Fast Computation of the OTFs for Various Computational Cameras

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1. Preliminary

In this report, we discuss how to compute exactly and efficiently the optical transfer function (OTF) of various computational cameras, under the assumptions of far-field wave optics and ideal thin lens. Approximations of the magnitudes of the OTFs have been previously given in [8], with derivations for cubic phase plates given in [12].

For ease and simplicity of derivation, let us stipulate that the aperture function of an imaging system is given by a complex function $P : \mathbb{R}^2 \to \mathbb{C}$ with the following constraints:

$$P(x, y) = 0, \qquad \text{if } |x| > \tfrac{1}{2} \text{ or } |y| > \tfrac{1}{2}, \tag{1.1}$$
$$|P(x, y)| \le 1, \qquad \text{otherwise.} \tag{1.2}$$

In other words, the aperture has a unit area, and transmits at most unit power per area. The aperture is paired with an ideal parabolic lens to form an incoherent imaging system.

A particular depth can be characterized by the misfocus parameter ψ, which is proportional to the deviation from the Lensmaker's equation: $\psi \propto \frac{1}{d_i} + \frac{1}{d_o} - \frac{1}{f}$, where d_i, d_o are the distances from the lens to the sensor and the object, respectively, and f is the focal length. (This quantity is proportional to the expression $\hat{W}_{20}$ in optics literature. See [3].) By convention, assuming a trivial zero-phase aperture is in place, $\psi = 0$ corresponds to the plane in focus.

The optical transfer function of an incoherent imaging system is a function of three variables, namely the spatial frequencies f_x, f_y and the depth of the scene ψ. It is known that the OTF can be expressed as an autocorrelation of the aperture function under our current assumptions[5]:

$$\begin{aligned} \mathit{OTF}_\psi(f_x, f_y) = \iint & P\left(t_1 + \frac{f_x}{2}, t_2 + \frac{f_y}{2}\right) \\ & \times P^*\left(t_1 - \frac{f_x}{2}, t_2 - \frac{f_y}{2}\right) \\ & \times e^{2i\psi(t_1 f_x + t_2 f_y)} dt_1 dt_2. \end{aligned} \tag{1.3}$$

Alternatively, one can first compute the point spread function (PSF) as the squared magnitude of the 2D Fourier transform of the defocused aperture, and exploit fact that the OTF at the given depth is simply the 2D Fourier transform of the PSF. For a thorough treatment of far-field wave optics with a standard lens, see Goodman[5] and Hopkins[6].

In the subsequent sections, we shall analytically evaluate (1.3) for several computational cameras without enlisting the help of Fourier transforms, and simplify them. In fact, we shall show that the OTF at given spatial frequencies and depth can be calculated in $O(1)$ for many computational cameras. This is considerably faster and numerically more accurate than is the Fourier-transform-based alternative.

Because the OTFs we study can be reduced into a couple of canonical integrals, we first prove two lemmata regarding how to compute those integrals.

Lemma 1. *Consider the complex quadratic exponential:*

$$Q(a, b, c, t_1, t_2) := \int_{t_1}^{t_2} \exp\left\{i(ax^2 + bx + c)\right\} dx.$$

Then, $Q(a, b, c, t_1, t_2)$ *equals,*

$$\begin{cases} e^{i\left(\frac{b(t_1+t_2)}{2}+c\right)}(t_2 - t_1)\mathit{sinc}\left(\frac{b(t_2-t_1)}{2\pi}\right), & a = 0, \\ e^{i\left(c-\frac{b^2}{4a}\right)}\sqrt{\frac{\pi}{2a}}\left[\mathcal{F}(t_2') - \mathcal{F}(t_1')\right], & a > 0, \\ Q^*(-a, -b, -c, t_1, t_2), & a < 0. \end{cases}$$

where $t_h' = \sqrt{\frac{2}{\pi}}\left(t_h\sqrt{a} + \frac{b}{2\sqrt{a}}\right)$ *for* $h \in \{1, 2\}$*, and* $\mathcal{F}$ *is the Fresnel integral[9]:*

