Introduction to Modal Vibration

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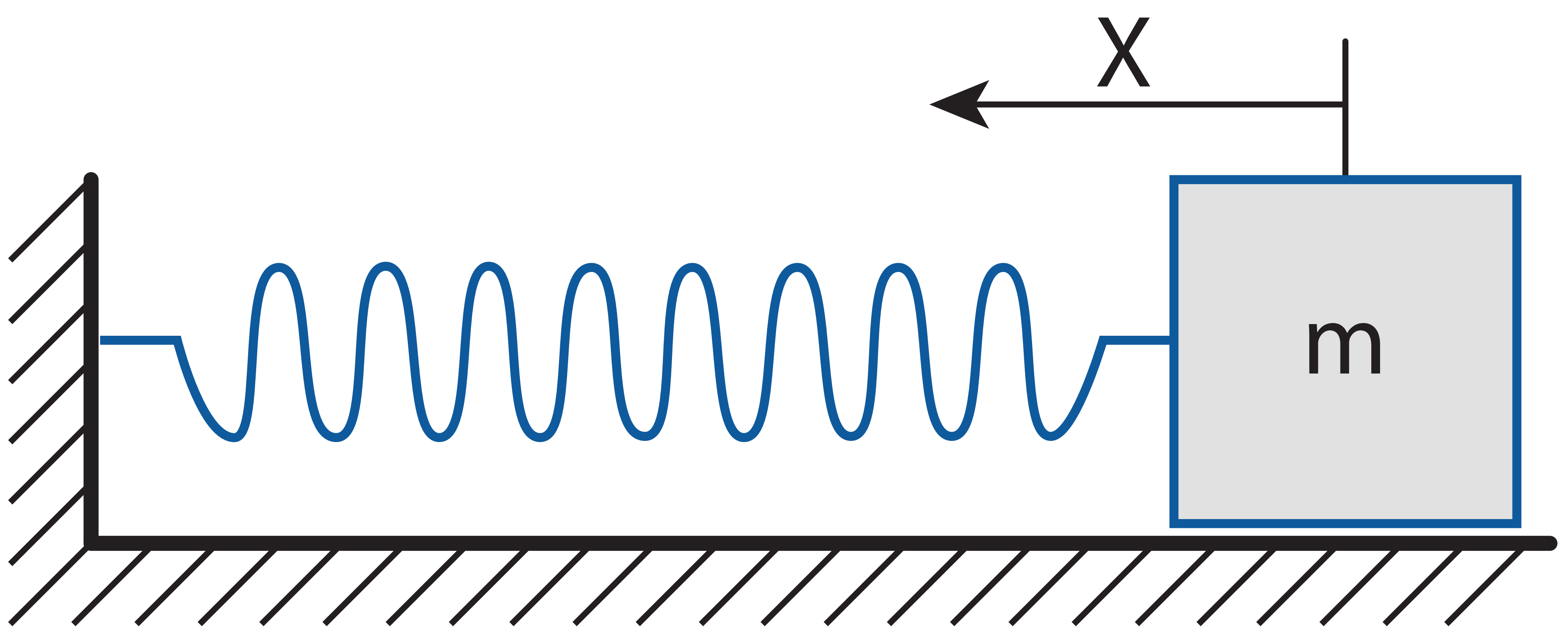
Render the Possibilities
SIGGRAPH2016

