \( \omega \):
\[ \omega(\vec{x}) = \vec{v} \cdot \vec{x} \]
Differentiating $k$-Forms

\[ d\omega(\bar{v}, \bar{v}_1, \ldots, \bar{v}_k) \]

gives derivatives of

\[ \omega(\bar{v}_1, \ldots, \bar{v}_k) \]
in the $\bar{v}$ direction

Yields $(k+1)$-forms!
Integrating $k$-Forms

$\int_{\gamma} \omega$

measures amount of $\omega$ parallel to $\gamma$

Integrate on $k$-dimensional objects
Stokes’ Theorem

$$\int_{\Omega} d\omega = \int_{\partial \Omega} \omega$$

Zoo of Operators

\[ \omega^# \] 1-form to vector

\[ \vec{\omega} \] Vector to 1-form

\[ \omega_1 \wedge \omega_2 \] Product of forms

\[ \star \omega \] Dual
**Zoo of Operators**

- $\omega^\#$: 1-form to vector
- $\vec{v}$: Vector to 1-form
- $\omega_1 \wedge \omega_2$: Product of forms
  - $k,p$-forms $\rightarrow (k+p)$-form (cross product!)
- $\star \omega$: Dual
  - $k$-forms $\rightarrow (n-k)$-form (plane to its normal)
</math>
Discrete Differential Geometry

Discrete rather than discretized notion of differential geometry.
Discrete Exterior Calculus (DEC)

Discrete version of exterior calculus.

\[ \omega^# \quad \vec{v} \quad \omega_1 \wedge \omega_2 \quad \star \omega \quad d\omega \]

\[ \ldots \]
Simplicial Complex
Oriented Simplicial Complex
Dual Complex
The Trick

Store *integrated* quantities!
The Trick

Discrete 0-form

\[ \int_{\omega} = f(v) \rightarrow \mathbb{R}^{|V|} \]

Store *integrated* quantities!
The Trick

Discrete 1-form

\[ \int_{e} \omega \rightarrow \mathbb{R}^{|E|} \]

Store *integrated* quantities!
The Trick

Discrete 2-form

\[ \int \omega \rightarrow \mathbb{R}^{|F|} \]

Store *integrated* quantities!
Stokes’ Theorem
Stokes’ Theorem

\[ \int_{\Omega} d\omega = \int_{\partial \Omega} \omega \]

\[ \int_{e} d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1 \]
Exterior Derivative

\[ d \in \mathbb{R}^{|E| \times |V|} \]

consists of 1, 0, -1

\[
\int_e d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1
\]
Exterior Derivative

\[ d \in \mathbb{R}^{\vert F \vert \times \vert E \vert} \]

consists of 1, 0, -1

\[
\int_t \int_{\partial t} d\omega = \int \omega = \omega_1 - \omega_2 + \omega_3
\]
\[ d \in \mathbb{R} \left| F \right| \times \left| E \right| \]

Haven’t made any approximations yet!

\[ \int d\omega = \int_\partial t \omega = \omega_1 - \omega_2 + \omega_3 \]
Moves to dual mesh
Hodge Star

Primal 2-form
Dual 0-form

Moves to dual mesh
Hodge Star

Primal 1-form
Dual 1-form

Moves to dual mesh
Hodge Star for 1-Forms

\[ \star \omega = \frac{|e_\ast|}{|e|} \omega \]
General Hodge Star Formulation

primal

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dual

http://brickisland.net/cs177/
General Hodge Star Formulation

primal

Decision: How big?

http://brickisland.net/cs177/
General Hodge Star Formulation

primal

can be a diagonal matrix!

dual

Decision: How big?

http://brickisland.net/cs177/
Discrete deRham Complex

0-forms (vertices) \[ \delta \rightarrow d \rightarrow \delta^{-1} \]

1-forms (edges) \[ \delta \rightarrow d \rightarrow \delta^{-1} \]

2-forms (faces) \[ \delta \rightarrow d \rightarrow \delta^{-1} \]

3-forms (tets) \[ \delta \rightarrow d \rightarrow \delta^{-1} \]
o-Form Laplacian

\[ \Delta = d \star d \star + \star d \star d \]
o-Form Laplacian

\[ \Delta = d \star d \star + \star d \star d \]

Cotangent Laplacian
o-Form Laplacian

\[ \Delta = d \star d \star + \star d \star d \]

Area weights
$\Delta = d \star d \star + \star d \star d$
Only Scratching the Surface...

- Additional operators
- Discrete DEC theorems
- Limits of this theory
Why Bother?

Treatment of tangent vector fields

Fisher et al. "Design of Tangent Vector Fields" (SIGGRAPH 2007)
Why Bother?

Ben Chen et al. “On Discrete Killing Vector Fields and Patterns on Surfaces” (SGP 2010)

Geometric PDEs
Why Bother?

Grinspun. "A Discrete Model of Thin Shells."
Elcott et al. "Stable, Circulation-Preserving, Simplicial Fluids." (TOG)

Simulation
Discrete Problems

Why Bother?

Mullen et al. "HOT: Hodge-Optimized Triangulations" (SIGGRAPH 2011)
Discrete Exterior Calculus

Questions?