Chapter 7

A Self-Adjoint Operator Formulation of Light Transport

As we have mentioned, it is very convenient for implementations to use the same scattering rules for light, importance, and particles. This is desirable for theoretical work as well, so that we may avoid the use of adjoint operators. However, the existing light transport models in graphics fail to achieve this, even when the scene model is physically valid. To obtain symmetry, the typical solution is to limit the scene to reflective surfaces (or more generally, to require that all media have the same refractive index). This is a major restriction, since it disallows materials such as glass and water, which occur frequently in graphics models.

In this chapter, we develop a framework where light, importance, and particles obey the same scattering rules, for any physically valid scene. Technically, this requires us to define operators that are *self-adjoint*, so that the same operators apply in all situations. The major issue, of course, is how to deal with transmission between media with different indices of refraction. The solution turns out to very simple and practical, and also reveals interesting connections with classical geometric optics.

7.1 Concepts of the new framework

We would like to find a framework where light and importance always obey the same transport equation, even when there are media with different refractive indices. We show how to achieve this by modifying the framework of Chapter 4. In this section, we discuss only the formal changes that are required, leaving interpretation and discussion for later. We assume throughout this chapter that only physically valid materials are used in the scene model.

The problem. The problem with the framework that we described in Chapter 4 is that sometimes the local scattering operator $\mathbf{K}$ is not self-adjoint. In this case, the light and importance transport operators are different, since they are given by $\mathbf{T}_L = \mathbf{KG}$ and $\mathbf{T}_W = \mathbf{K}^*\mathbf{G}$ respectively.

To fix this, recall that $\mathbf{K}$ is defined by

$$(\mathbf{K}L)(\mathbf{x}, \omega_{\mathrm{o}}) \;=\; \int_{\mathcal{S}^2} f_{\mathrm{s}}(\mathbf{x}, \omega_{\mathrm{i}} \!\to\! \omega_{\mathrm{o}})\, L(\mathbf{x}, \omega_{\mathrm{i}})\, d\sigma^{\perp}_{\mathbf{x}}(\omega_{\mathrm{i}})\,,$$

and that $\mathbf{K} = \mathbf{K}^*$ whenever f_{s} is symmetric. From Section 6.2, we also know that

$$\frac{f_{\mathrm{s}}(\omega_{\mathrm{i}} \!\to\! \omega_{\mathrm{o}})}{\eta_{\mathrm{o}}^2} \;=\; \frac{f_{\mathrm{s}}(\omega_{\mathrm{o}} \!\to\! \omega_{\mathrm{i}})}{\eta_{\mathrm{i}}^2} \tag{7.1}$$

for any physically valid BSDF; that is, $f_{\mathrm{s}}(\omega_{\mathrm{i}} \!\to\! \omega_{\mathrm{o}})/\eta_{\mathrm{o}}^2$ is a symmetric function. Thus, if we could change the definition of $\mathbf{K}$ to use this symmetric function $f_{\mathrm{s}}/\eta_{\mathrm{o}}^2$, rather than f_{s} itself, then we would have $\mathbf{K} = \mathbf{K}^*$ for any physically valid scene model.

The solution. There is a very simple way to achieve this. The idea is to define a new solid angle measure σ', which *replaces* the usual measure σ in all radiometric quantities and definitions. We call the new measure *basic solid angle*, and it is defined by[1]

$$d\sigma'_{\mathbf{x}}(\omega) \;=\; \eta^2(\mathbf{x}, \omega)\, d\sigma(\omega)\,, \tag{7.2}$$

where $\eta(\mathbf{x}, \omega)$ is the refractive index of the medium that is adjacent to $\mathbf{x}$ in the direction ω. Note that unlike the usual solid angle measure σ, the basic solid angle measure $\sigma'_{\mathbf{x}}$ is

[1] As always, when measures are defined using "infinitesimals", it should be understood as shorthand for a formal definition involving integration. For example, a more precise definition of equation (7.2) would be

$$\sigma'_{\mathbf{x}}(D) \;=\; \int_D \eta^2(\mathbf{x}, \omega)\, d\sigma(\omega)\,,$$

where $D \subset \mathcal{S}^2$ is a σ-measurable set of directions.

a function of position, since its value depends on the refractive indices of the surrounding media.

By mechanically substituting this new measure into all of our old definitions, we obtain a framework with very desirable symmetry properties. The symbols for the new quantities are obtained by appending a prime symbol (e.g. L'), while their names are obtained by prefixing the word *basic* (e.g. basic radiance). This naming convention is justified by the unique invariance properties of these quantities, which will be studied further in Appendix 7.A. It also extends the terminology of Nicodemus, who first introduced the ideas of *basic radiance* and *basic throughput* [Nicodemus 1976]. By replacing the solid angle measure as outlined above, we obtain these concepts along with a variety of new ones (which generally differ from their original definitions by a factor of η^2):

- The *basic projected solid angle* measure ($\sigma_{\mathbf{x}}^{\perp\prime}$) on $\mathcal{S}^2$:

$$\begin{aligned} d\sigma_{\mathbf{x}}^{\perp\prime}(\omega) &= |\omega \cdot \mathbf{N}(\mathbf{x})|\, d\sigma'_{\mathbf{x}}(\omega) \\ &= \eta^2(\mathbf{x}, \omega)\, d\sigma_{\mathbf{x}}^{\perp}(\omega)\,. \end{aligned} \tag{7.3}$$

