On the relationship between Radiance and Irradiance: Determining the illumination from images of a convex Lambertian object

Ravi Ramamoorthi and Pat Hanrahan

Computer Science Dept., Stanford University

Gates Wing 3B-386, Stanford, CA 94305 USA

{*ravir,hanrahan*}@*graphics.stanford.edu*

http://graphics.stanford.edu/papers/invlamb/

We present a theoretical analysis of the relationship between incoming radiance and irradiance. Specifically, we address the question of whether it is possible to compute the incident radiance from knowledge of the irradiance at all surface orientations. This is a fundamental question in computer vision and inverse radiative transfer. We show that the irradiance can be viewed as a simple convolution of the incident illumination, i.e. radiance and a *clamped cosine* transfer function. Estimating the radiance can then be seen as a deconvolution operation. We derive a simple closed-form formula for the irradiance in terms of spherical-harmonic coefficients of the incident illumination and demonstrate that the odd-order modes of the lighting with order greater than one are completely annihilated. Therefore, these components cannot be estimated from the irradiance, contradicting a theorem due to Preisendorfer.

A practical realization of the radiance-from-irradiance problem is the estimation of the lighting from images of a homogeneous convex curved Lambertian surface of known geometry under distant illumination, since a Lambertian object reflects light equally in all directions proportional to the irradiance. We briefly discuss practical and physical considerations, and describe a simple experimental test to verify our theoretical results.

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1 Introduction

This paper presents a theoretical analysis of the relationship between incoming radiance and irradiance. Radiance and irradiance are basic optical quantities, and their relationship is of fundamental interest to many fields, including computer vision, radiative transfer, and computer graphics. Physically, we are interested in analyzing the properties of the light field generated when a homogeneous convex curved Lambertian surface of known geometry reflects a distant illumination field. A Lambertian surface reflects light proportional to the incoming irradiance, so analysis of this physical system is equivalent to a mathematical analysis of the relationship between incoming radiance and irradiance.

The specific question of interest to us in this paper is the estimation of the incident radiance from the irradiance, i.e. estimation of the lighting from observations of a Lambertian surface. We are able to derive a closed form formula in terms of the spherical harmonic coefficients for the irradiance, and to thereby show that odd modes of the lighting with order greater than one cannot be estimated.

The rest of this paper is organized as follows. In section 2, we briefly discuss some previous work. In section 3, we introduce the mathematical and physical preliminaries. In section 4, we obtain the equation for the irradiance in terms of spherical harmonic coefficients. Section 5 applies this result to the main problem of this paper, recovering the input radiance from the irradiance, and derives the main results of this paper. Section 6 briefly discusses some practical considerations and describes a simple experimental verification. We discuss some of the broader implications of our results and conclude the paper in section 7.

2 Previous Work

The *radiance from irradiance* problem as discussed in this paper is addressed by Preisendorfer[1] in his treatise on hydrologic optics. He considers the recovery of radiance, given irradiance at all surface orientations. Preisendorfer's conclusion is that irradiance and radiance are equivalent and irradiance can be inverted to give the input radiance. By deriving a simple closed-form formula, we will show that this assertion is not true. More recently, Marschner and Greenberg[2] have considered the inverse lighting problem, assuming curved Lambertian surfaces, and have proposed a practical solution method. However, they have noted the problem to be ill-conditioned, and have made extensive use of regularization. By deriving a closed-form formula for the case of convex objects, we can explain the ill-conditioning observed by Marschner and Greenberg[2] and propose alternative algorithms.

Sato et al.[3] use shadows instead of curvature to recover the lighting in a scene, and also estimate the reflectance parameters of a planar surface. Inverse problems in transport theory have also been studied in other areas such as radiative transfer and neutron scattering. See McCormick[4] for a review. However, to the best of our knowledge, these researchers have not addressed the specific radiance from irradiance problem treated here, wherein we use the varying reflected radiance over a curved Lambertian surface to estimate the input lighting.

The work presented here is related to efforts in many different areas of computer graphics and vision, and is likely to be of interest to these communities. Within the context of rendering surfaces under distant illumination (referred to as *environment mapping* within computer graphics), many previous authors like Miller and Hoffman[5] and Cabral et al.[6] have qualitatively described reflection as a convolution, and have empirically demonstrated that a Lambertian BRDF behaves like a low-pass filter. In computer vision, similar observations have been made by many researchers such as Haddon and Forsyth,[7] and Jacobs et al.[8] Our main contribution is in formalizing these previous qualitative results by deriving analytic quantitative formulae relating the incoming radiance to the irradiance.

Although our goals are very different, our work is also related to efforts in the object recognition community to characterize the appearance of a Lambertian surface under all possible illumination conditions, and it is likely that our results will be applicable to this problem. Belhumeur and Kriegman[9] have theoretically described this set of images in terms of an illumination cone, while empirical results have been obtained by Epstein et al.[10] In independent work simultaneous with our own, Basri and Jacobs[11] have described Lambertian reflection as a convolution, and have applied the results to face recognition.

In order to derive an analytic formula for the input illumination, we must analyze the properties of the reflected light field from a homogeneous Lambertian surface. The light field[12] is a fundamental quantity in light transport and therefore has wide applicability for both forward and inverse problems in a number of fields. A good introduction to the various radiometric quantities derived from light fields that we will use in this paper is provided by McCluney.[13] Light fields have been used directly for rendering images from photographs in computer graphics, without considering the underlying geometry,[14,15] or by parameterizing the light field on the object surface.[16] In previous work, we[17] have performed a theoretical analysis of 2D or *flatland* light fields, which is similar in spirit to the analysis in this paper for 3D Lambertian surfaces.

To derive our results, we will represent quantities using spherical harmonics.[18–20] In previous work, D'Zmura[21] has qualitatively analyzed reflection as a linear operator in terms of spherical harmonics. Basis functions have also been used in representing BRDFs for computer graphics. A number of authors[22–24] have used spherical harmonics, while Koenderink and van Doorn[25] have described BRDFs using Zernike polynomials.

3 Preliminaries

Assumptions: Mathematically, we are simply considering the relationship between the irradiance, expressed as a function of surface orientation, and the incoming radiance, expressed as a function of incident angle. The corresponding physical system is a curved convex homogeneous Lambertian surface reflecting a distant illumination field. For the physical system, we will assume that the surfaces under consideration are convex, so they may be parameterized by the surface orientation, as described by the surface normal, and so that interreflection and shadowing can be ignored. Also, surfaces will be assumed to be Lambertian and homogeneous, so the reflectivity can be characterized by a constant albedo. We will further assume here that the illuminating light sources are distant, so the illumination or incoming radiance can be represented as a function of direction only. This also means the incident illumination does not depend directly upon surface position, but only on surface orientation. Finally, for the purposes of experimental measurement, we will assume that the geometry of the object and its location with respect to the camera is known, so that we can relate each image pixel to a particular location on the object surface.

Notation used in the paper is listed in table 1. A diagram of the local geometry of the situation is shown in figure 1. We will use two types of coordinates. Unprimed global coordinates denote angles with respect to a global reference frame. On the other hand, primed local coordinates denote angles with respect to the local reference frame, defined by the local surface normal and an arbitrarily chosen tangent vector. These two coordinate systems are related simply by a rotation, and this relationship will be detailed shortly.

The Reflection Equation: In local coordinates, we can relate the irradiance to the incoming radiance by

$$E(\mathbf{x}) = \int_{\Omega'} L(\mathbf{x}, \theta_i', \phi_i') \cos\theta_i' \, d\Omega' \tag{1}$$

where E is the irradiance, as a function of position $\mathbf{x}$ on the object surface, and L is the radiance of the incident light field. As noted earlier, primes denote quantities in local coordinates. The integral is over the upper hemisphere with respect to the local surface normal.

Practically, we may observe the irradiance by considering the brightness of a homogeneous Lambertian reflector. For a Lambertian surface with constant reflectance, we can relate the reflected radiance to the irradiance by

$$B(\mathbf{x}) = \rho E(\mathbf{x}) \tag{2}$$

where B is the *radiant exitance* (i.e. brightness) as a function of position $\mathbf{x}$ on the object surface, ρ is the surface reflectance, which lies between 0 and 1, and E is the irradiance defined in equation 1. In computer graphics, the radiant exitance B is usually referred to by the older term, *radiosity*.