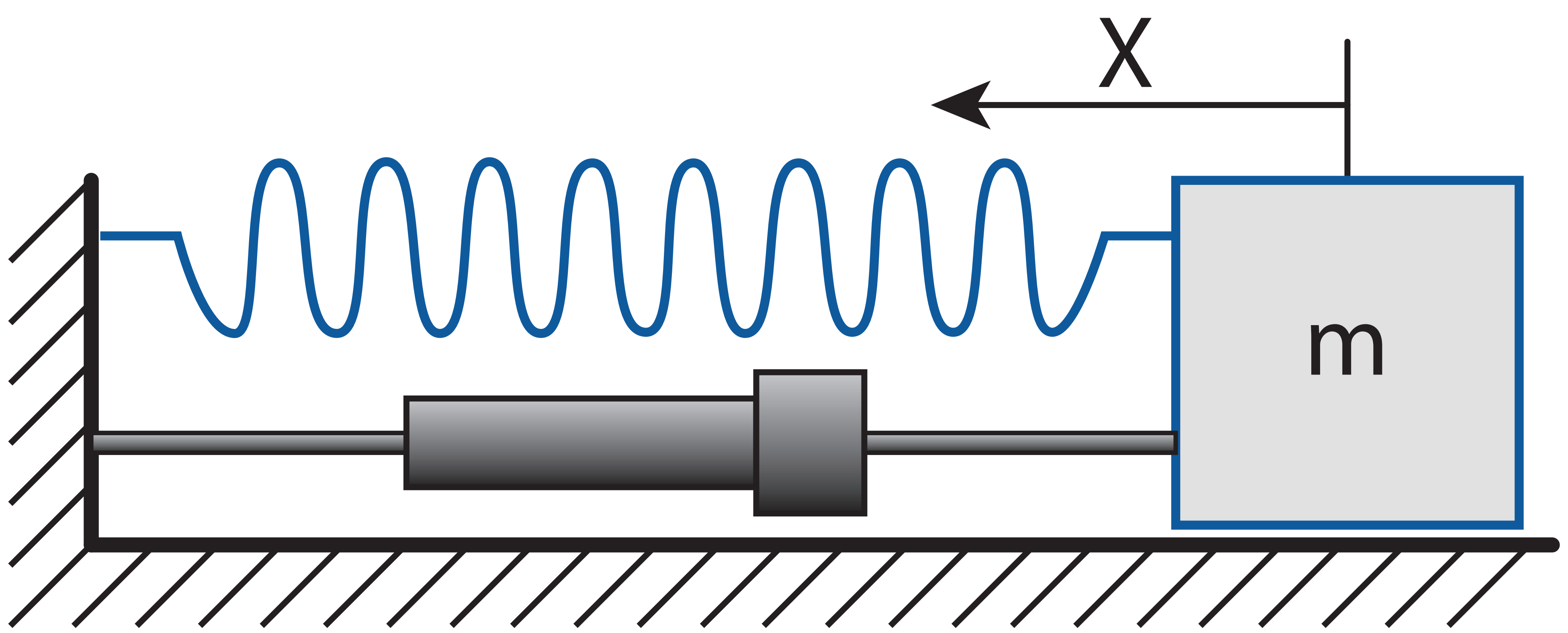
1D Vibration



$$m\ddot{x} + kx = 0$$

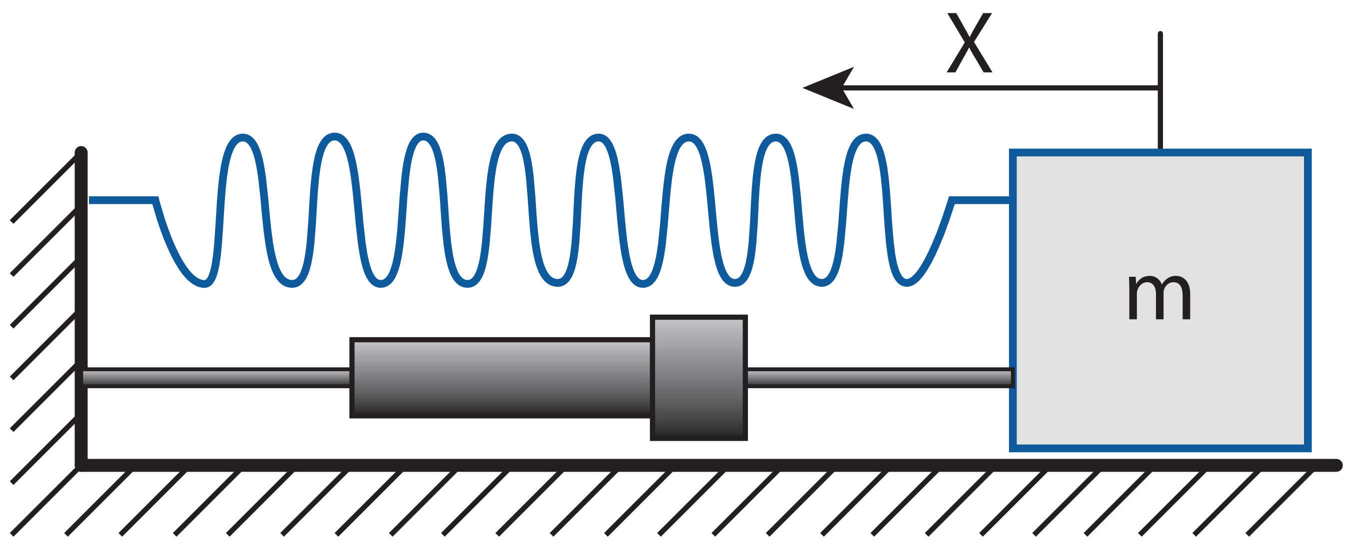
$$x(t) = Ae^{-i\omega t} \qquad \omega = \sqrt{\frac{k}{m}}$$