- The *basic throughput* measure (μ') on the ray space $\mathcal{R}$:

$$\begin{aligned} d\mu'(\mathbf{x}, \omega) &= dA(\mathbf{x})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega) \\ &= \eta^2(\mathbf{x}, \omega)\, dA(\mathbf{x})\, d\sigma_{\mathbf{x}}^{\perp}(\omega)\,, \end{aligned}$$

  or in other words,
$$d\mu'(\mathbf{r}) = \eta^2(\mathbf{r})\, d\mu(\mathbf{r})\,. \tag{7.4}$$

- *Basic radiance* (L'):

$$L'(\mathbf{r}) = \frac{d\Phi(\mathbf{r})}{d\mu'(\mathbf{r})} = \frac{L(\mathbf{r})}{\eta^2(\mathbf{r})}\,. \tag{7.5}$$

  Basic spectral radiance (L'_ν) is defined in a similar way.

- The *basic inner product* on $L_2(\mathcal{R})$:

$$\begin{aligned} \langle f, g \rangle' &= \int_{\mathcal{R}} f(\mathbf{r})\, g(\mathbf{r})\, d\mu'(\mathbf{r}) \\ &= \int_{\mathcal{R}} f(\mathbf{r})\, g(\mathbf{r})\, \eta^2(\mathbf{r})\, d\mu(\mathbf{r})\,. \end{aligned} \tag{7.6}$$

This inner product will be used to define adjoint operators in this chapter, i.e. two operators $\mathbf{H}$ and $\mathbf{H}^*$ are adjoint if $\langle \mathbf{H}^* f, g\rangle' = \langle f, \mathbf{H} g\rangle'$ for all f and g.

- The *basic BSDF* (f_{s}'):

$$f_{\mathrm{s}}'(\mathbf{x}, \omega_{\mathrm{i}} \to \omega_{\mathrm{o}}) \;=\; \frac{dL_{\mathrm{o}}'(\mathbf{x}, \omega_{\mathrm{o}})}{dE(\mathbf{x}, \omega_{\mathrm{i}})} \;=\; \frac{f_{\mathrm{s}}(\mathbf{x}, \omega_{\mathrm{i}} \to \omega_{\mathrm{o}})}{\eta_{\mathrm{o}}^2}\,, \tag{7.7}$$

where we have once again used the convention that $\eta_{\mathrm{o}} = \eta(\mathbf{x}, \omega_{\mathrm{o}})$.

Note that we have only redefined quantities whose definitions depend on solid angle. All other quantities (e.g. irradiance) are left unchanged.

These new quantities have interesting symmetry properties. Most importantly, the basic BSDF of any physically valid material is guaranteed to be symmetric:

$$f_{\mathrm{s}}'(\mathbf{x}, \omega_{\mathrm{i}} \to \omega_{\mathrm{o}}) \;=\; f_{\mathrm{s}}'(\mathbf{x}, \omega_{\mathrm{o}} \to \omega_{\mathrm{i}})\,.$$

This follows directly from the general reciprocity principle (7.1) that was proven in Chapter 6. (As a special case of this, Appendix 7.C derives the basic BSDF for perfect specular refraction and shows how to express it in a symmetric form.)

The other quantities defined above also have interesting symmetry properties, some of which take the form of *optical invariants* (a notion from classical geometric optics). These properties are discussed in Appendix 7.A.

7.2 The self-adjoint operator formulation

We now show how to put these concepts together into a framework of self-adjoint transport operators. The main idea is to use *basic radiance* (L') for all light transport calculations, while for importance transport the standard definitions are used.[2] As we will see, this leads to the desired symmetry properties, because basic radiance and importance satisfy the same transport equation. Furthermore, this framework computes exactly the same value for every measurement as before.

[2] Recall that importance has units of $[S \cdot \mathrm{W}^{-1}]$, and thus it is not affected by the new solid angle measure.

We now give the details of our framework. Measurements are computed using the basic inner product (7.6):

$$I \;=\; \langle W_{\mathrm{e}}, L_{\mathrm{i}}' \rangle' \,. \tag{7.8}$$

This equation gives the same results as the original measurement equation (4.20), since

$$\langle W_{\mathrm{e}}, L_{\mathrm{i}}' \rangle' \;=\; \int_{\mathcal{R}} W_{\mathrm{e}}(\mathbf{r})\, \frac{L_{\mathrm{i}}(\mathbf{r})}{\eta^2(\mathbf{r})} \left[\eta^2(\mathbf{r})\, d\mu'(\mathbf{r})\right] \;=\; \langle W_{\mathrm{e}}, L_{\mathrm{i}} \rangle \,.$$

The propagation operator $\mathbf{G}$ is unchanged; however, we define the new *basic local scattering operator* ($\mathbf{K}'$) by

$$(\mathbf{K}'h)(\mathbf{x}, \omega_{\mathrm{o}}) \;=\; \int_{\mathcal{S}^2} f_{\mathrm{s}}'(\mathbf{x}, \omega_{\mathrm{i}} \!\to\! \omega_{\mathrm{o}})\, h(\mathbf{x}, \omega_{\mathrm{i}})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_{\mathrm{i}}) \,. \tag{7.9}$$