Note that the radiant exitance B is simply a scaled version of the irradiance E, and is exactly equal to E if the Lambertian surface has reflectance $\rho = 1$. Furthermore, for a Lambertian object, the reflected radiance is the same in all directions, and is therefore directly proportional to the radiant exitance, being given by B/π. In the rest of this paper, we will analyze equation 1, relating the irradiance E to the incident radiance L. It should be understood that the practically measurable quantities for a Lambertian surface—the radiant exitance B, or the corresponding reflected radiance—are simply scaled versions of the irradiance, and can be derived trivially from E and the surface reflectance ρ by using equation 2.

We now manipulate equation 1 by performing a number of substitutions. First, the assumption of distant illumination means the illumination field is homogeneous over the surface, i.e. independent of surface position $\mathbf{x}$, and depends only on the *global* incident angle (θ_i, ϕ_i). This allows us to replace $L(\mathbf{x}, \theta_i', \phi_i') \rightarrow L(\theta_i, \phi_i)$. Secondly, consider the assumption of a curved convex surface. This ensures there is no shadowing or interreflection, so the irradiance is only because of the distant illumination field L. This fact is implicitly assumed in equation 1. Furthermore, since the illumination is distant, we may reparameterize the surface simply by the surface normal $\mathbf{n}$. Equation 1 now becomes

$$E(\mathbf{n}) = \int_{\Omega'} L(\theta_i, \phi_i) \cos\theta_i' \, d\Omega' \tag{3}$$

The goal of this paper is to determine what we can learn about the incident radiance L from measuring the functional dependence of the irradiance E on the surface normal $\mathbf{n}$.

To proceed further, we will parameterize $\mathbf{n}$ by its spherical angular-coordinates (α, β, γ). Here, (α, β) define the angular coordinates of the local normal vector, i.e.

$$\mathbf{n} = [\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha] \tag{4}$$

γ defines the local tangent frame, i.e. rotation of the coordinate axes about the normal vector. For isotropic surfaces—those where there is no preferred tangential direction, i.e. where rotation of the tangent frame about the surface normal has no physical effect—the parameter γ has no physical significance, and we have therefore not explicitly considered it in equation 3. We will include γ for completeness in the ensuing discussion on rotations, but will eventually eliminate it from our equations after showing mathematically that it does in fact have no effect on the final results. Finally, for convenience, we will define a transfer function $A(\theta_i') = \cos\theta_i'$. With these modifications, equation 3 becomes

$$E(\alpha, \beta, \gamma) = \int_{\Omega'} L(\theta_i, \phi_i) A(\theta_i') \, d\,\Omega' \tag{5}$$

Note that local and global coordinates are mixed. The lighting is expressed in global coordinates, since it is constant over the object surface when viewed with respect to a global reference frame, while the transfer function $A = \cos\theta_i'$ is expressed naturally in local coordinates. To analyze this equation further, we will need to apply a rotation corresponding to the surface orientation (α, β, γ) in order to convert the lighting into local coordinates.

Rotating the Lighting: We have assumed that the incoming light field remains constant (in a global frame) over the object. To convert to the local coordinates in which (θ_i', ϕ_i') are expressed, we must perform the appropriate rotation on the lighting. Let $L(\theta_i, \phi_i)$ be the global incoming radiance and (α, β, γ) the parameters corresponding to the local surface orientation. We define $R^{\alpha,\beta,\gamma}$ to be a rotation operator that rotates (θ_i', ϕ_i') into global coordinates (θ_i, ϕ_i). $R^{\alpha,\beta,\gamma}$ can be expressed in terms of the standard Euler-Angle representation, and is given by $R^{\alpha,\beta,\gamma} = R_z(\beta) R_y(\alpha) R_z(\gamma)$, where R_z is a rotation about the Z axis and R_y about the Y axis. Refer to figure 2 for an illustration. It is easy to verify that this rotation correctly transforms the local coordinates $(0', 0')$—corresponding to the local representation of the Z-axis, i.e. surface normal—to the global coordinates (α, β)—corresponding to the global representation of the surface normal. The relevant transformations are given below.

$$
\begin{aligned}
R^{\alpha,\beta,\gamma} &= R_z(\beta)R_y(\alpha)R_z(\gamma) \\
(\theta_i, \phi_i) &= R^{\alpha,\beta,\gamma}\left(\theta_i', \phi_i'\right) \\
L(\theta_i, \phi_i) &= L\left(R^{\alpha,\beta,\gamma}\left(\theta_i', \phi_i'\right)\right) \qquad (6)
\end{aligned}
$$

Note that the angular parameters are rotated as if they were a unit vector pointing in the appropriate direction. It should also be noted that this rotation of parameters is equivalent to an inverse rotation of the function, with R^{-1} being given by $R_z(-\gamma)R_y(-\alpha)R_z(-\beta)$.

Finally, we can plug equation 6 into equation 5 to derive

$$
\boxed{E(\alpha, \beta, \gamma) = \int_{\Omega'} L\left(R^{\alpha,\beta,\gamma}\left(\theta_i', \phi_i'\right)\right) A(\theta_i')\, d\,\Omega'} \qquad (7)
$$

Note that this equation is essentially a convolution, although we have a rotation operator rather than a translation. The irradiance can be viewed as a convolution of the incident illumination L and the transfer function $A = \cos\theta_i'$. Different observations of the irradiance E, at points on the object surface with different orientations, correspond to different rotations of the transfer function—since the local upper hemisphere is rotated—which can also be thought of as different rotations of the incident light field. In the next section, we will see that this integral becomes a simple product when transformed to spherical harmonics, further stressing the analogy with convolution. Our goal will then be to *deconvolve* the irradiance in order to recover the incident illumination.

4 Spherical Harmonic Representation

We now proceed to construct a closed-form description of the irradiance in terms of spherical harmonic coefficients. Spherical harmonics are the analogue on the sphere to the Fourier basis on the line or circle. The spherical harmonic $Y_{l,m}$ is given by

$$Y_{l,m}(\theta, \phi) = N_{l,m} P_{l,m}(\cos\theta) \exp(Im\phi) \tag{8}$$

where $N_{l,m}$ is a normalization factor. In the above equation, the azimuthal dependence is expanded in terms of Fourier basis functions. The θ dependence is expanded in terms of the associated Legendre functions $P_{l,m}$. The indices obey $l \geq 0$ and $-l \leq m \leq l$. Thus, there are $2l+1$ basis functions for given order l. Inui et al.[18] is a good reference for background on spherical harmonics and their relationship to rotations.

We begin by expanding the lighting in global coordinates in terms of spherical harmonics.

$$L(\theta_i, \phi_i) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} L_{l,m} Y_{l,m}(\theta_i, \phi_i) \tag{9}$$

We will then transform to local coordinates by applying the appropriate rotations.

Rotation of Spherical Harmonics: Let us now build up the rotation operator on the spherical harmonics. We will use notation of the form $R()\,\{Y_{l,m}()\}$ to stand for a rotation by the rotation operator R of the parameters of the spherical harmonic $Y_{l,m}$. For instance,

$$R_z(\beta)\,\{Y_{l,m}(\theta_i', \phi_i')\} = Y_{l,m}\left(R_z(\beta)\,\{\theta_i', \phi_i'\}\right) \tag{10}$$

First, from the form of the spherical harmonics, rotation about z is simple. Specifically,

$$R_z(\beta)\,\{Y_{l,m}(\theta_i', \phi_i')\} = Y_{l,m}(\theta_i', \phi_i' + \beta) = \exp(Im\beta) Y_{l,m}(\theta_i', \phi_i') \tag{11}$$

Rotation about y is more complicated and is given by

$$R_y(\alpha)\,\{Y_{l,m}(\theta_i', \phi_i')\} = \sum_{m'=-l}^{l} D^l_{m,m'}(\alpha) Y_{l,m'}(\theta_i', \phi_i') \tag{12}$$

where D^l is a $(2l+1) \times (2l+1)$ matrix that tells us how a spherical harmonic transforms under rotation about the y-axis, i.e. how to rewrite a rotated spherical harmonic as a linear combination of all the spherical harmonics of the same order. The important thing to note here is that the m indices are *mixed*—a spherical harmonic after rotation must be expressed as a combination of other spherical harmonics with different m indices. However, the l indices are not mixed; rotations of spherical harmonics with order l are composed entirely of other spherical harmonics with order l.