$$\mathcal{F}(x) := \int_0^x \exp\left\{\frac{i\pi t^2}{2}\right\} dt.$$

Proof. First, when the leading coefficient a is zero, the complex exponential is directly integrable:

$$\begin{aligned} \int_{t_1}^{t_2} \exp\{i(bx + c)\}\, dx &= \left.\frac{\exp\{i(bx + c)\}}{ib}\right|_{t_1}^{t_2} \\ &= \frac{\exp\{i(bt_2 + c)\} - \exp\{i(bt_1 + c)\}}{ib} \\ &= e^{i\left(\frac{b(t_1+t_2)}{2}+c\right)}(t_2 - t_1)\mathrm{sinc}\left(\frac{b(t_2 - t_1)}{2\pi}\right), \end{aligned}$$

as desired. Otherwise, a sequence of changes of variables can be applied to re-express the integral as a Fresnel integral. Assuming $a > 0$, let $x' = \sqrt{\frac{2}{\pi}}\left(x\sqrt{a} + \frac{b}{2\sqrt{a}}\right)$. Then,

$$\begin{aligned}
&\int_{t_1}^{t_2} \exp\left\{i(ax^2+bx+c)\right\} dx \\
&= \int_{t_1}^{t_2} \exp\left\{ i\left(\left(x\sqrt{a} - \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a}\right)\right\} dx \\
&= e^{i\left(c-\frac{b^2}{4a}\right)} \int_{t_1}^{t_2} \exp\left\{ i\left(x\sqrt{a} - \frac{b}{2\sqrt{a}}\right)^2 \right\} dx \\
&= e^{i\left(c-\frac{b^2}{4a}\right)} \int_{t_1'}^{t_2'} \exp\left\{\frac{i\pi x'^2}{2}\right\} \frac{dx'}{\sqrt{2a/\pi}} \\
&= e^{i\left(c-\frac{b^2}{2a}\right)} \sqrt{\frac{\pi}{2a}} \left[\mathcal{F}(t_2') - \mathcal{F}(t_1')\right].
\end{aligned}$$

Lastly, when $a < 0$, we first compute the complex conjugate of the integral. To do so, we may evaluate the integral with the coefficients a, b, c negated. Then, the integral falls under the case we have previously covered. Simply taking the conjugate of the result yields the answer. □

Lemma 2. *Consider the following integral of two sinc functions multiplied together:*

$$\int \textit{sinc}\,(ax/\pi)\,\textit{sinc}\,(bx/\pi)\,dx.$$

Then, the integral is equivalent to,

$$\begin{aligned}
T(a,b,x) =& \frac{(b-a)\mathit{Si}((a-b)x) + (a+b)\mathit{Si}((a+b)x)}{2ab} \\
& - x \cdot \textit{sinc}\left(\frac{ax}{\pi}\right)\textit{sinc}\left(\frac{bx}{\pi}\right),
\end{aligned}$$

where Si is the sine integral[14]:

$$\mathit{Si}(x) := \int_0^x \textit{sinc}\left(\frac{t}{\pi}\right) dt.$$

Proof. The validity of the lemma can be checked by differentiating $T(a,b,x)$ with respect to x and verifying that the derivative equals the original integral of two sinc functions.

