4.3 Linear, Homogeneous Equations with Constant Coefficients

3.2. Let $y = e^{\lambda t}$ in $y'' + 5y' + 6y = 0$ to obtain

$$\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6e^{\lambda t} = 0,$$
$$e^{\lambda t}(\lambda^2 + 5\lambda + 6) = 0.$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\lambda^2 + 5\lambda + 6 = 0,$$
$$(\lambda + 3)(\lambda + 2) = 0,$$

3.6. Let $y = e^{\lambda t}$ in $6y'' + 5y' - 6y = 0$ to obtain

$$6\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} - 6e^{\lambda t} = 0,$$
$$e^{\lambda t}(6\lambda^2 + 5\lambda - 6) = 0.$$

Because $e^{\lambda t} \neq 0$, we arrive at the logistic equation

$$6\lambda^2 + 5\lambda - 6 = 0$$
$$(3\lambda - 2)(2\lambda + 3) = 0.$$

3.10. If $y'' + 2y' + 17y = 0$, then the characteristic equation is

$$\lambda^2 + 2\lambda + 17 = 0.$$

The roots of the characteristic equation are $-1 \pm 4i$, leading to the complex solutions

$$z(t) = e^{(-1+4i)t} \quad \text{and} \quad \overline{z}(t) = e^{(-1-4i)t}.$$

However, by Euler's identity,

$$z(t) = e^{-t}e^{4it} = e^{-t}(\cos 4t + i \sin 4t),$$

and the real and imaginary parts of z lead to a fundamental set of real solutions $y_1(t) = e^{-t}\cos 4t$ and $y_2(t) = e^{-t}\sin 4t$. Hence the general solution is

$$y(t) = C_1 e^{-t}\cos 4t + C_2 e^{-t}\sin 4t.$$

3.14. If $y'' - 6y' + 9y = 0$, then the characteristic equation is

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0.$$

Hence the characteristic equation has a repeated root, $\lambda = 3$. Therefore, $y_1(t) = e^{3t}$ and $y_2(t) = te^{3t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{3t} + C_2 te^{3t} = (C_1 + C_2 t)e^{3t}.$$

3.20. If $y'' - 2y' + 17y = 0$, then the characteristic equation is

$$\lambda^2 - 2\lambda + 17 = 0,$$

with roots $1 \pm 4i$. The complex solution,

$$z(t) = e^{(1+4i)t} = e^t(\cos 4t + i \sin 4t)$$

leads to a fundamental set of real solutions and the general solution

$$y(t) = e^t(C_1 \cos 4t + C_2 \sin 4t).$$

The initial condition $y(0) = -2$ provides

$$-2 = C_1.$$

Differentiating the general solution,

$$y'(t) = e^t(C_1 \cos 4t + C_2 \sin 4t) + e^t(-4C_1 \sin 4t + 4C_2 \cos 4t),$$

3.22. If $y'' + 10y' + 25y = 0$, then the characteristic equation is

$$\lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0,$$

with repeated root $\lambda = -5$. This leads to the fundamental set of solutions $y_1(t) = e^{-5t}$ and $y_2(t) = te^{-5t}$ and the general solution is

$$y(t) = C_1 e^{-5t} + C_2 t e^{-5t} = (C_1 + C_2 t)e^{-5t}.$$

Using the initial condition $y(0) = 2$ leads to

$$2 = C_1.$$

Differentiating the general solution,

$$y'(t) = C_2 e^{-5t} - 5(C_1 + C_2 t)e^{-5t},$$

then the initial condition $y'(0) = -1$ leads to

$$-1 = C_2 - 5C_1.$$

Thus, $C_1 = 2$ and $C_2 = 9$ and the final solutions is

$$y(t) = (2 + 9t)e^{-5t}.$$

4.4 Harmonic Motion

4.11. Substitute $m = 0.2\,\text{kg}$ and $k = 5\,\text{kg/s}^2$ in $my'' + ky = 0$ to obtain $0.2y'' + 5y = 0$ or

$$y'' + 25y = 0.$$

The characteristic equation is $\lambda^2 + 25 = 0$, with zeros $\lambda = \pm 5i$, so

$$z(t) = e^{5it} = \cos 5t + i \sin 5t$$

is a complex solution. The real and imaginary parts of this solution form a fundamental set of real solutions, giving the general solution

$$y(t) = C_1 \cos 5t + C_2 \sin 5t.$$

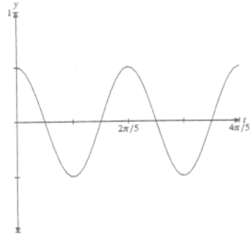
The initial displacement is 0.5 m, so $y(0) = 0.5$ and $C_1 = 0.5$. Differentiating,