Finally, we can combine the above two equations, with a similar result for the z rotation by γ to derive the required rotation formula:

$$\begin{aligned}
R^{\alpha,\beta,\gamma}\left\{Y_{l,m}(\theta_i',\phi_i')\right\} &= R_z(\beta)R_y(\alpha)R_z(\gamma)\left\{Y_{l,m}(\theta_i',\phi_i')\right\} \\
&= \sum_{m'=-l}^{l} \tilde{D}^l_{m,m'}(\alpha,\beta,\gamma)Y_{l,m'}(\theta_i',\phi_i') \\
\tilde{D}^l_{m,m'}(\alpha,\beta,\gamma) &= D^l_{m,m'}(\alpha)\exp(Im\beta)\exp(Im'\gamma) \qquad (13)
\end{aligned}$$

In terms of group theory, the matrix $\tilde{D}$ can be viewed as the $2l+1$-dimensional representation of the rotation group $SO(3)$, with the interesting α-dependence being encapsulated by D. Equation 13 is simply the standard rotation formula for spherical harmonics.

Since the transfer function $A = \cos\theta_i'$ has no azimuthal dependence, terms with $m' \neq 0$ will vanish when we perform the integral in equation 7. Therefore, we will be most interested in the coefficient of the term with $m' = 0$ i.e. $\tilde{D}^l_{m,0} = D^l_{m,0}(\alpha)\exp(Im\beta)$. It can be shown that this is simply equal to $\left(\sqrt{4\pi/(2l+1)}\right) Y_{l,m}(\alpha,\beta)$. This result will be assumed in equation 23.

Another way to derive the result just stated without appealing to the properties of the matrix D is to realize that we simply want the coefficient of the term with no azimuthal dependence. At $\theta_i' = 0$, the rotated function is determined only by the term with $m' = 0$. In fact, it can be shown (Jackson[19] eq. 3.59) that with $\tilde{D}^l_{m,0}$ equal to the desired coefficient.

$$R^{\alpha,\beta,\gamma}\left\{Y_{l,m}(0',\phi_i')\right\} = \sqrt{\frac{2l+1}{4\pi}}\tilde{D}^l_{m,0}(\alpha,\beta,\gamma) \tag{14}$$

Noting that $R^{\alpha,\beta,\gamma}\left\{Y_{l,m}(0',\phi_i')\right\}$ corresponds to evaluating $Y_{l,m}$ at the local Z-axis or normal, i.e. global coordinates of (α,β), we see that $\tilde{D}^l_{m,0} = \left(\sqrt{4\pi/(2l+1)}\right)Y_{l,m}(\alpha,\beta)$.

Representing the Transfer Function: We switch our attention now to representing the transfer function $A(\theta_i') = \cos\theta_i'$. Since an object only reflects the upper hemisphere, $A(\theta_i')$ is nonzero only when $\cos\theta_i' > 0$. The transfer function $A(\theta_i') = 0$ over the lower hemisphere where $\cos\theta_i' < 0$. We may refer to the transfer function as the *clamped cosine* function since it is equal to the cosine function over the upper hemisphere, but is clamped to 0 when $\cos\theta_i' < 0$ over the lower hemisphere. We will need to use many formulas for representing integrals of spherical harmonics, for which a reference[20] will be useful. First, we expand the transfer function in terms of spherical harmonics without azimuthal dependence.

$$A(\theta_i') = \cos\theta_i' = \sum_{n=0}^{\infty} A_n Y_{n,0}(\theta_i') \tag{15}$$

The coefficients are given by

$$A_n = 2\pi\int_0^{\frac{\pi}{2}} Y_{n,0}(\theta_i')\cos\theta_i'\sin\theta_i' d\,\theta_i' \tag{16}$$

The factor of 2π comes from integrating 1 over the azimuthal dependence. It is important to note that the limits of the integral range from 0 to $\pi/2$ and not π because we are considering only the upper hemisphere. The expression above may be simplified by writing in terms of Legendre polynomials $P(\cos\theta_i')$. Putting $u = \cos\theta_i'$ in the above integral and noting that

$P_1(u) = u$ and that $Y_{n,0}(\theta_i') = \sqrt{(2n+1)/(4\pi)}P_n(\cos\theta_i')$, we obtain

$$A_n = 2\pi\sqrt{\frac{2n+1}{4\pi}}\int_0^1 P_n(u)P_1(u)\,du \tag{17}$$

To gain further insight, we need some facts regarding the Legendre polynomials. P_n is odd if n is odd, and even if n is even. The Legendre polynomials are orthogonal over the domain $[-1,1]$ with the orthogonality relationship being given by

$$\int_{-1}^1 P_a(u)P_b(u) = \frac{2}{2a+1}\delta_{a,b} \tag{18}$$

From this, we can establish some results about equation 17. When n is equal to 1, the integral evaluates to half the norm above, i.e. 1/3. When n is odd but greater than 1, the integral in equation 17 vanishes. This is because for $a = n$ and $b = 1$, we can break the left-hand side of equation 18 using the oddness of a and b into two equal integrals from $[-1,0]$ and $[0,1]$. Therefore, both of these integrals must vanish, and the latter integral is the right-hand integral in equation 17. When n is even, the required formula is given by manipulating equation 20 in chapter 5 of MacRobert.[20] Putting it all together, we have:

$$\boxed{\begin{array}{rl} n=1 & A_n = \sqrt{\dfrac{\pi}{3}} \\[2ex] n>1, odd & A_n = 0 \\[2ex] n \;\; even & A_n = 2\pi\sqrt{\dfrac{2n+1}{4\pi}}\dfrac{(-1)^{\frac{n}{2}-1}}{(n+2)(n-1)}\left[\dfrac{n!}{2^n(\frac{n}{2}!)^2}\right] \end{array}} \tag{19}$$

We can determine the asymptotic behavior of A_n for large even n by using Stirling's formula. The bracketed term goes as $n^{-1/2}$, which cancels the term in the square root. Therefore, the asymptotic behavior for even terms is $A_n \sim n^{-2}$. A plot of A_n for the first few terms is shown in figure 3, and approximation of the *clamped cosine* by spherical harmonic terms as n increases is shown in figure 4.

Spherical Harmonic version of the Reflection Equation We now have the necessary tools to write equation 7 in terms of spherical harmonics. Substituting equation 13 and equation 15, we obtain

$$\begin{aligned} E(\alpha,\beta,\gamma) &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l}^{l}\left(L_{l,m}A_n D^l_{m,m'}(\alpha)\exp(Im\beta)\exp(Im'\gamma)T_{n,l,m'}\right) \\ T_{n,l,m'} &= \int_{\phi_i'=0}^{2\pi}\int_{\theta_i'=0}^{\pi} Y_{l,m'}(\theta_i',\phi_i')Y_{n,0}(\theta_i',\phi_i')\sin\theta_i'\,d\theta_i'\,d\phi_i' \end{aligned} \tag{20}$$

Note that we have summed over all indices, and have removed the restriction on the integral to the upper hemisphere, because that restriction has now already been folded into the coefficients A_n. By orthonormality of the spherical harmonics,

$$\int_{\phi_i'=0}^{2\pi}\int_{\theta_i'=0}^{\pi} Y_{l,m'}(\theta_i',\phi_i')Y_{n,0}(\theta_i',\phi_i')\sin\theta_i'\,d\theta_i'\,d\phi_i' = \delta_{l,n}\delta_{m',0} \tag{21}$$

Therefore, terms that do not satisfy $n = l, m' = 0$ will vanish. Making these substitutions in equation 20, we obtain

$$E(\alpha,\beta,\gamma) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} L_{l,m}A_l D^l_{m,0}(\alpha)\exp(Im\beta) \tag{22}$$

As noted earlier, it can be shown that

$$D^l_{m,0}(\alpha)\exp(Im\beta) = \sqrt{\frac{4\pi}{2l+1}}Y_{l,m}(\alpha,\beta) \tag{23}$$

With this relation, we can write

$$E(\alpha,\beta,\gamma) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\sqrt{\frac{4\pi}{2l+1}}A_l L_{l,m}Y_{l,m}(\alpha,\beta)\right) \tag{24}$$

To complete the expansion in terms of spherical harmonics, we first drop the γ dependence of E. We can see that the right hand side of the equation above does not depend on γ, as required by physical considerations, since γ has no physical significance. Then, we complete the expansion in terms of spherical harmonics by also expanding the irradiance $E(\alpha, \beta)$, i.e.

$$E(\alpha, \beta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} E_{l,m} Y_{l,m}(\alpha, \beta) \tag{25}$$

We can now equate coefficients to obtain the reflection equation in terms of spherical harmonic coefficients:

$$\boxed{E_{l,m} = \sqrt{\frac{4\pi}{2l+1}} A_l L_{l,m}} \tag{26}$$

Discussion: This equation states that the standard direct illumination integral in equation 7 can be viewed as a simple product in terms of spherical harmonic coefficients. This is not really surprising, considering that equation 7 can be interpreted as showing that the irradiance is a *convolution* of the incident illumination and the transfer function, with different observations E corresponding to different rotations of the incident light field. Since equation 26 is in terms of spherical harmonic coefficients, it can be viewed in signal processing terms as a filtering operation; the output light field can be obtained by filtering the input lighting using the *clamped-cosine* transfer function.