$$\begin{aligned}
\frac{\partial T(a,b,x)}{\partial x} &= \tfrac{-(a-b)^2}{2ab}\operatorname{sinc}\left(\tfrac{(a-b)x}{\pi}\right) + \tfrac{(a+b)^2}{2ab}\operatorname{sinc}\left(\tfrac{(a+b)x}{\pi}\right) \\
&\quad - xa\cdot\operatorname{sinc}\left(\frac{bx}{\pi}\right)\left(\frac{\cos(ax)}{ax} - \frac{\sin(ax)}{a^2x^2}\right) \\
&\quad - xb\cdot\operatorname{sinc}\left(\frac{ax}{\pi}\right)\left(\frac{\cos(bx)}{bx} - \frac{\sin(bx)}{b^2x^2}\right) \\
&\quad - \operatorname{sinc}\left(\frac{ax}{\pi}\right)\operatorname{sinc}\left(\frac{bx}{\pi}\right) \\
= &\frac{(b-a)\sin((a-b)x)}{2abx} + \frac{(a+b)\sin((a+b)x)}{2abx} \\
&- \frac{\sin(bx)}{b}\left(\frac{\cos(ax)}{x} - \frac{\sin(ax)}{ax^2}\right) \\
&- \frac{\sin(ax)}{a}\left(\frac{\cos(bx)}{x} - \frac{\sin(bx)}{bx^2}\right) - \frac{\sin(ax)\sin(bx)}{abx^2}.
\end{aligned}$$

Expanding the first two sine terms in the last line and canceling out the resulting terms yields $\operatorname{sinc}\left(\frac{ax}{\pi}\right)\operatorname{sinc}\left(\frac{bx}{\pi}\right)$ as desired. □

2. The OTFs of Computational Cameras

In this section, the exact OTFs of a standard lens, a cubic phase plate[3], focus sweep[10], a lattice focal lens[8], and a coded aperture[7, 13] are derived from their definitions. The OTFs we derive in this section are free of integral signs, and are expressed in terms of simple primitives, such as the sinc function, the Fresnel integral and the sine integral. Numerically evaluating these expressions are discussed in Section 3.

2.1. Standard Lens

Under the far-field assumption, the aperture function of a standard lens is a mask whose phase varies quadratically with respect to spatial coordinates, where the coefficient of the quadratic corresponds to the depth ψ_f at which the lens is focused:

$$P : (x,y) \mapsto \begin{cases} \exp\left\{-i\psi_f(x^2+y^2)\right\}, & \text{if } |x|,|y| \le \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

By convention, ψ_f is set to zero, in which case P is simply a binary function that is uniformly 1 inside the unit square centered at the origin and uniformly 0 outside.

Claim 1. *The optical transfer function of a standard lens at frequency (f_x, f_y) and misfocus ψ is given by*

$$\begin{aligned}
\mathit{OTF}_\psi(f_x,f_y) &= (1-|f_x|)(1-|f_y|) \\
&\times \textit{sinc}\left(\frac{(|f_x|-f_x^2)\psi}{\pi}\right)\textit{sinc}\left(\frac{(|f_y|-f_y^2)\psi}{\pi}\right).
\end{aligned} \tag{2.2}$$

Proof. Consider the integrand of (1.3). Because P has a bounded support, the domain of integration in which the integrand is nonzero must obey the following condition:

$$|t_1| \le \tfrac{1}{2} - \tfrac{|f_x|}{2}, \qquad |t_2| \le \tfrac{1}{2} - \tfrac{|f_y|}{2}. \tag{2.3}$$

Since the value of the aperture function restricted to this domain is uniformly 1, we can rewrite the integral as,

$$\int_{-\frac{1}{2}+\frac{|f_x|}{2}}^{\frac{1}{2}-\frac{|f_x|}{2}} \int_{-\frac{1}{2}+\frac{|f_y|}{2}}^{\frac{1}{2}-\frac{|f_y|}{2}} \exp\{2i\psi(t_1 f_x + t_2 f_y)\} dt_1 dt_2.$$

This integral is separable, and each of the two separated integrals is a complex linear exponential. Then Lemma 1 applies, and we can obtain the desired expression. □

The OTF of a standard lens is a fundamental result in wave optics. Proofs of the same result can be found in [5, 8].

2.2. Cubic Phase Plate

The cubic phase plate[3] modulates the phase of incoming light with a cubic polynomial, and is parametrized by a single parameter α, the coefficient of the cubic:

$$P : (x, y) \mapsto \begin{cases} \exp\left\{i\alpha(x^3 + y^3)\right\}, & \text{if } |x|, |y| \le \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Do note that other variations of this “wavefront-coding” technique exist[2, 4, 11], featuring circular aperture and/or non-cubic phase functions. Unfortunately, they are less conducive to mathematical analysis.