It is helpful to expand the basic scattering equation $L_{\mathrm{o}}' = \mathbf{K}' L_{\mathrm{i}}'$, to see how the old and new quantities are related:

$$\frac{L_{\mathrm{o}}}{\eta_{\mathrm{o}}^2} \;=\; \int_{\mathcal{S}^2} \frac{f_{\mathrm{s}}(\mathbf{x}, \omega_{\mathrm{i}} \!\to\! \omega_{\mathrm{o}})}{\eta_{\mathrm{o}}^2}\, \frac{L_{\mathrm{i}}(\mathbf{x}, \omega_{\mathrm{i}})}{\eta_{\mathrm{i}}^2} \left[\eta_{\mathrm{i}}^2\, d\sigma_{\mathbf{x}}^{\perp}(\omega_{\mathrm{i}})\right] . \tag{7.10}$$

We see that the η_{i}^2 and η_{o}^2 factors are handled consistently. The most important difference is that the BSDF has been replaced by a symmetric quantity (the basic BSDF f_{s}'). Because of this, it is straightforward to check that the scattering operator $\mathbf{K}'$ is self-adjoint (see Appendix 7.B for details). Also notice that when there is only a single medium at $\mathbf{x}$, then $\mathbf{K}'$ is identical to the original operator $\mathbf{K}$ (since the hidden factor of η_{o}^2 in f_{s}' cancels the hidden factor of η_{i}^2 in $\sigma^{\perp\prime}$).

Light and importance transport operators. We define the transport operators $\mathbf{T}_X$ and solution operators $\mathbf{S}_X$ in the same way as before (see Section 4.7). These can be summarized as follows:

	Exitant	Incident
Light	$\mathbf{T}_{L_{\mathrm{o}}'} = \mathbf{K}'\mathbf{G}$	$\mathbf{T}_{L_{\mathrm{i}}'} = \mathbf{G}\mathbf{K}'$
Importance	$\mathbf{T}_{W_{\mathrm{o}}} = \mathbf{K}'^{*}\mathbf{G}$	$\mathbf{T}_{W_{\mathrm{i}}} = \mathbf{G}\mathbf{K}'^{*}$

However, because $\mathbf{G} = \mathbf{G}^*$ and $\mathbf{K}' = \mathbf{K}'^*$ for physically valid scenes, these definitions simplify to

$$\mathbf{T}_{L'_\mathrm{o}} = \mathbf{T}_{W_\mathrm{o}} = \mathbf{K}'\mathbf{G}\,,$$
$$\mathbf{T}_{L'_\mathrm{i}} = \mathbf{T}_{W_\mathrm{i}} = \mathbf{G}\mathbf{K}'\,.$$

Thus, basic radiance and importance obey the same transport equation.

We should emphasize that only the light transport operators have been changed in this framework; the importance transport operators have the same definitions as before. This is not immediately obvious, since we originally defined $\mathbf{T}_{W_\mathrm{o}} = \mathbf{K}^*\mathbf{G}$, and now we have defined $\mathbf{T}_{W_\mathrm{o}} = \mathbf{K}'^*\mathbf{G}$. However, $\mathbf{K}^*$ and $\mathbf{K}'^*$ are actually the same operator (as will be shown in Appendix 7.B). Because of this fact (which holds in all environments, physically valid or not), we continue to use the same symbols $\mathbf{T}_{W_\mathrm{i}}$ and $\mathbf{T}_{W_\mathrm{o}}$ for the importance transport operators.

To summarize, the main idea of the self-adjoint framework is to use basic radiance rather than radiance for light transport calculations, and to compensate for this by including a factor of η^2 in the measurement equation. With these simple changes, light, importance, and particles can be scattered and propagated in the same way. Further details of the framework are described in Appendix 7.B.

7.3 Consequences for implementations

We show how this framework affects the implementation of path tracing and bidirectional rendering algorithms. It is actually very simple to use the self-adjoint framework, since no scaling factors are required for transmission between different media, and the same scattering rules apply to light, importance, and particles.

Consider the structure of an ordinary path tracing algorithm. The calculation starts at the viewpoint, where a particular pixel value $I_j = \langle W_\mathrm{e}^{(j)}, L_\mathrm{o}\rangle$ is estimated by sampling a ray that contributes to this integral. The initial ray lies in a medium with some refractive index η_1. We then proceed by following a path backward, through a sequence of media with indices $\eta_2, \ldots, \eta_k$, until finally a light source is reached, and the emitted radiance L_e is computed. With a standard framework (e.g. that of Chapter 4), a scaling factor of η_i^2/η_{i+1}^2 is required between each pair of media i and $i+1$ to account for the change in radiance.

With the self-adjoint framework, we obtain the same result in a simpler way. Starting again at the viewpoint, we sample a ray $\mathbf{r}$ to estimate the basic inner product $I_j = \langle W_e^{(j)}, L_o' \rangle'$. Since $\mathbf{r}$ lies in medium η_1, the weight for this ray has an extra factor of η_1^2 compared to the standard path tracing implementation. However, we now evaluate the *basic* radiance along this ray, which means that no special scaling factors are required as we follow a path backward through media $\eta_2, \ldots, \eta_k$. When the path finally reaches a light source, we must divide its emitted radiance by η_k^2 to obtain basic radiance.[3] Thus the combined scaling factor for this path is η_1^2/η_k^2, which is identical to the product of all the scaling factors above.