Since A_l vanishes for odd values of $l > 1$, as seen in equation 19, the irradiance has zero projection onto odd-order modes, i.e. $E_{l,m} = 0$ when $l > 1$ and odd. In terms of the filtering analogy, since the filter corresponding to $A = \cos\theta_i'$ destroys high-frequency odd terms in the spherical harmonic expansion of the lighting, the corresponding terms are not found in the irradiance. Further, for large even l, the asymptotic behavior of $E_{l,m} \sim l^{-5/2}$ since $A_l \sim l^{-2}$. The transfer function A acts as a low pass filter, causing the rapid decay of high-frequency terms in the lighting.

We are now ready to answer the question motivating this paper—to what extent can we estimate the incoming radiance or illumination distribution, given the irradiance at all orientations or surface normals. This can be viewed as a problem of deconvolution. Equation 26 makes it trivial to formulate a closed-form solution.

5 Radiance from Irradiance

In this section, we discuss the inverse lighting problem. We want to find $L_{l,m}$ given the functional dependence of the irradiance on the surface normal. A practical realization is that we want to find the incident illumination from observations of a homogeneous convex curved Lambertian surface. From equation 26, it is trivial to derive a simple closed-form relation:

$$L_{l,m} = \sqrt{\frac{2l+1}{4\pi}} \frac{E_{l,m}}{A_l} \tag{27}$$

We will have difficulty solving for $L_{l,m}$ only if for all m, $E_{l,m}$ and A_l both vanish, in which case the right-hand side cannot be determined. We have already seen that this happens when l is odd and $l > 1$. This brings us to our main result.

Theorem 1 *In general, it is not possible to recover the odd-order spherical harmonic modes with order > 1 of a distant radiance distribution from information about the irradiance at every surface orientation. In practical terms, observations of a homogeneous convex curved Lambertian surface do not determine the odd order modes with order > 1 of the incoming distant illumination field.*

This theorem is simple to understand in terms of signal processing. The filter A has no amplitude along certain modes, and annihilates the corresponding lighting coefficients when convolved with the incident illumination. Therefore, a deconvolution method cannot recover the corresponding original components of the lighting. A stronger version of the theorem is that adding a perturbation to the incident light field comprising only of odd modes with order > 1 does not change the irradiance.

Theorem 2 *A perturbation of the incident distant illumination field consisting entirely of a linear superposition of odd-order spherical harmonic modes with order > 1 has no effect on the irradiance at any surface orientation, and hence no effect on the reflected radiance from, or on the appearance of, a convex curved Lambertian object.*

As a simple example, consider adding a perturbation $\triangle L = Y_{3,m}(\theta_i, \phi_i)$ for any m to the incident lighting L. We consider the change in irradiance $\triangle E$ at a point with surface normal in spherical coordinates (α, β). To evaluate this, we need to rotate the lighting into the local coordinate frame. We know that for spherical harmonics, rotation does not change the order, but merely mixes the indices. Since the cosine in the irradiance integral does not have azimuthal dependence, the only term that can affect the irradiance is the term without azimuthal dependence i.e. $Y_{3,0}$ in local coordinates. The precise coefficient of this term is not important, and for similar reasons, we ignore the factor of 2π that comes from integrating over the azimuthal angle. We can simply denote $C(\alpha, \beta)$ as a constant factor and write:

$$\triangle E(\alpha, \beta) = C(\alpha, \beta) \int_0^{\pi/2} Y_{3,0}(\theta_i') \cos\theta_i' \sin\theta_i' \, d\,\theta_i' \tag{28}$$

Substituting for $Y_{3,0}$, ignoring the pre-multiplying constant which we absorb into C, and then substituting $u = \cos\theta'_i$ we obtain:

$$
\begin{aligned}
\triangle E(\alpha,\beta) &= C(\alpha,\beta)\int_0^{\pi/2}\left(\frac{5}{3}\cos^3\theta'_i - \cos\theta'_i\right)\cos\theta'_i\sin\theta'_i\,d\,\theta'_i \\
&= C(\alpha,\beta)\int_0^{1}\left(\frac{5}{3}u^3 - u\right)u\,d\,u \\
&= C(\alpha,\beta)\int_0^{1}\left(\frac{5}{3}u^4 - u^2\right)d\,u \\
&= C(\alpha,\beta)\left(\frac{5}{3}\frac{1}{5}u^5\,|_0^1 - \frac{1}{3}u^3\,|_0^1\right) \\
&= C(\alpha,\beta)\left(\frac{1}{3} - \frac{1}{3}\right) \\
&= 0 \qquad (29)
\end{aligned}
$$

To make matters concrete, we can now claim that the following two lighting functions are equivalent in that they produce the same irradiance at all surface orientations, and therefore cannot be distinguished by observations of a convex Lambertian surface. The constants have been chosen to ensure that the lighting remains nonnegative everywhere.

$$
\begin{aligned}
L_1(\theta_i,\phi_i) &= 1 \\
L_2(\theta_i,\phi_i) &= 1 + \frac{5}{3}\cos^3\theta_i - \cos\theta_i \qquad (30)
\end{aligned}
$$

Comparison to Preisendorfer: Preisendorfer[1] (volume 2, pages 143-151) concludes that radiance and irradiance are equivalent, with radiance always being recoverable from measurements of irradiance at all surface orientations. His argument is that a positive sum lighting perturbation must result in a positive norm change in the integrated irradiance over all surface orientations. In fact, he shows that the norm of the change in the integral of the irradiance is proportional to the sum of the lighting perturbation. Therefore, he concludes that any lighting perturbation must result in a corresponding perturbation of the reflected light field. He neglects to consider that while overall, the lighting must be nonnegative, it is possible to devise a zero-sum perturbation to the lighting since the perturbation can have both positive and negative components, with the only physical requirement being that the net lighting is nowhere negative. Indeed, all the odd-order modes, including the example above, have zero sum since their integral with Y_{00}, the constant term, must be 0 by orthogonality. Nevertheless, the condition that the lighting must be positive to be physically realizable is important and provides a further constraint on allowable perturbations. We will see that because of the constraint of positivity, there are several important special cases where the irradiance distribution does in fact uniquely determine the radiance or input illumination distribution. However, in general, as evidenced by the example above, the irradiance distribution fails to completely specify the radiance, as summarized in theorems 1 and 2.

Constraining the Lighting to be Positive: We have so far not considered the physical requirement that the lighting be everywhere positive. To be physically realizable, any perturbations of the incident illumination must be small enough that the lighting remains nonnegative everywhere. Here, we will show how this constraint somewhat restricts the set of *allowable* perturbations—perturbations that do not affect the irradiance. Our analysis is fairly straightforward, and we leave for future work a more complete characterization. We start by enumerating two important properties that allowable perturbations must satisfy. Here, we will use $\triangle L$ to denote a perturbation.