Claim 2. *The optical transfer function of a cubic phase plate at frequency (f_x, f_y) and misfocus ψ is given by*

$$\begin{aligned} OTF_\psi(f_x, f_y) =& Q\left(3\alpha f_x, 2\psi f_x, \alpha\tfrac{f_x^2}{4}, \tfrac{|f_x|-1}{2}, \tfrac{1-|f_x|}{2}\right) \\ &\times Q\left(3\alpha f_y, 2\pi\psi f_y, \alpha\tfrac{f_y^2}{4}, \tfrac{|f_y|-1}{2}, \tfrac{1-|f_y|}{2}\right), \end{aligned} \tag{2.5}$$

where $Q(\cdot)$ is as defined in Lemma 1.

Proof. The definition of OTF in (1.3) gives us:

$$\begin{aligned} OTF_\psi(f_x, f_y) &= \iint \exp i\alpha\left[\left(t_1 + \tfrac{f_x}{2}\right)^3 + \left(t_2 + \tfrac{f_y}{2}\right)\right] \\ &\quad \times \exp -i\alpha\left[\left(t_1 - \tfrac{f_x}{2}\right)^3 - \left(t_2 - \tfrac{f_y}{2}\right)\right] \\ &\quad \times \exp 2i\psi(t_1 f_x + t_2 f_y) dt_1 dt_2 \\ &= \iint \exp i\alpha\left\{3f_x t_1^2 + 3f_y t_2^2 + \tfrac{f_x^2}{4} + \tfrac{f_y^2}{4}\right\} \\ &\quad \times \exp 2i\psi(t_1 f_x + t_2 f_y) dt_1 dt_2 \\ &= \int \exp i\left(3f_x\alpha t_1^2 + 2\psi f_x t_1 + \alpha f_x^2/4\right) dt_1 \\ &\quad \times \int \exp i\left(3f_y\alpha t_2^2 + 2\psi f_y t_2 + \alpha f_y^2/4\right) dt_2, \end{aligned}$$

where the domain of integration is again given by (2.3). Hence we have the product of two complex quadratic exponentials, and Lemma 1 applies to yield the desired expression. □

2.3. Focus Sweep

Focus sweep[10] moves the lens along the optical axis during the integration time, causing the depth at which the lens focuses to vary as a function of time, so the OTF at each time slice is effectively that of a defocused lens. The resulting OTF is then obtained by averaging these OTFs over time. We assume that a standard lens is used, and that the motion of the lens causes each misfocus parameter $\gamma \in \psi + [-S, S]$ to contribute equally to the final OTF. This is indeed the case if the lens moves at a constant speed.

Claim 3. *The optical transfer function of focus sweep at frequency (f_x, f_y) and misfocus ψ is given by*

$$\begin{aligned} OTF_\psi(f_x, f_y) &= \tfrac{1}{2S}(1 - |f_x|)(1 - |f_y|) \\ &\quad \times \left[T(K_1, K_2, \psi + S) - T(K_1, K_2, \psi - S)\right], \end{aligned} \tag{2.6}$$

where $T(\cdot)$ is as defined in Lemma 2.

Proof. Sweeping through depths $\gamma \in \psi + [-S, S]$ is equivalent to averaging the OTF of the standard lens in the given depth range. Claim 1 immediately yields,

$$\begin{aligned} OTF_\psi(f_x, f_y) &= \frac{1}{2S}\int_{-S}^{S}(1 - |f_x|)(1 - |f_y|) \\ &\qquad \times \operatorname{sinc}\left(\frac{K_1(\psi + \gamma)}{\pi}\right)\operatorname{sinc}\left(\frac{K_2(\psi + \gamma)}{\pi}\right) d\gamma \\ &= \frac{1}{2S}(1 - |f_x|)(1 - |f_y|) \\ &\quad \times \int_{-S}^{S}\operatorname{sinc}\left(\frac{K_1(\psi + \gamma)}{\pi}\right)\operatorname{sinc}\left(\frac{K_2(\psi + \gamma)}{\pi}\right) d\gamma \\ &= \frac{1}{2S}(1 - |f_x|)(1 - |f_y|) \\ &\quad \times \int_{\psi - S}^{\psi + S}\operatorname{sinc}\left(\frac{K_1\gamma'}{\pi}\right)\operatorname{sinc}\left(\frac{K_2\gamma'}{\pi}\right) d\gamma'. \end{aligned}$$