With bidirectional algorithms, some of the calculations are carried out by propagating information forward from the light sources. For example, consider the "pool of water" scene from Section 5.2. Suppose that a particle tracing pass is used to accumulate the caustics on the pool bottom in a view-independent form (e.g. a texture map), which is then rendered using a ray tracing pass. With the self-adjoint framework, the particle tracing pass does not require any changes. The ray tracing pass is similar to the path tracing algorithm described above, except that now the "emission function" consists of a texture map on the pool bottom, which must be expressed in the form of basic radiance before it is used. This is done looking up the irradiance value in the texture map, and dividing it by the η^2 value of the surrounding medium (i.e. water).

Similarly, the self-adjoint framework can be used with algorithms such as density estimation [Shirley et al. 1995], the photon map [Jensen 1996], and bidirectional path tracing, by making changes of a similar nature.

We should mention that it is also possible to obtain a symmetric transport framework by working with the quantities L/η and W/η (rather than L and W), and computing measurements using the ordinary inner product. With this convention, light and importance are both scaled by the same factor of η_t/η_i when they enter a new medium. However, this scheme only gives correct results when all sources and sensors are located in media whose refractive index is $\eta = 1$.

[3] Alternatively, the emission from light sources can be expressed using basic radiance in the first place.

Appendix 7.A Classical optical invariants

The quantities defined in Section 7.1 have several important symmetry properties. Some of these correspond to the classical notion of an optical invariant, a topic that we explore here.

Optical invariants are defined within the framework of classical geometric optics, which studies the formation of images by systems of mirrors and lenses. An *optical invariant* is a quantity that preserved by such systems, that is, a numerical measurement that has the same value for any real object and its image.

7.A.1 The Smith-Helmholtz invariant

Of the classical optical invariants, the most famous is the *Smith-Helmholtz* or *Lagrange* invariant, which was first stated by Smith in his *Compleat System of Opticks* (Cambridge, 1738), and subsequently rediscovered by Lagrange (1803) and von Helmholtz [1856, p. 74].

Consider a lens system that has rotational symmetry about the lens axis, so that it can be represented by a planar diagram (see Figure 7.1). Given some object and its corresponding image, the Smith-Helmholtz invariant states that

$$\eta h \alpha \;=\; \eta' h' \alpha' \,, \tag{7.11}$$

where η is the refractive index of the medium containing the object, h is the object height, and α is the angle over which light is radiated from the object toward the lens system. The quantities η', h', and α' denote the corresponding quantities for the image (where α' is now the angle over which light is received from the lens system, at a given point of the image). The ratio h'/h is called the *linear magnification* of the lens system, while α'/α is called the *angular magnification*; equation (7.11) shows that these quantities are related in a simple way.

7.A.2 The invariance of basic throughput

Another classical invariant is *basic throughput* [Nicodemus 1976, p. 37], which is also known as *etendue* [Steel 1974]. This quantity has already been defined (Section 7.1), but we repeat its definition here:

$$\mu'(D) \;=\; \int_D \eta^2(\mathbf{r})\, d\mu(\mathbf{r})\,,$$

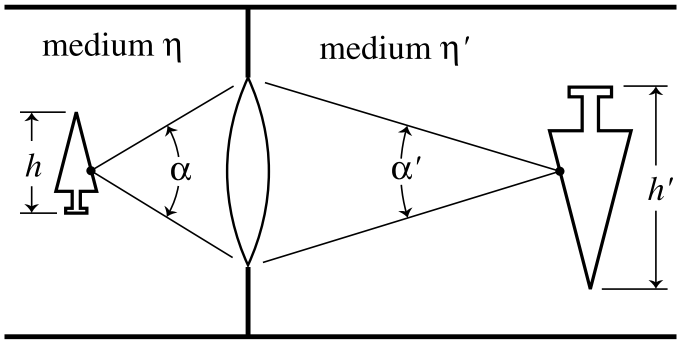

Figure 7.1: The Smith-Helmholtz invariant relates the geometry of an object and its corresponding image, according to the equation $\eta h \alpha = \eta' h' \alpha'$.

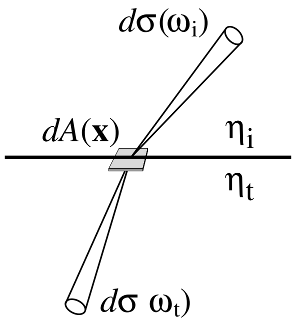

Figure 7.2: When a beam of light is refracted, its basic throughput is preserved.

where $D \subset \mathcal{R}$ is a set of rays. With respect to classical geometric optics, D would represent the beam of rays that leave an object toward the lens system, eventually forming an image. The invariance of this quantity implies that as this light beam propagates through an optical system, its basic throughput μ' is preserved.