1. We know that an allowable perturbation must be constructed of odd-order spherical harmonic modes. These modes have the property of being *odd* over the sphere—the value of a function is negated at the antipodal point on the sphere. This can be written

as

$$\triangle L(\theta_i, \phi_i) = - \triangle L(\pi - \theta_i, \pi + \phi_i) \tag{31}$$

2. The remaining condition is that the perturbation's projection onto order 1 modes must be 0 (since the order 1 modes can be recovered). The order 1 modes are simply the linear terms x, y and z. These modes are linear and odd—their value negates at the antipodal point. Therefore, in conjunction with condition 1, an allowable perturbation must have zero linear moment over any hemisphere—since the antipodal hemisphere is negated for both the function and the order 1 mode. Condition 2 is important in showing that for a directional source, there is no allowable nonzero norm perturbation.

We first consider allowable values of a perturbation, as described by the following lemma.

Lemma 1 *If L is the true value of the incident illumination, the values of an allowable perturbation $\triangle L$ satisfy*

$$-L(\theta_i, \phi_i) \leq \triangle L(\theta_i, \phi_i) \leq L(\pi - \theta_i, \pi + \phi_i) \tag{32}$$

The norm of the maximum allowable perturbation therefore satisfies

$$\mid \triangle L(\theta_i, \phi_i) \mid \leq \max[L(\theta_i, \phi_i), L(\pi - \theta_i, \pi + \phi_i)] \tag{33}$$

The first inequality holds trivially to maintain positivity of L. The second inequality follows from the need to maintain positivity at the antipodal point since $\triangle L(\pi - \theta_i, \pi + \phi_i) = - \triangle L(\theta_i, \phi_i)$. This simple lemma leads to a number of corollaries for various special cases, where we show that increasingly complex lighting conditions admit no nonzero norm perturbation.

Corollary 1 *Allowable perturbations* $\triangle L(\theta_i, \phi_i)$ *must be* 0 *at points where the true lighting value* $L(\theta_i, \phi_i)$ *and the true lighting value for the antipodal point* $L(\pi - \theta_i, \pi + \phi_i)$ *are both* 0*. In particular, if the true lighting* L *is everywhere* 0*, there is no allowable perturbation.*

This follows trivially from equation 33. Note that if the true lighting is a single directional source, this condition forces the allowable perturbation to be 0 everywhere except at the source and its antipodal point.

Corollary 2 *If the true lighting is a single directional source, there is no nonzero norm allowable perturbation, and therefore, the true radiance can be recovered from the irradiance distribution at all surface orientations.*

From the discussion below corollary 1, the allowable perturbation can have a nonzero (negative) value only at the source—with a corresponding negation (positive value) at the antipodal point. A perturbation $-\triangle L$ at the source and $+\triangle L$ at the antipodal point yields a nonzero moment with respect to one or both of $\cos\theta_i$ or $\sin\theta_i$—the order 1 modes. In other words, since we are on a sphere, one of x, y or z has to be nonzero, so the linear moments cannot vanish. Note that by the second required property, these moments must vanish in an allowable perturbation. Therefore, a perturbation consisting of a negative spike—with an equal positive spike at the antipodal point—cannot be constructed using only odd-order spherical harmonics with order > 1. In particular, this allows the corollary to be extended to the case of two antipodal sources, since an allowable perturbation must have the same form—a negative spike at one of the sources along with an equal positive spike at the other source. A much stronger statement is found below.

Corollary 3 *If the true lighting consists of 2 distinct directional sources, or a non-degenerate arrangement of 3 directional sources, there is no allowable nonzero norm perturbation.*

From the previous discussions, we know that allowable perturbations can be nonzero only at the sources and their antipodal points. The condition of vanishing moments with respect to the linear order 1 modes leads to a set of 3 simultaneous equations requiring us to choose perturbation intensities at the the sources so that the moments in x, y, and z vanish. This is not possible for non-degenerate configurations of 3 sources unless all perturbation intensities are 0. It is also not possible for any two distinct sources, since we can reparameterize so one of the sources is on the +Z axis. To make all linear moments vanish, the other source would need to have $x = y = 0$, i.e. be antipodal at -Z—in which case, by the argument below corollary 2, there is no allowable perturbation—or coincide with the first source—violating the requirement that the sources be distinct.

The corollary does not extend to more than 3 directional sources. For four directional sources, we may set the (negative) perturbation arbitrarily at one of the sources to fix the scale. We can then solve the vanishing moment equations for the remaining three perturbations, and the condition for a nonzero norm perturbation is that all three perturbations at the sources are negative, as required by equation 32.

With more than four directional sources, we have much more freedom in choosing the perturbations, and it is likely that there always exist allowable perturbations if the sources are far enough apart, i.e. do not satisfy the conditions of corollary 4 below. Our primary consideration in this paper has been continuous lighting distributions or area sources. For these sources (which can be thought of as made of an infinite number of directional sources), corollary 3 clearly does not apply.

Corollary 4 *If there exists a hemispherical region in which the true lighting $L = 0$ everywhere, i.e. all the sources are strictly confined to one hemisphere, there is no allowable nonzero norm perturbation.*

We can always reparameterize so that all the sources lie in +Z for instance and since $\triangle L \leq 0$ over this hemisphere (since L at the antipode is 0), the linear moment over Z must be negative, violating the condition that the linear moments of the perturbation must vanish. This corollary may have implications for natural lighting when the upper hemisphere is the major contributor. Note that the sources must be *strictly* confined to one hemisphere—a collection of sources at the "equator" or any other great circle does not satisfy the requirements of the corollary.

In the next section, we will consider practical issues with lighting recovery. It will be shown that regardless of the theoretical results derived here, inverse lighting is in practice poorly conditioned, so even in cases where a unique solution exists, a numerical algorithm is unlikely to find it. Nevertheless, the results of this set of corollaries tells us that it will be helpful to try to explicitly ensure positivity of the recovered solutions in any numerical algorithm; this makes possible the solution of some cases—or reduces the norm of the maximum allowable perturbation—that are otherwise ambiguous by theorems 1 and 2.

6 Practical Considerations

This section briefly discusses some practical considerations with respect to the radiance-from-irradiance problem, and describes a simple experiment verifying the formulas derived.

Equation 26 and the following discussion considering asymptotic forms show that the coefficients of the irradiance fall off rapidly i.e. $E_{l,m} \sim l^{-5/2} L_{l,m}$. This indicates that in practice the inverse lighting problem is very poorly conditioned. In fact, we can explicitly write out numerically the first few terms for the irradiance.

$$\begin{aligned}
E_{0,0} &= 3.142 L_{0,0} \\
E_{1,m} &= 2.094 L_{1,m} \\
E_{2,m} &= 0.785 L_{2,m} \\
E_{3,m} &= 0 \\
E_{4,m} &= -0.131 L_{4,m} \\
E_{5,m} &= 0 \\
E_{6,m} &= 0.049 L_{6,m}
\end{aligned} \tag{34}$$

We see that already for $E_{4,m}$, the coefficient is only about 4% of what it is for $E_{0,0}$. Therefore, in real applications—where surfaces are only approximately Lambertian, and there are errors in measurement—we only expect to robustly measure the irradiance up to order 2, and this is the maximum order up to which we can recover the incident illumination. Since there are $2l+1$ indices (values of m, which ranges from $-l$ to $+l$) for order l, this corresponds to 9 coefficients for $l \leq 2$—1 term with order 0, 3 terms with order 1, and 5 terms with order 2. Note that the single order 0 mode $Y_{0,0}$ is a constant, the 3 order 1 modes are linear functions of the Cartesian coordinates—in real form, they are simply x, y, and z—while the 5 order 2 modes are quadratic functions of the Cartesian coordinates. Therefore, the irradiance—or equivalently, the reflected light field from a convex Lambertian object—can be well approximated as a quadratic polynomial of the Cartesian coordinates of the surface normal vector.

Enforcement of positivity constraints and consideration of error metrics based on higher-order derivatives may improve the results somewhat and make them physically more plausible, but it will still be virtually impossible to recover higher order coefficients of the lighting.