1D Vibration



$$m\ddot{x} + d\dot{x} + kx = 0$$

1D Vibration



$$m\ddot{x} + d\dot{x} + kx = f(t)$$

Impulse Response $R(t)$:

$$m\ddot{x} + d\dot{x} + kx = \delta(t)$$

Time convolution:

$$f(t) \star R(t) = \int_0^t f(\tau)R(t-\tau)\mathrm{d}\tau$$

Simple Derivation of Impulse Response

Laplace Transform: $$F(s) = \int_0^\infty e^{-st} f(t) dt$$

$$f'(t) \Longrightarrow sF(s) - f(0)$$

$$f''(t) \Longrightarrow s^2F(s) - sf(0) - f'(0)$$

$$\delta(t) \Longrightarrow 1$$

$$(ms^2 + ds + k)\tilde{x}(s) = 1$$

Simple Derivation of Impulse Response

$$\tilde{x}(s) = \frac{1}{ms^2 + ds + k}$$

Inverse Laplace Transform:

$$x(t) = \frac{1}{m\omega_d} e^{-\xi\omega t} \sin \omega_d t$$

$$\omega = \sqrt{\frac{k}{m}}, \quad \xi = \frac{c}{2m\omega} \quad \omega_d = \omega\sqrt{1-\xi^2}.$$

Numerical Integration

Explicit Euler

Implicit Euler

Runge-Kutta Method

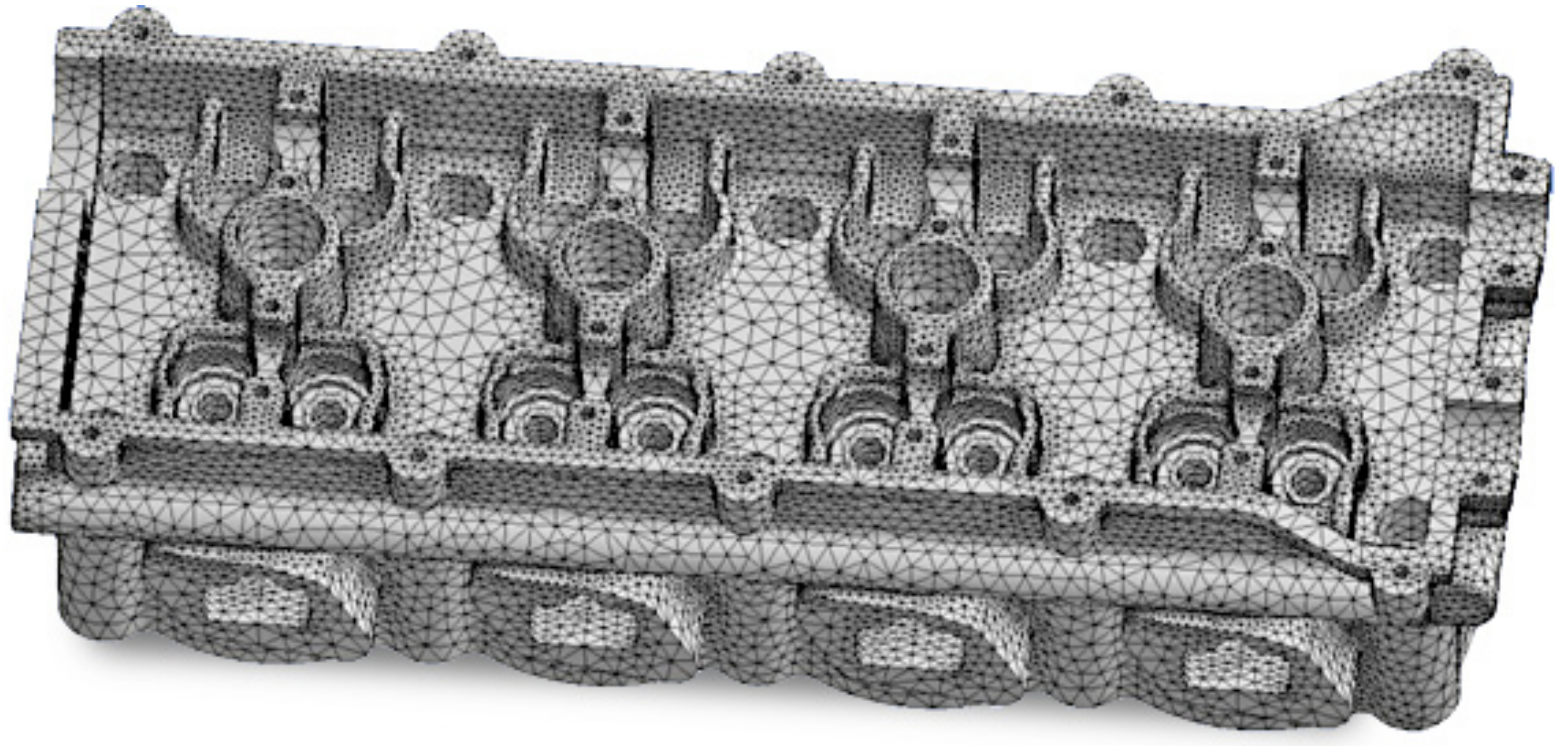
Digital IIR Filter:

$$x_k = 2\varepsilon \cos\theta x_{k-1} - \varepsilon^2 x_{k-2} + \frac{2 f_{k-1}[\varepsilon \cos(\theta + \gamma) - \varepsilon^2 \cos(2\theta + \gamma)]}{3 m \omega \omega_d}$$

$$\varepsilon = e^{-\xi \omega h}, \theta = \omega_d h \quad \gamma = \arcsin \xi$$

Elastic Vibration

$$\mathsf{M}\ddot{\boldsymbol{u}} + \mathsf{D}\dot{\boldsymbol{u}} + \mathsf{K}\boldsymbol{u} = \boldsymbol{f}(t)$$

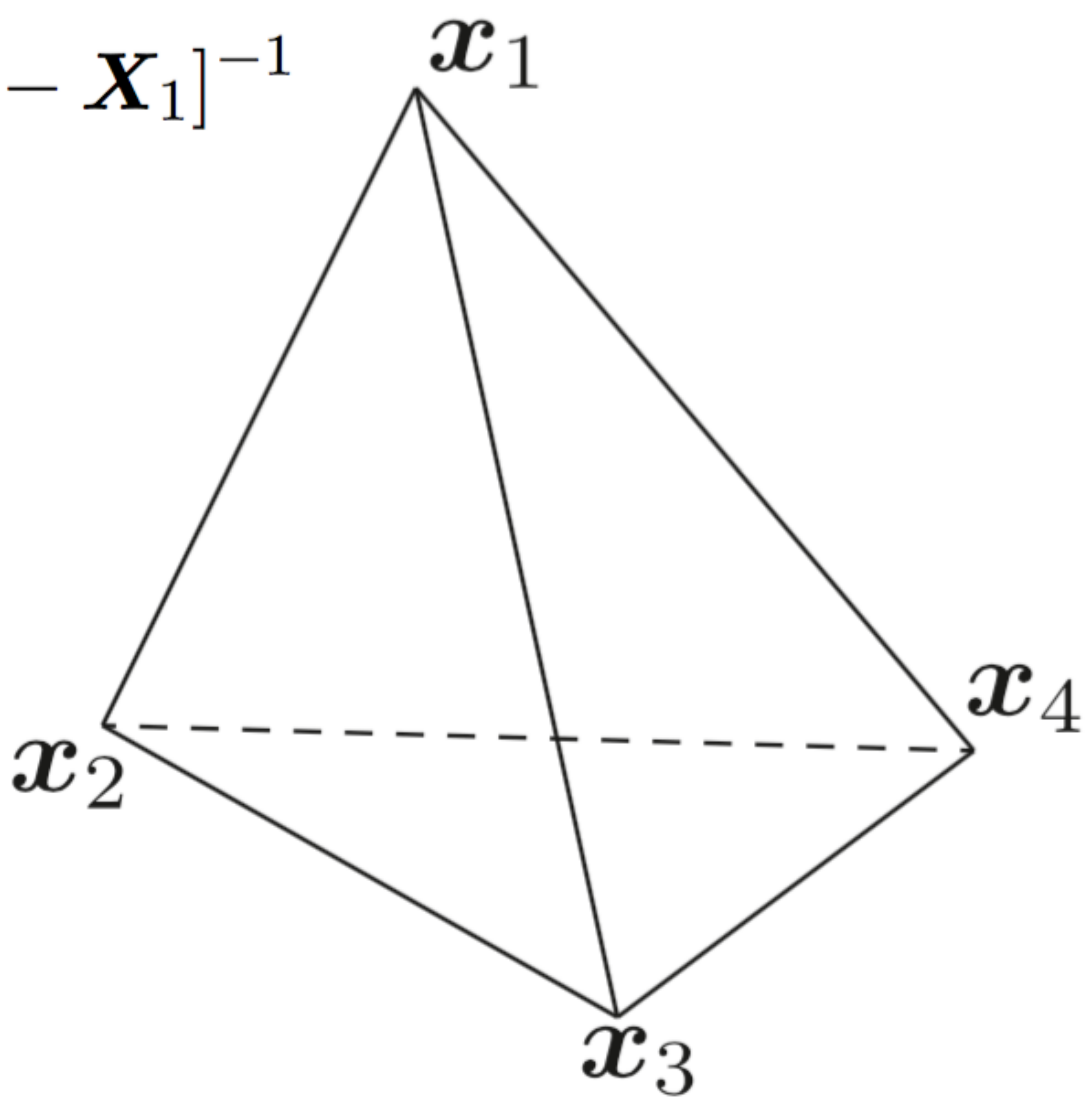


- M: mass matrix
- D: damping matrix
- K: stiffness matrix

$3n \times 3n$ sparse, symmetric matrices

Stiffness Matrix

$$\mathsf{F} = [\boldsymbol{x}_2 - \boldsymbol{x}_1 \;\; \boldsymbol{x}_3 - \boldsymbol{x}_1 \;\; \boldsymbol{x}_4 - \boldsymbol{x}_1][\boldsymbol{X}_2 - \boldsymbol{X}_1 \;\; \boldsymbol{X}_3 - \boldsymbol{X}_1 \;\; \boldsymbol{X}_4 - \boldsymbol{X}_1]^{-1}$$