An example: perfect specular refraction. We show how this invariance can be proven, for the special case of perfect specular refraction. Consider a beam of light that strikes small surface patch $dA(\mathbf{x})$, occupying a solid angle of $d\sigma(\omega_\mathrm{i})$ (see Figure 7.2). Let ω_t be the direction of the refracted beam, which occupies a solid angle of $d\sigma(\omega_\mathrm{t})$. In Section 5.2.1.1, we have already shown that the incident and transmitted beams are related by

$$\eta_{\rm i}^2 \, d\sigma^\perp(\omega_{\rm i}) \;=\; \eta_{\rm t}^2 \, d\sigma^\perp(\omega_{\rm t}) \,, \tag{7.12}$$

recalling that $\sigma^\perp$ denotes the projected solid angle. Since these beams travel through the same area $dA(\mathbf{x})$ on the surface, we thus have

$$\begin{aligned}
\eta_{\rm i}^2 \, dA(\mathbf{x}) \, d\sigma^\perp(\omega_{\rm i}) &= \eta_{\rm t}^2 \, dA(\mathbf{x}) \, d\sigma^\perp(\omega_{\rm t}) \\
\Longrightarrow \qquad \eta_{\rm i}^2 \, d\mu(\mathbf{r}_{\rm i}) &= \eta_{\rm t}^2 \, d\mu(\mathbf{r}_{\rm t}) \\
\Longrightarrow \qquad d\mu'(\mathbf{r}_{\rm i}) &= d\mu'(\mathbf{r}_{\rm t}) \,,
\end{aligned} \tag{7.13}$$

where $\mathbf{r}_{\rm i} = (\mathbf{x}, \omega_{\rm i})$ and $\mathbf{r}_{\rm t} = (\mathbf{x}, \omega_{\rm t})$ represent the incident and transmitted rays. By integrating this relationship, we can show that basic throughput μ' is invariant for an arbitrary set of rays D that strike the surface. This affirms the invariance of μ' in the special case of refraction. It is also straightforward to show that basic throughput is preserved when light is reflected, or when it propagates through a constant medium (see Appendix 7.B). Using more advanced techniques, it is possible to show that basic throughput is actually preserved in any system that obeys the laws of geometric optics [Nicodemus 1963].

Note that the Smith-Helmholtz invariant can be derived as a special case of this law. To see this, observe that for rotationally symmetric lens systems, the area dA of an object is proportional to y^2, while the solid angle $d\sigma$ over which light radiates is proportional to α^2. Thus the Smith-Helmholtz invariant follows immediately from (7.13).[4]

7.A.3 The invariance of basic radiance

If we assume that each light beam follows a single path through an optical system (i.e. partial reflection is not allowed), and that there are no losses due to absorption, we can also show the invariance of *basic radiance* [Nicodemus 1976, p. 26]. That is, as a beam of light propagates through an optical system, its basic radiance $L' = L/\eta^2$ is preserved (this is known as *Abbe's law* [Keitz 1971, p. 195]).

The invariance of basic radiance can be derived directly from its definition,

$$L'(\mathbf{r}) \;=\; \frac{d\Phi(\mathbf{r})}{d\mu'(\mathbf{r})} \,.$$

[4] Note that the Smith-Helmholtz equation (7.11) is strictly valid only for infinitesimally small objects that are aligned with the optical axis.

That is, as a beam of light propagates through an optical system, its power $d\Phi$ and its basic throughput $d\mu'$ are both preserved (by conservation of energy, and the invariance of basic throughput). Thus, basic radiance is invariant as well. This can be shown under very general conditions by using thermodynamic principles [Liebes 1969].

Basic spectral radiance L_ν/η^2 is an optical invariant as well, when it is parameterized by frequency. However, if spectral radiance is parameterized by wavelength, then L_λ/η^3 is invariant instead [Nicodemus 1976, p. 52], since wavelengths (unlike frequencies) are modified at the interface (see Section 6.2).

The other "basic" quantities we have defined also possess symmetry properties, however they do not take the form of optical invariants. For example, the basic BSDF is symmetric, but does not correspond to any property that is preserved by beams propagating through an optical system. Similarly, equation (7.12) implies that the basic projected solid angle is preserved at a refractive interface $(d\sigma^{\perp\prime}(\omega_{\mathrm{i}}) = d\sigma^{\perp\prime}(\omega_{\mathrm{t}}))$.

Appendix 7.B Properties of the new operators

We consider the adjoints and norms of the operators defined in this chapter. (The invertibility properties are unchanged from Section 4.B.1.)

7.B.1 Adjoints

Recall that adjoint operators in this chapter are defined with respect to the basic inner product, i.e. two operators $\mathbf{H}$ and $\mathbf{H}^*$ are adjoint if $\langle \mathbf{H}^* f, g\rangle' = \langle f, \mathbf{H} g\rangle'$ for all f and g.

Lemma 7.1. *The reversal map M preserves the measure μ'. In other words, $\mu'(M(D)) = \mu'(D)$ for any measurable set $D \subset \tilde{\mathcal{R}}$.*

The proof depends on the fact that the refractive indices of $\mathbf{r}$ and $M(\mathbf{r})$ are always equal. It is otherwise similar to the proof of Lemma 4.4.

Theorem 7.2. *The operator $\mathbf{G}$ is self-adjoint (with respect to the basic inner product).*

The proof is similar to Theorem 4.5, but requires the preceding lemma.