Applying Lemma 2 completes the proof. □

2.4. Lattice Focal Lens

The lattice focal lens[8] is a regular n-by-n grid of square lenslets with varying focal power. Its aperture can be expressed as the sum over all its elements:

$$P(x, y) = \sum_{j_1=0}^{n-1}\sum_{j_2=0}^{n-1} P_{(j_1, j_2)}(x, y), \tag{2.7}$$

where $P_{(j_1,j_2)}(x, y)$ corresponds to a $\frac{1}{n}$-by-$\frac{1}{n}$ standard lens at row j_1 and column j_2, focused at depth $\psi_{(j_1,j_2)}$:

$$\begin{aligned} P_{(j_1,j_2)}(x, y) &= \exp\left\{-i\psi_{(j_1,j_2)}(x^2 + y^2)\right\}, \\ &\text{if } \left|x - \tfrac{2j_1-n+1}{2n}\right|, \left|y - \tfrac{2j_2-n+1}{2n}\right| \le \tfrac{1}{2n}, \\ &\text{zero otherwise.} \end{aligned} \tag{2.8}$$

Let us incorporate this into the definition of $OTF_\psi(f_1, f_2)$ in (1.3). Then the integrand can be distributed into n^4 correlation terms:

$$\begin{aligned} OTF = \iint & P\left(t_1 + \tfrac{f_1}{2}, t_2 + \tfrac{f_2}{2}\right) P^*\left(t_1 - \tfrac{f_1}{2}, t_2 - \tfrac{f_2}{2}\right) \\ &\times \exp\left\{2i\psi(t_1 f_1 + t_2 f_2)\right\} dt_1 dt_2 \end{aligned}$$

$$= \iint \sum_{j_1,j_2} P_{(j_1,j_2)}\left(t_1 + \tfrac{f_1}{2}, t_2 + \tfrac{f_2}{2}\right)$$
$$\times \sum_{k_1,k_2} P^*_{(k_1,k_2)}\left(t_1 - \tfrac{f_1}{2}, t_2 - \tfrac{f_2}{2}\right)$$
$$\times \exp\{2i\psi(t_1 f_1 + t_2 f_2)\} dt_1 dt_2$$
$$= \sum_{j_1,j_2}\sum_{k_1,k_2} \iint P_{(j_1,j_2)}\left(t_1 + \tfrac{f_1}{2}, t_2 + \tfrac{f_2}{2}\right)$$
$$\times P^*_{(k_1,k_2)}\left(t_1 - \tfrac{f_1}{2}, t_2 - \tfrac{f_2}{2}\right)$$
$$\times \exp\{2i\psi(t_1 f_1 + t_2 f_2)\} dt_1 dt_2. \quad (2.9)$$

Now for $h \in \{1, 2\}$, define

$$\Delta_h = f_h + \frac{k_h - j_h}{n}, \qquad t'_h = t_h - \frac{k_h + j_h - (n-1)}{2n},$$

and apply change of variables to obtain

$$\sum_{\substack{j_1,j_2\\k_1,k_2}} \iint P_{(j_1,j_2)}\left(t'_1 + \tfrac{2j_1-n+1}{2n} + \tfrac{\Delta_1}{2}, t'_2 + \tfrac{2j_2-n+1}{2n} + \tfrac{\Delta_2}{2}\right)$$
$$\times P^*_{(k_1,k_2)}\left(t'_1 + \tfrac{2k_1-n+1}{2n} - \tfrac{\Delta_1}{2}, t'_2 + \tfrac{2k_2-n+1}{2n} - \tfrac{\Delta_2}{2}\right)$$
$$\times \prod_{h=1}^{2} e^{2i\psi\left(t'_h + \frac{k_h+j_h-(n-1)}{2n}\right)\left(\Delta_h - \frac{k_h-j_h}{n}\right)} dt'_1 dt'_2.$$