Discussion: Thus, even though in theory we can recover all the even modes of the lighting, in practice we only expect to recover the first 9 coefficients of the lighting—modes up to order 2. For practical purposes, the irradiance in general can be characterized by only its first 9 coefficients; the others vanish or are too small to be accurately measured. Thus, the Lambertian BRDF acts as a very low pass filter, passing through only the first 9 coefficients of the lighting, with the irradiance effectively restricted to being at most quadratic in the Cartesian coordinates of the surface normal vector. These observations help explain the results of Marschner and Greenberg.[2] In that paper, an attempt was made to solve the inverse lighting problem, treating the surfaces as Lambertian. The authors noted that the problem appeared ill-conditioned, and not amenable to accurate solution. Therefore, they had to rely heavily on a regularizing term that preserved the smoothness of the solution. The results in this paper show why the problem is ill-conditioned, and suggest a different regularization scheme. We can assume the high-frequency lighting coefficients to be inaccurate, so we do not attempt to recover them, and merely set them to 0. This indicates that a spherical-harmonic basis is ideal for recovering the lighting.

The fact that the irradiance is sufficiently slowly varying across the surface that it can be described by so few parameters has implications for a number of research areas, a fact that is explored in the next section. Since Lambertian surfaces are a reasonably close approximation to many real world objects, and are a widely used approximation in computer graphics and vision, we expect this result to have wide applicability.

Experimental Verification: We describe a simple experiment to verify the results of the paper. Using a camera mounted on a spherical gantry, we took a few calibrated grayscale images of a teflon sphere with known radius and position from different viewing positions using the same illumination—primarily from a distant ceiling light and an umbrella lamp. The lighting was measured in high dynamic range by using an almost perfectly specular mirror sphere (gazing ball). The spherical harmonic coefficients of the lighting were then computed. We were able to compute $B(\alpha, \beta)$, the radiant exitance as a function of the surface normal, on the teflon sphere by discarding specularities and averaging measurements of the same surface location on the teflon sphere as seen from different viewing directions, We then directly used equation 27 to determine the first 9 spherical harmonic coefficients of the lighting. These could then be compared to those obtained from the lighting measured using the mirror sphere. Since we did not know the relative reflectances of the teflon and mirror spheres—or equivalently the precise scaling factor relating the radiant exitance B to the irradiance E—there was a scale factor that we did not recover. Therefore, we uniformly scaled one set of lighting coefficients to be able to make meaningful comparisons with the other.

Figure 5 shows results from our experimental test. In (A), we show an image of the mirror and teflon spheres. We see that the image of the teflon sphere is a low-pass filtered version of the lighting, retaining essentially none of the high-frequency content that is visible in the image of the mirror sphere. Image (B) shows the high-resolution "real" illumination distribution, as recovered from the mirror sphere. Images (C)-(F) compare recovered and real illumination distributions. Note that since we do not explicitly enforce positivity, there are some darker negative regions in images (C)-(F). The image (F) represents a failed attempt; it is included to show the futility of attempting to recover the higher order modes of the lighting. However, a comparison of images (C) and (D) indicates that the first 9 coefficients of the lighting can be well recovered from observation of a curved convex Lambertian surface. Thus, we see that we are able to recover the first 9 coefficients of the lighting. But, as predicted by our theory, we fail to recover higher order coefficients.

Table 2 shows a numerical comparison of real and recovered lighting coefficients (in real form) for the first two orders of spherical harmonics. The second column shows the (scaled) irradiance coefficients $E_{l,m}$. These are divided by the third column, as per equation 27, to obtain the recovered lighting coefficients $L_{l,m}$, given in the penultimate column. The final column has the "real" lighting coefficients, as found using a gazing sphere. We see that the real and recovered values match closely. We also see that the irradiance coefficients are lower for $l = 2$ than for $l = 0$ or $l = 1$, since there is greater attenuation by the BRDF filter (A_l is smaller). Finally, table 2 shows the irradiance coefficients $E_{l,m}$ for $l = 3$. According to the theory, these coefficients should be identically 0 since $A_3 = 0$. We see that the experimental values are indeed very close to zero.

7 Conclusions

We have presented a theoretical analysis of the relationship between radiance and irradiance. We have shown that the operation of reflection is analogous to convolution of the illumination and a *clamped-cosine* function, and have derived a simple closed-form formula in terms of spherical harmonic coefficients. We have further demonstrated that the *clamped cosine*—or equivalently, the Lambertian BRDF—acts as a very low pass filter, making deconvolution to recover the lighting difficult. In fact, we have demonstrated that odd-order modes of the incident illumination with order > 1 cannot be recovered from the irradiance, i.e. from observation of a Lambertian surface. In other words, the radiance-from-irradiance problem is ill-posed or ambiguous. We have also presented evidence showing that in practical terms, the reflected light field from a Lambertian surface is characterized only by spherical harmonic modes up to order 2, and we can therefore reliably estimate only the first 9 coefficients of the incident illumination. These results confirm some previous empirical results, and also open up the possibility of novel algorithms for problems in many areas of computer graphics and vision.

For instance, in the general context of inverse rendering to recover lighting and BRDFs, our results suggest the use of a spherical harmonic basis as a suitable representation, with regularization obtained by limiting the number of modes used. In rendering images in computer graphics with complex illumination represented by environment maps, our results indicate that for largely diffuse surfaces, an accurate lighting description is not necessary. Efficient algorithms might result from rendering in frequency space, considering only the first few spherical harmonic coefficients of the lighting. For object recognition under varying illumination, our results indicate that the space of all possible images of an object can be easily described by a small basis set of images, corresponding to the lowest order modes of the lighting. Our work may also have applications in visual perception. Since our results indicate that lighting cannot lead to rapid variation of intensity over a Lambertian surface, such variation must be because of specularity or texture, and this result may be useful in explaining how one can perceive these quantities independently of the illumination.

Further work must be done on developing this theory for non-Lambertian surfaces, and in considering other effects such as shadows and interreflections. Further practical work is required on the problems just mentioned. We believe that the techniques developed in this paper are of fundamental interest and may provide a firm theoretical foundation for novel algorithms in many different research areas.