strain tensor: $\mathsf{E} = \frac{1}{2}(\mathsf{F} + \mathsf{F}^T)$

stress tensor: $\mathsf{S} = \mathsf{C} : \mathsf{E}$

$$\mathsf{S}_{ab} = \sum_{i,j} \mathsf{C}_{abij} \mathsf{E}_{ij}$$

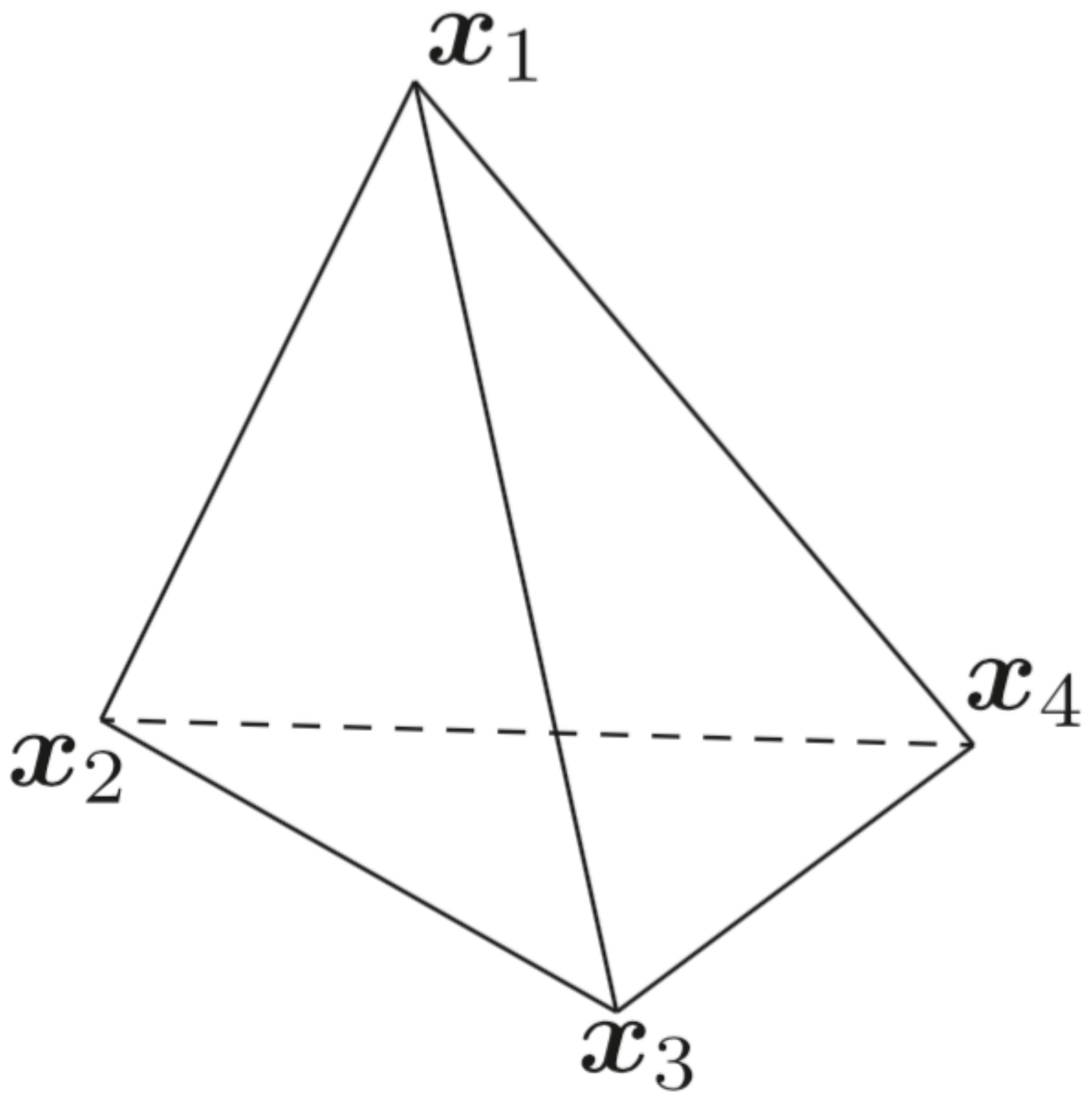
Stiffness Matrix

Stress tensor: S = C : E

For isotropic, homogeneous material:

$$C_{ijkl} = K\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})$$

i,j

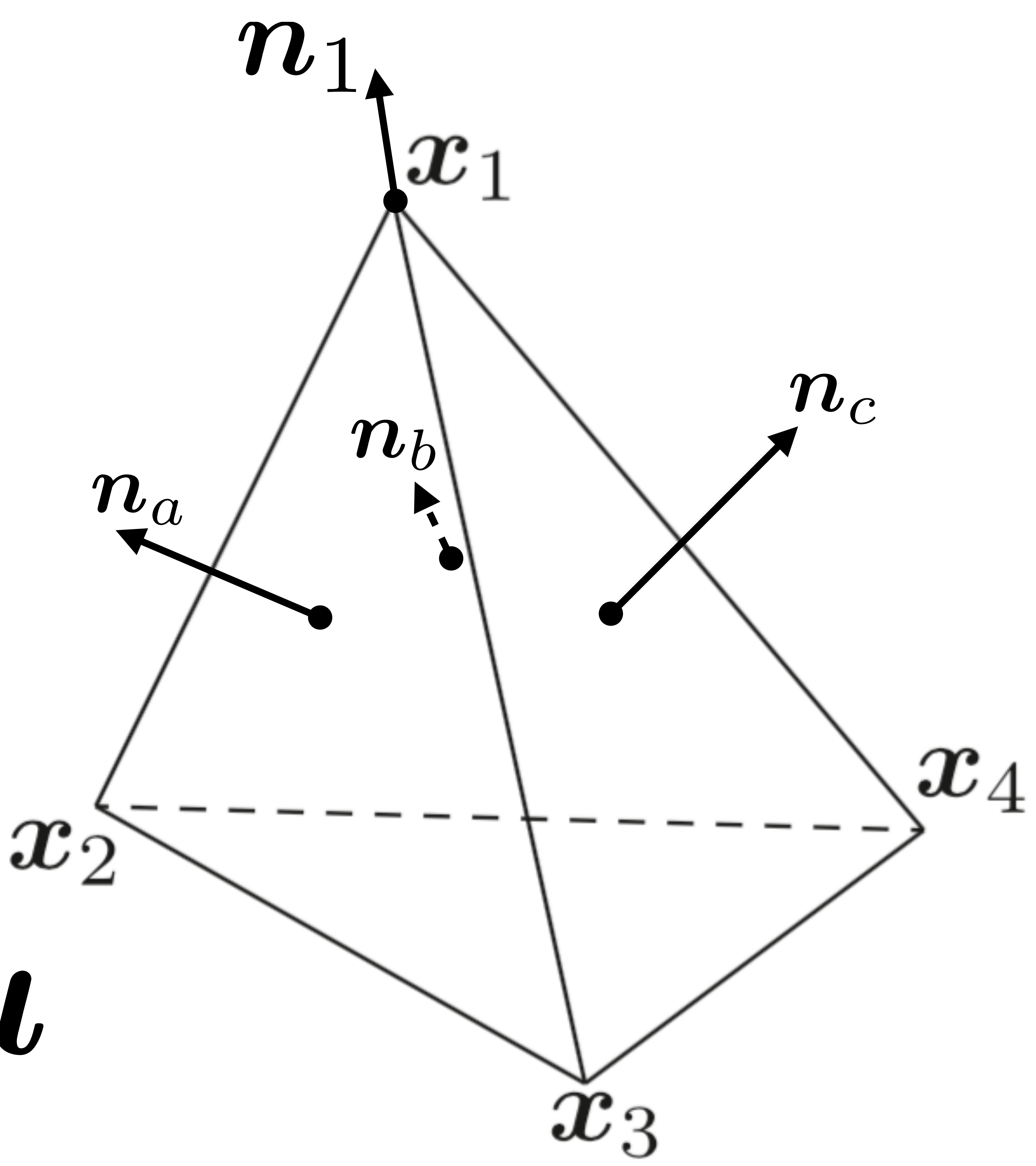


Stiffness Matrix

Internal elastic force

$$f = \mathsf{S}\boldsymbol{n}$$

$$= (\mathsf{C} : \mathsf{E})\boldsymbol{n}$$



force is linear w.r.t. displacement $\boldsymbol{u}$

$$\boldsymbol{f} = \mathsf{K}\boldsymbol{u}$$

Rayleigh Damping

$$D = \alpha M + \beta K$$

- Empirical model
- Lack proper understanding of energy dissipation and damping behavior

Linear Modal Analysis

Goal: solve

$$\mathsf{M}\ddot{\boldsymbol{u}} + \mathsf{D}\dot{\boldsymbol{u}} + \mathsf{K}\boldsymbol{u} = \boldsymbol{f}(t)$$

Generalized eigenvalue decomposition

$$\mathsf{MU} = \mathsf{MUS}$$

with

$$\mathsf{U}^T\mathsf{MU} = \mathsf{I} \quad \text{and} \quad \mathsf{U}^T\mathsf{KU} = \mathsf{S}$$

Change of Variables

Goal: solve

$$\mathsf{M}\ddot{\boldsymbol{u}} + \mathsf{D}\dot{\boldsymbol{u}} + \mathsf{K}\boldsymbol{u} = \boldsymbol{f}(t)$$

Let $\boldsymbol{u} = \mathsf{U}\boldsymbol{q}$

$$\mathsf{U}^T\mathsf{M}\mathsf{U}\ddot{\boldsymbol{q}} + \mathsf{U}^T\mathsf{D}\mathsf{U}\dot{\boldsymbol{q}} + \mathsf{U}^T\mathsf{K}\mathsf{U}\boldsymbol{q} = \mathsf{U}^T\boldsymbol{f}(t)$$

$$\Rightarrow \quad \ddot{\boldsymbol{q}} + (\alpha\mathsf{I} + \beta\mathsf{S})\dot{\boldsymbol{q}} + \mathsf{S}\boldsymbol{q} = \mathsf{U}^T\boldsymbol{f}(t)$$

Pipeline

- Generate volumetric mesh
- Construct the stiffness & mass matrices
- Run generalized eigen decomposition
- Construct the decoupled modal vibration equations
- Numerically integrate individual modes

Thank you!