In this form, it is clear from (2.8) that the domain of integration for each correlation term is constrained by

$$\left|t'_h \pm \tfrac{\Delta_h}{2}\right| \le \tfrac{1}{2n}, \quad (2.10)$$

which is equivalent to $|t'_h| \le \frac{1}{2n} - \frac{|\Delta_h|}{2}$. Since each summand is an integral of a complex quadratic exponential in two separable variables (with a known rectangular support), Lemma 1 applies, eventually yielding:

Claim 4. *The optical transfer function of an n-by-n lattice focal lens parametrized by $\psi_{(j_1,j_2)}$ is given by*

$$\sum_{\substack{j_1,j_2\\k_1,k_2}} \prod_{h=1}^{2} Q\Big(- \psi_{(j_1,j_2)} + \psi_{(k_1,k_2)},\ 2\psi\left(\Delta_h + \tfrac{j_h-k_h}{n}\right)$$
$$- \psi_{(j_1,j_2)}\left(\tfrac{2j_h-n+1}{n} + \Delta_h\right) + \psi_{(k_1,k_2)}\left(\tfrac{2k_h-n+1}{n} - \Delta_h\right),$$
$$- \psi_{(j_1,j_2)}\left(\tfrac{2j_h-n+1}{2n} + \tfrac{\Delta_h}{2}\right)^2 + \psi_{(k_1,k_2)}\left(\tfrac{2k_h-n+1}{2n} - \tfrac{\Delta_h}{2}\right)^2$$
$$+ 2\psi\left(\tfrac{k_h+j_h-(n-1)}{2n}\right)\left(\Delta_h - \tfrac{k_h-j_h}{n}\right),$$
$$- \tfrac{1}{2n} + \tfrac{|\Delta_h|}{2}, \tfrac{1}{2n} - \tfrac{|\Delta_h|}{2}\Big). \quad (2.11)$$

As before, $Q(\cdot)$ is as defined in Lemma 1.

Note that unless $\frac{1}{2n} > \frac{|\Delta_h|}{2}$ holds for $h \in \{1, 2\}$, the domain of integration is empty. From this, we deduce that for every index (j_1, j_2), there exist only a few indices (k_1, k_2) for which the two lenslets have nonzero correlation term. The indices of those lenslets are easily computable:

$$\frac{1}{2n} > \frac{|\Delta_h|}{2} \iff \frac{1}{2n} > \frac{|f_h + \frac{k_h - j_h}{n}|}{2}$$
$$\iff 1 > |n \cdot f_h + k_h - j_h|$$
$$\iff -1 + j_h - n \cdot f_h < k_h < 1 + j_h - n \cdot f_h$$
$$\implies k_h = \lfloor j_h - n \cdot f_h \rfloor \text{ or } \lceil j_h - n \cdot f_h \rceil.$$

Because each of k_1, k_2 is constrained to two choices, we conclude that there are only 4 pairs of indices (k_1, k_2) for which the correlation term is nonzero. Another way to demonstrate this fact is to imagine shifting the entire lens by (f_x, f_y) and comparing it to the untranslated copy. Each lenslet in the original copy can intersect at most four lenslets from the translated copy, and vice versa.

Hence, the OTF of an n-by-n lattice focal lens can be computed by evaluating a complex quadratic exponential at most $8n^2$ times.