Theorem 7.3. *The adjoint of $\mathbf{K}'$ is given by*

$$(\mathbf{K}'^* h)(\mathbf{x}, \omega_{\mathrm{o}}) \;=\; \int_{\mathcal{S}^2} f_{\mathrm{s}}'^*(\mathbf{x}, \omega_{\mathrm{i}} \!\to\! \omega_{\mathrm{o}})\, h(\mathbf{x}, \omega_{\mathrm{i}})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_{\mathrm{i}})\,.$$

In particular, $\mathbf{K}'$ is self-adjoint for any physically valid scene model.

The proof is similar to Theorem 4.6. The last statement follows from the fact that $f_{\mathrm{s}}' = f_{\mathrm{s}}'^*$ for physically valid scenes (7.1).

Corollary 7.4. *The operators $\mathbf{K}'^*$ and $\mathbf{K}^*$ are the same (for all scenes).*

Proof. We have

$$\begin{aligned}
(\mathbf{K}'^* h)(\mathbf{x}, \omega_{\mathrm{o}}) &= \int_{\mathcal{S}^2} f_{\mathrm{s}}'^*(\mathbf{x}, \omega_{\mathrm{i}} \!\to\! \omega_{\mathrm{o}})\, h(\mathbf{x}, \omega_{\mathrm{i}})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_{\mathrm{i}}) \\
&= \int_{\mathcal{S}^2} f_{\mathrm{s}}'(\mathbf{x}, \omega_{\mathrm{o}} \!\to\! \omega_{\mathrm{i}})\, h(\mathbf{x}, \omega_{\mathrm{i}})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_{\mathrm{i}}) \\
&= \int_{\mathcal{S}^2} \frac{f_{\mathrm{s}}(\mathbf{x}, \omega_{\mathrm{o}} \!\to\! \omega_{\mathrm{i}})}{\eta_{\mathrm{i}}^2}\, h(\mathbf{x}, \omega_{\mathrm{i}})\, \left[\eta_{\mathrm{i}}^2\, d\sigma_{\mathbf{x}}^{\perp}(\omega_{\mathrm{i}})\right] \\
&= \int_{\mathcal{S}^2} f_{\mathrm{s}}(\mathbf{x}, \omega_{\mathrm{o}} \!\to\! \omega_{\mathrm{i}})\, h(\mathbf{x}, \omega_{\mathrm{i}})\, d\sigma_{\mathbf{x}}^{\perp}(\omega_{\mathrm{i}}) \\
&= (\mathbf{K}^* h)(\mathbf{x}, \omega_{\mathrm{o}})\,. \quad \blacksquare
\end{aligned}$$

Thus importance obeys the same scattering rules in both frameworks. In physically valid scenes, we also have $\mathbf{K}' = \mathbf{K}'^*$, so that $\mathbf{K}'$, $\mathbf{K}'^*$, and $\mathbf{K}^*$ are all the same operator.

7.B.2 Norms

Since the new operators are defined using the basic throughput measure μ', it will be convenient to define a new set of L_p norms, denoted $\|\cdot\|'_p$:

$$\|f\|'_p \;=\; \left(\int_{\mathcal{R}} |f(\mathbf{r})|^p \, d\mu'(\mathbf{r})\right)^{\frac{1}{p}} . \tag{7.14}$$

(The norm $\|\cdot\|_\infty$ defined by equation (4.9) is not affected by this change, but we will relabel it $\|\cdot\|'_\infty$ for consistency.)

The old and new norms are always within a constant factor of each other, as stated by the following lemma:

Lemma 7.5. *Let $\eta_{\min}$ and $\eta_{\max}$ denote the minimum and maximum refractive indices in the given scene. Then for any $1 \le p \le \infty$ and any $f \in L_p(\mathcal{R})$, we have*

$$(\eta^2_{\min})^{1/p} \|f\|_p \;\le\; \|f\|'_p \;\le\; (\eta^2_{\max})^{1/p} \|f\|_p \, . \tag{7.15}$$

Furthermore if $\mathbf{H}$ is any bounded operator on $L_p(\mathcal{R})$, then

$$\|\mathbf{H}\|_p \;\le\; \left(\frac{\eta^2_{\max}}{\eta^2_{\min}}\right)^{\frac{1}{p}} \|\mathbf{H}\|'_p \, . \tag{7.16}$$

The proofs follow directly from the corresponding definitions.

As a corollary, note that the space $L_p(\mathcal{R})$ contains the same functions when it is defined using either of the norms $\|\cdot\|_p$ or $\|\cdot\|'_p$, since the two norms are always within a constant factor. Thus we can refer to $L_p(\mathcal{R})$ in either case without ambiguity.

Theorem 7.6. $\|\mathbf{K}'\|'_p < 1$ *for any physically valid scene, and for any* $1 \le p \le \infty$.