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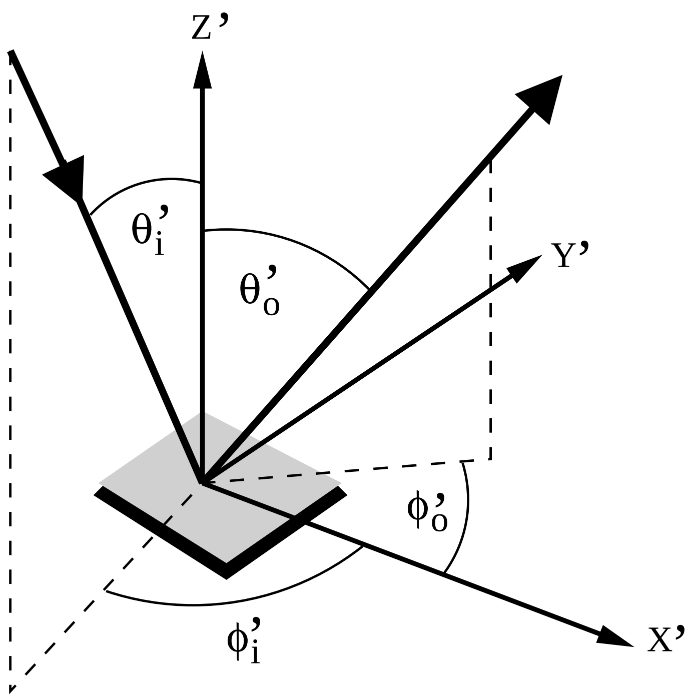
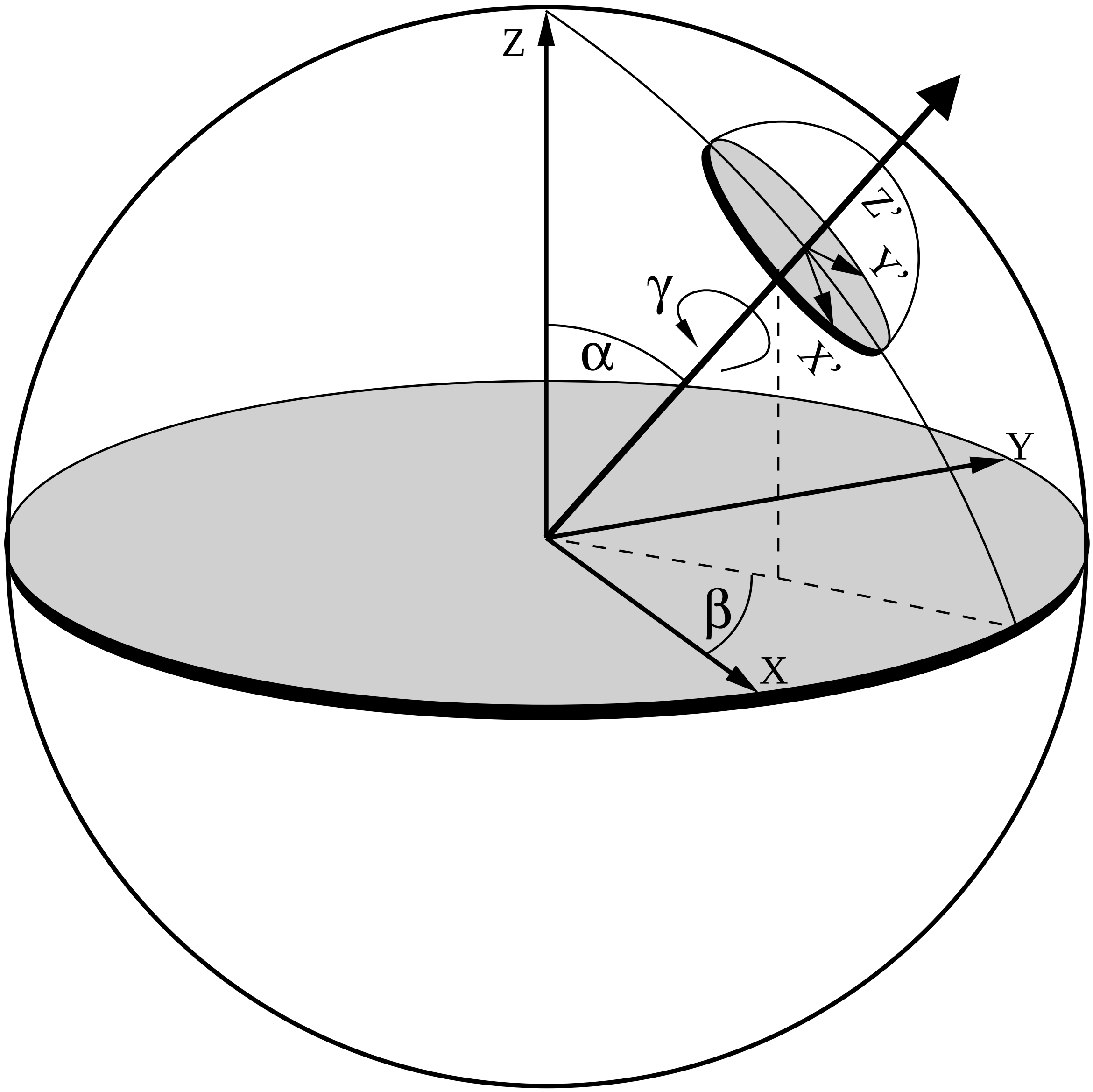


Fig. 1. A7901 Ramamoorthi and Hanrahan



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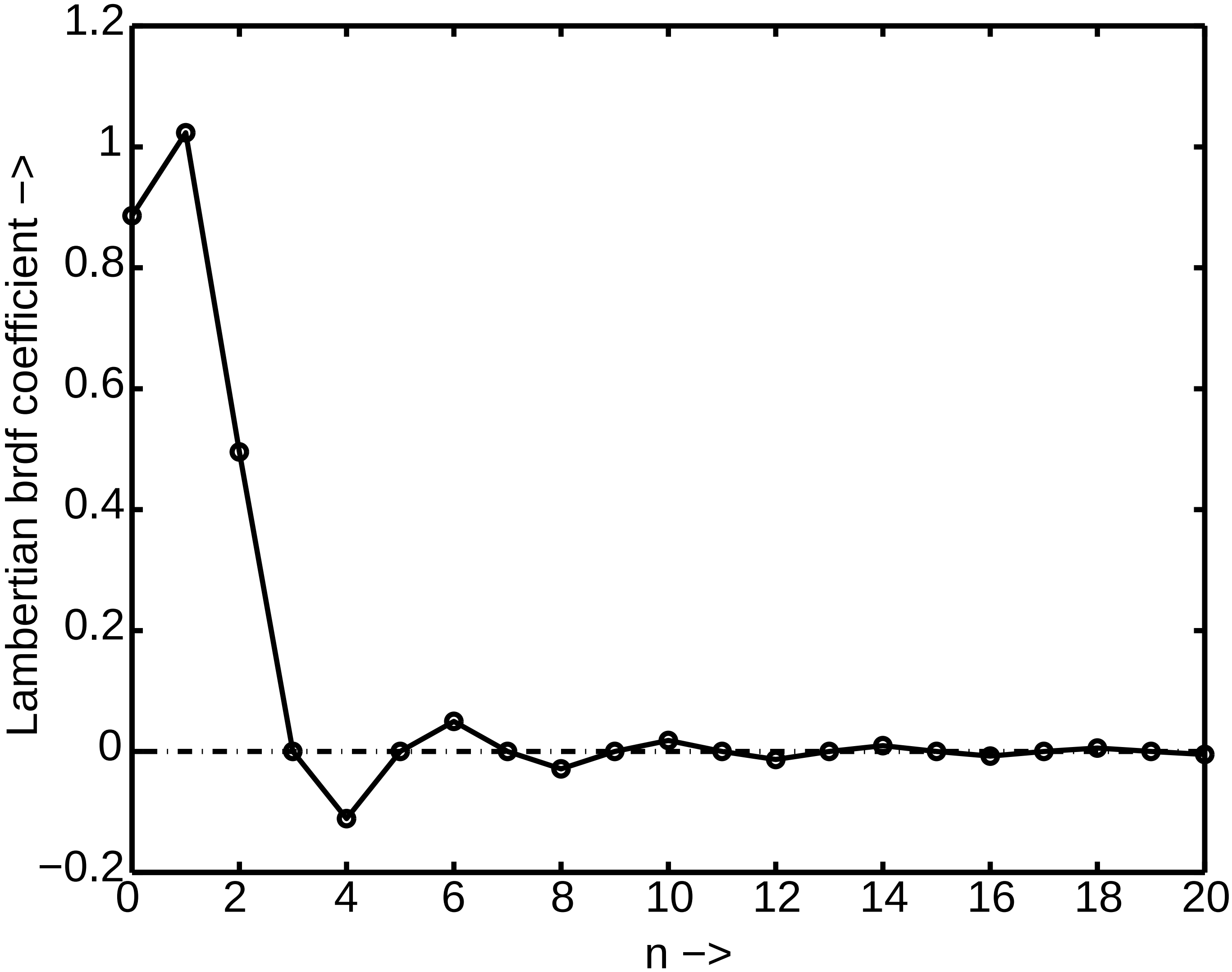