2.5. Coded Aperture

A binary coded aperture[7, 13] can be analyzed in a manner similar to the lattice focal lens: the whole aperture can be partitioned into squares each of which either fully transmits or blocks light. More generally, for a coded aperture with non-binary transmittance, each subsquare can be modeled as,

$$P_{(j_1,j_2)}(x, y) = I(j_1, j_2),$$
$$\text{if } \left|x - \tfrac{2j_1-n+1}{2n}\right|, \left|y - \tfrac{2j_2-n+1}{2n}\right| \le \tfrac{1}{2n},$$
$$\text{zero otherwise.} \quad (2.12)$$

Note the close similarity to (2.8): whereas each subsquare of a lattice focal lens exhibits its own focal power, here it has its own transmittance. Now we may again compute the expression in Claim 4 with this new definition for subsquares. The resulting correlation terms are weighted by the transmittance of the two subsquares being correlated, and the terms $\psi_{(j_1,j_2)}$ are identically zero.

Claim 5. *The optical transfer function of an n-by-n coded aperture is given by*

$$\sum_{\substack{j_1,j_2\\k_1,k_2}} \prod_{h=1}^{2} Q\Big(0,\ 2\psi\left(\Delta_h + \tfrac{j_h-k_h}{n}\right),$$
$$2\psi\left(\tfrac{k_h+j_h-(n-1)}{2n}\right)\left(\Delta_h - \tfrac{k_h-j_h}{n}\right),$$
$$- \tfrac{1}{2n} + \tfrac{|\Delta_h|}{2}, \tfrac{1}{2n} - \tfrac{|\Delta_h|}{2}\Big) I(j_1, j_2) I(k_1, k_2), \quad (2.13)$$

where $k_h = \lfloor j_h - n \cdot f_h \rfloor$ or $\lceil j_h - n \cdot f_h \rceil$ and $I(\cdot)$ is the transmittance at the given subsquare.

3. Discussion

The OTFs of computational cameras derived thus far may contain the Fresnel integrals or the sine integrals. Both of them can be calculated very efficiently and accurately using a small number of additions and multiplications[9, 14]. This enables very fast sampling OTFs of the said computational cameras.

Claim 6. *The optical transfer function at a given frequency* (f_x, f_y) *and misfocus* ψ *can be computed in constant time for the standard lens, cubic phase plate, focus sweep, lattice focal lens and coded aperture.*

Proof. Claims 1-4 give an analytic integral-free expression for the relevant OTFs, each of which is comprised of a finite number of terms that can be computed in $O(1)$. It is true that the OTFs for lattice focal lens and coded aperture have runtime that grows quadratically as a function of the number of subsquares in each dimension, but typically the number of subsquares is very small. □

Let us discuss the consequences of the above claim. A typical task in analyzing the OTF might require evaluating the said OTF at *all* relevant locations. The spatial frequencies f_x, f_y vary between the diffraction limit (± 1 under our assumptions), and the misfocus ψ varies in some finite range of interest. Hence, we would like to evaluate OTF inside this 3D bounding box. In practice, we can voxelize the space and sample it with sufficiently high resolution.

To sample the OTF at $N \times N \times N$ locations, the conventional approach is to discretely sample the aperture, defocus it for each of the N depths, take the squared magnitude of its FFT (with respect to the spatial coordinates) to obtain the PSF, and then take the FFT with respect to the spatial coordinates again to obtain the OTF. This algorithm runs in $O(N^3 \log N)$ and is susceptable to aliasing unless the original sampling of the aperture includes large enough padding by zero. Furthermore, because the PSF is a discrete sample, taking its Fourier transform introduces numerical error (since the Fourier transform, an integral, is being approximated by a summation.)

On the other hand, knowing the analytic formula for the OTF allows one to sample the OTF in $O(N^3)$, and gives an exact answer (up to the fidelity of the subroutines that calculate $\mathcal{F}(\cdot)$ and *Si*$(\cdot)$.) These improvements come at no cost to parallelizability: each OTF of the said computational cameras can be obtained under a second at a resolution of $512 \times 512 \times 512$ on commodity GPUs like Nvidia GTX 260. This enables the large-scale analyses, such as the ones performed in [1], with the necessary numerical precision.

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