Proof. The proof is very similar to [Arvo 1995, Appendix A.8]. First, we consider the case $p = 1$. To bound the operator norm $\|\mathbf{K}'\|'_1$, we must find a number m such that

$$\|\mathbf{K}'h\|'_1 \;\le\; m\,\|h\|'_1 \qquad \text{for any function} \quad h \in L_1(\mathcal{R}) \, .$$

To obtain such a bound, we compute

$$\begin{aligned}
\|\mathbf{K}'h\|_1' &= \int_{\mathcal{R}} (\mathbf{K}'h)(\mathbf{r})\, d\mu'(\mathbf{r}) \\
&= \int_{\mathcal{S}^2} \int_{\mathcal{M}} (\mathbf{K}'h)(\mathbf{x}, \omega_\mathrm{o})\, dA(\mathbf{x})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_\mathrm{o}) \\
&= \int_{\mathcal{S}^2} \int_{\mathcal{M}} \int_{\mathcal{S}^2} f_\mathrm{s}'(\mathbf{x}, \omega_\mathrm{i} \to \omega_\mathrm{o})\, h(\mathbf{x}, \omega_\mathrm{i})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_\mathrm{i})\, dA(\mathbf{x})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_\mathrm{o}) \\
&= \int_{\mathcal{M}} \int_{\mathcal{S}^2} \left[\int_{\mathcal{S}^2} f_\mathrm{s}'(\mathbf{x}, \omega_\mathrm{i} \to \omega_\mathrm{o})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_\mathrm{o}) \right] h(\mathbf{x}, \omega_\mathrm{i})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_\mathrm{i})\, dA(\mathbf{x})
\end{aligned}$$

(where we have dropped the absolute value signs, since all quantities are positive).

To obtain an upper bound on this expression, we let m denote the maximum value attained by the bracketed quantity over the entire domain of the outer integrals:

$$m = \operatorname*{ess\,sup}_{(\mathbf{x}, \omega_\mathrm{i}) \in \mathcal{R}} \int_{\mathcal{S}^2} f_\mathrm{s}'(\mathbf{x}, \omega_\mathrm{i} \to \omega_\mathrm{o})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_\mathrm{o})\,.$$

We thus have

$$\|\mathbf{K}'h\|_1' \;\le\; m \int_{\mathcal{M}} \int_{\mathcal{S}^2} h(\mathbf{x}, \omega_\mathrm{i})\, d\sigma^{\perp\prime}(\omega_\mathrm{i})\, dA(\mathbf{x}) \;=\; m\, \|h\|_1'\,,$$

so that m is an upper bound on the operator norm $\|\mathbf{K}'\|_1'$.

To better understand the meaning of this bound, we re-express it in terms of the ordinary BSDF f_s:

$$\begin{aligned}
m &= \operatorname*{ess\,sup}_{(\mathbf{x}, \omega_\mathrm{i}) \in \mathcal{R}} \int_{\mathcal{S}^2} f_\mathrm{s}'(\mathbf{x}, \omega_\mathrm{i} \to \omega_\mathrm{o})\, d\sigma_{\mathbf{x}}^{\perp\prime}(\omega_\mathrm{o}) \qquad (7.17) \\
&= \operatorname*{ess\,sup}_{(\mathbf{x}, \omega_\mathrm{i}) \in \mathcal{R}} \int_{\mathcal{S}^2} \frac{f_\mathrm{s}(\mathbf{x}, \omega_\mathrm{i} \to \omega_\mathrm{o})}{\eta^2(\mathbf{x}, \omega_\mathrm{o})} \left[\eta^2(\mathbf{x}, \omega_\mathrm{o})\, d\sigma_{\mathbf{x}}^{\perp}(\omega_\mathrm{o}) \right] \\
&= \operatorname*{ess\,sup}_{(\mathbf{x}, \omega_\mathrm{i}) \in \mathcal{R}} \int_{\mathcal{S}^2} f_\mathrm{s}(\mathbf{x}, \omega_\mathrm{i} \to \omega_\mathrm{o})\, d\sigma_{\mathbf{x}}^{\perp}(\omega_\mathrm{o})\,.
\end{aligned}$$

Comparing this to the BSDF energy-conservation condition (6.5) derived in Chapter 6, we see that if the scene model uses only physically valid BSDF's, then we are guaranteed that $m \le 1$. Furthermore, real materials will always have at least a small amount of absorption, so that for physically valid scenes we may assume that $m < 1$.[5] This establishes the theorem in the case $p = 1$.

[5]Even for situations such as total internal reflection, or reflection from metals at grazing angles, there will always be some absorption due to tiny imperfections and impurities in the materials.

The case $p = \infty$ is very similar; it is straightforward to show that

$$\|\mathbf{K}'\|'_\infty \;\leq\; \operatorname*{ess\,sup}_{(\mathbf{x},\omega_\mathrm{i})\in\mathcal{R}} \int_{\mathcal{S}^2} f'_\mathrm{s}(\mathbf{x}, \omega_\mathrm{o} \to \omega_\mathrm{i})\, d\sigma^{\perp\prime}_{\mathbf{x}}(\omega_\mathrm{o})$$

.

Notice that this expression is identical to (7.17), except that the directional arguments to the basic BSDF f'_s have been exchanged. Since f'_s is guaranteed to be symmetric for physically valid scenes, we obtain the same bound as for $p = 1$, namely

$$\|\mathbf{K}'\|'_\infty \;\leq\; m \;<\; 1\,.$$

For values of p with $1 < p < \infty$, we use the fact that

$$\|\mathbf{K}'\|'_p \;\leq\; \max\{\|\mathbf{K}'\|'_1, \|\mathbf{K}'\|'_\infty\}\,.$$

This was shown by Arvo [1995, Theorems 12 and 13], whose results apply to any operator of the form $\mathbf{K}$ or $\mathbf{K}'$. ∎

From this result and the bound (7.16), we obtain the following:

Corollary 7.7. *For any physically valid scene, and for any* $1 \leq p \leq \infty$*,*

$$\|\mathbf{K}\|_p \;<\; \frac{\eta^2_{\max}}{\eta^2_{\min}}\,,$$

where $\eta_{\min}$ *and* $\eta_{\max}$ *denote the minimum and maximum refractive indices in the environment.*

This was previously stated as Theorem 4.12.