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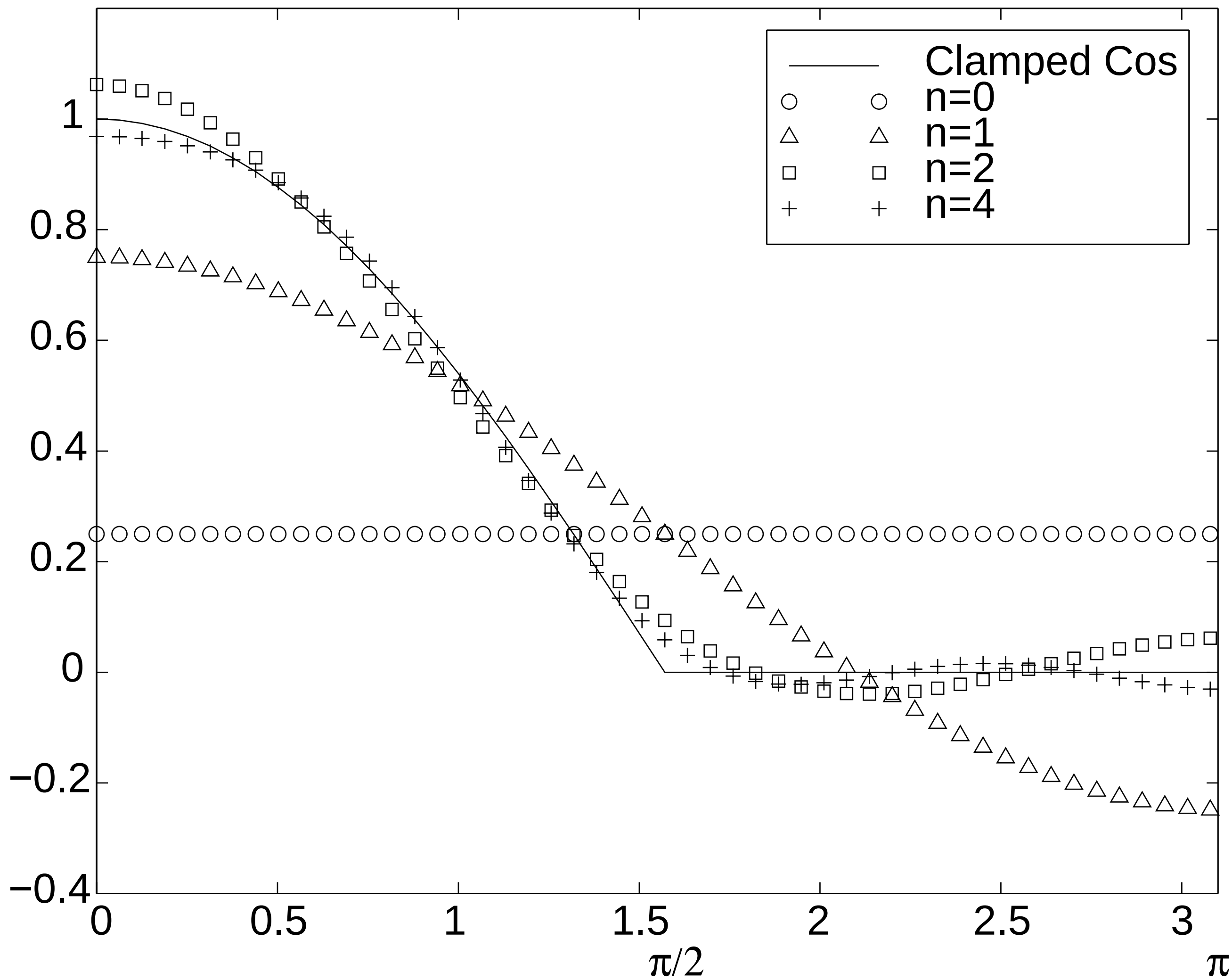

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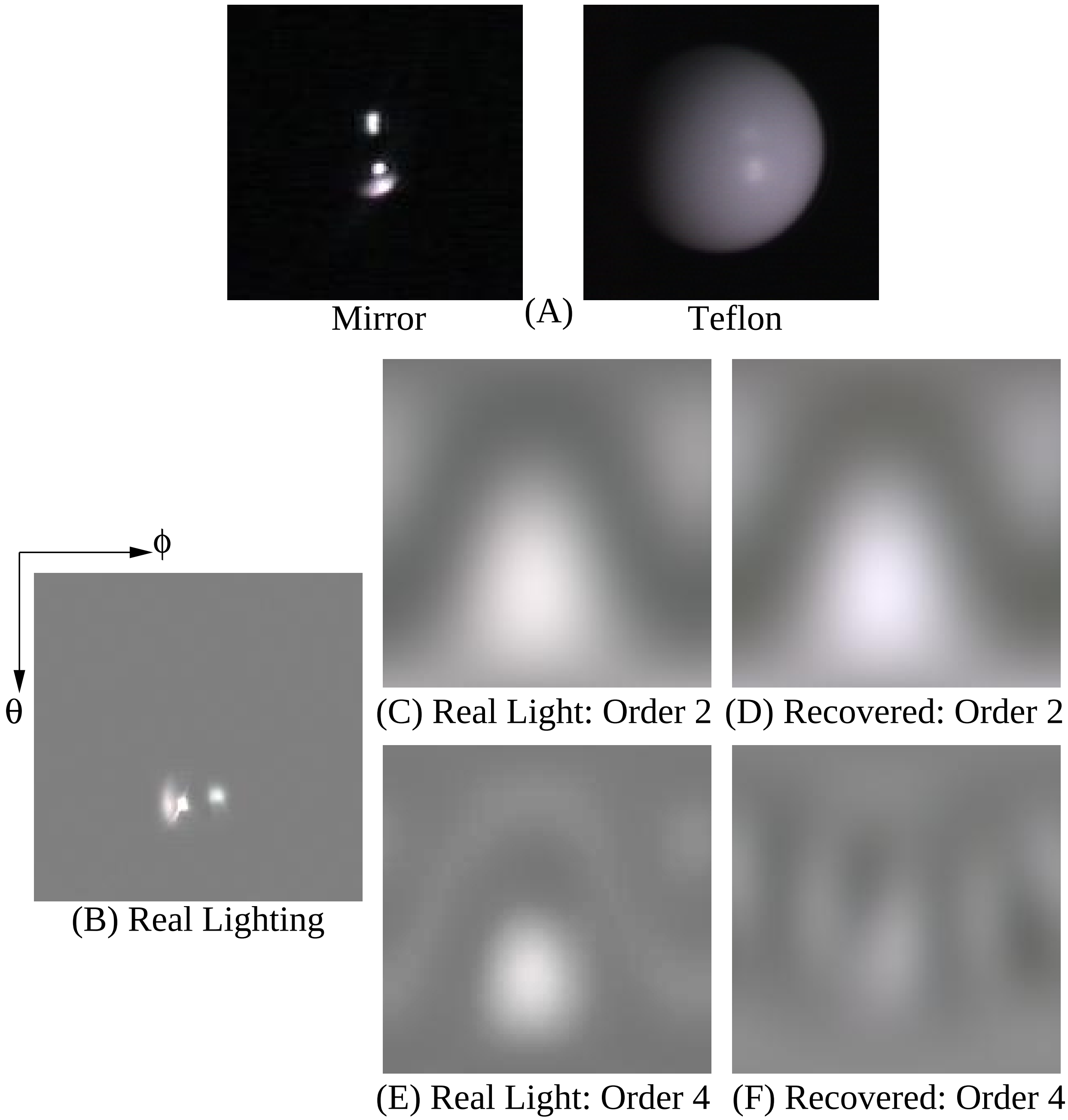

Fig. 5. A7901 Ramamoorthi and Hanrahan

Table 1. Notation used in the paper

L	Incoming radiance
$L_{l,m}$	Coefficients of spherical-harmonic expansion of L
E	Irradiance
$E_{l,m}$	Coefficients of spherical-harmonic expansion of E
B	Radiant exitance
ρ	Surface reflectance
A	Cosine of local incident angle $A = \cos\theta_i'$
A_l	Coefficients of spherical-harmonic expansion of A.
θ_i', θ_i	Incident elevation angle in *local*, *global* coordinates
ϕ_i', ϕ_i	Incident azimuthal angle in *local*, *global* coordinates
θ_o', θ_o	Outgoing elevation angle in *local*, *global* coordinates
ϕ_o', ϕ_o	Outgoing azimuthal angle in *local*, *global* coordinates
Ω', Ω	Hemisphere of integration in *local*,*global* coordinates
$\mathbf{x}$	Surface position
$\mathbf{n}$	Surface normal
α	Surface normal parameterization—elevation angle
β	Surface normal parameterization—azimuthal angle
γ	Tangent frame angle
$\tilde{D}^l_{m,m'}$	Representation matrices of $SO(3)$
$Y_{l,m}$	Spherical Harmonic
I	$\sqrt{-1}$

Table 2. Comparison of Recovered and Real Lighting Coefficients

(l,m)	E_{lm}	$\sqrt{4\pi/(2l+1)}A_l$	L_{lm} (Rec.)	L_{lm} (Real)
(0,0)	3.39	3.14	1.08	0.96
(1,-1)	-0.63		-0.30	-0.31
(1,0)	-1.99	2.09	-0.95	-0.97
(1,1)	2.55		-1.22	-1.20
(2,-2)	-0.38		-0.48	-0.44
(2,-1)	0.37		0.47	0.42
(2,0)	0.05	0.79	0.06	0.05
(2,1)	-1.21		-1.54	-1.55
(2,2)	0.50		0.64	0.74
(3,-3)	-0.02			
(3,-2)	0.00			
(3,-1)	-0.03			
(3,0)	0.02	0		
(3,1)	0.03			
(3,2)	0			
(3,3)	-0.04			