Finally, we can put these results together to show that the solution operators $\mathbf{S}_X$ are well defined, i.e. that the operators $(\mathbf{I} - \mathbf{T}_X)$ are invertible.

Theorem 7.8. *For any physically valid scene, the solution operators* $\mathbf{S}_X$ *exist and are well-defined, where* X *is any of* L'_i, L'_o, L_i, L_o, W_i, *or* W_o.

This was previously stated as Theorem 4.13.

Proof. When X is one of L'_i, L'_o, W_i, or W_o, then $\mathbf{T}_X$ is a composition of $\mathbf{K}'$ and $\mathbf{G}$. Therefore

$$\|\mathbf{T}_X\|' \;\leq\; \|\mathbf{K}'\|'\,\|\mathbf{G}\|' \;<\; 1\,,$$

and thus $(\mathbf{I} - \mathbf{T}_X)$ is invertible.

For the cases $X = L_{\mathrm{i}}$ and $X = L_{\mathrm{o}}$, it is sufficient to show that $\|\mathbf{T}_X^k\| < 1$ for some integer $k \geq 1$. To do this, observe that since $\|\mathbf{T}_X\|' < 1$, there is some integer k such that

$$\|\mathbf{T}_X^k\|' \;<\; (\eta_{\min}/\eta_{\max})^2 \,.$$

Applying the relationship (7.16) between the operator norms $\|\cdot\|$ and $\|\cdot\|'$, we obtain the desired result. ∎

Appendix 7.C The basic BSDF for refraction

In Section 5.2.2 we showed that the BSDF for perfect specular refraction is

$$f_{\mathrm{s}}(\omega_{\mathrm{i}} \!\rightarrow\! \omega_{\mathrm{t}}) \;=\; \frac{\eta_{\mathrm{t}}^2}{\eta_{\mathrm{i}}^2}\, \delta_{\sigma^\perp}(\omega_{\mathrm{i}} - R(\omega_{\mathrm{t}}))$$

where R is the refraction mapping, and δ_μ is the Dirac distribution with respect to the given measure μ. Thus the corresponding basic BSDF is

$$f_{\mathrm{s}}'(\omega_{\mathrm{i}} \!\rightarrow\! \omega_{\mathrm{t}}) \;=\; \frac{1}{\eta_{\mathrm{t}}^2} f_{\mathrm{s}}(\omega_{\mathrm{i}} \!\rightarrow\! \omega_{\mathrm{t}}) \;=\; \frac{1}{\eta_{\mathrm{i}}^2}\, \delta_{\sigma^\perp}(\omega_{\mathrm{i}} - R(\omega_{\mathrm{t}}))\,. \tag{7.18}$$

We have given two separate arguments showing that this quantity is symmetric: the first was a direct mathematical derivation (Appendix 5.B), while the second was based on a general reciprocity principle (Chapter 6).

In this appendix, we show how the basic BSDF for refraction can be rewritten to make its symmetry more obvious. The idea is to rewrite it as a Dirac distribution with respect to the basic projected solid angle measure ($\sigma^{\perp\prime}$). Using the relationship (7.3) between basic and ordinary projected solid angle, and the identity (5.35) between Dirac distributions with respect to different measures, we have

$$f_{\mathrm{s}}'(\omega_{\mathrm{i}} \!\rightarrow\! \omega_{\mathrm{t}}) \;=\; \frac{1}{\eta_{\mathrm{i}}^2}\, \delta_{\sigma^\perp}(\omega_{\mathrm{i}} - R(\omega_{\mathrm{t}})) \;=\; \delta_{\sigma^{\perp\prime}}(\omega_{\mathrm{i}} - R(\omega_{\mathrm{t}}))\,.$$

The symmetry of this quantity can then be expressed as

$$\delta_{\sigma^{\perp\prime}}(\omega_{\mathrm{i}} - R(\omega_{\mathrm{t}})) \;=\; \delta_{\sigma^{\perp\prime}}(\omega_{\mathrm{t}} - R(\omega_{\mathrm{i}}))\,,$$

which follows from the fact that the mapping R is a bijection, and that it preserves the basic projected solid angle measure (see Section 5.2 and equation (5.36)).

With respect to the angular parameterization (θ, ϕ), the basic BSDF for refraction can be written as

$$f_{\mathrm{s}}'(\theta_{\mathrm{i}}, \phi_{\mathrm{i}}, \theta_{\mathrm{t}}, \phi_{\mathrm{t}}) \;=\; 2\,\delta(\eta_{\mathrm{i}}^2 \sin^2\theta_{\mathrm{i}} - \eta_{\mathrm{t}}^2 \sin^2\theta_{\mathrm{t}})\,\delta(\phi_{\mathrm{i}} - (\phi_{\mathrm{t}} \pm \pi))\,,$$

which follows from equations (5.38), (7.7), and (5.30). In this form, the symmetry of the basic BSDF is clear.