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t.$$

The system is released from rest, so $y'(0) = 0$ and $C_2 = 0$. Thus, the solution is

$$y(t) = 0.5 \cos 5t,$$

which has amplitude 0.5, frequency 5 rad/s, and zero phase.



4.14. The system $mx'' + kx = 0$, or $x'' + (k/m)x = 0$, is equivalent to

$$x'' + \omega_0^2 x = 0,$$

with $\omega_0^2 = k/m$. The characteristic equation is $\lambda^2 + \omega_0^2 = 0$, with roots $\lambda = \pm\omega_0 i$. Thus, we have complex solution

$$z(t) = e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t.$$

The real and imaginary parts of this solution form a fundamental set of solutions and provide the general solution

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t.$$

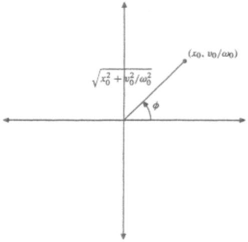
The initial condition $x(0) = x_0$ gives $C_1 = x_0$. Differentiate.

$$x'(t) = -C_1\omega_0\sin\omega_0 t + C_2\omega_0\cos\omega_0 t.$$

The initial condition $x'(0) = v_0$ gives $C_2 = v_0/\omega_0$. Thus, the solution is

$$x(t) = x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t.$$

Plot the coefficients, calculate the magnitude of the vector, and mark the angle.



The tangent of the angle ϕ is easily calculated.

$$\tan\phi = \frac{v_0}{x_0\omega_0}$$

Factor out the magnitude as follows.

$$x(t) = \sqrt{x_0^2 + v_0^2/\omega_0^2}\left(\frac{x_0}{\sqrt{x_0^2 + v_0^2/\omega_0^2}}\cos\omega_0 t + \frac{v_0/\omega_0}{\sqrt{x_0^2 + v_0^2/\omega_0^2}}\sin\omega_0 t\right)$$

But $\cos\phi = x_0/\sqrt{x_0^2 + v_0^2/\omega_0^2}$ and $\sin\phi = (v_0/\omega_0)/\sqrt{x_0^2 + v_0^2/\omega_0^2}$, so we can write

$$x(t) = \sqrt{x_0^2 + v_0^2/\omega_0^2}(\cos\phi\omega_0 t + \sin\phi\sin\omega_0 t)$$
$$x(t) = \sqrt{x_0^2 + v_0^2/\omega_0^2}\cos(\omega_0 t - \phi).$$

Thus, the amplitude of the motion is

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}$$
$$A = \sqrt{x_0^2 + \frac{v_0^2}{k/m}}$$
$$A = \sqrt{x_0^2 + \frac{mv_0^2}{k}}.$$

4.18. By Hooke's Law,

$$k = \frac{F}{y} = \frac{mg}{y}$$
$$k = \frac{(0.05\,\text{kg})(9.8\,\text{m/s}^2)}{(0.2\,\text{m})}$$
$$k = 2.45\,\text{N/m}.$$

Hence, $my'' + \mu y' + ky = 0$ becomes $0.05y'' + \mu y' + 2.45y = 0$, or

$$y'' + 20\mu y' + 49y = 0.$$

This system has characteristic equation $\lambda^2 + 20\mu\lambda + 49 = 0$, with zeros given by

$$\lambda = \frac{-20\mu \pm \sqrt{400\mu^2 - 196}}{2}.$$

The system is critically damped if it has one single, repeated root. The happens only if

$$400\mu^2 - 196 = 0$$
$$\mu^2 = \frac{196}{400}$$
$$\mu = \frac{14}{20}.$$

Thus, $\mu = 7/10$, and our equation becomes

$$y'' + 14y' + 49y = 0,$$

whose characteristic equation $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2 = 0$ has a repeated root $\lambda = -7$. Thus, the general solution is

$$y(t) = (C_1 + C_2 t)e^{-7t}.$$

If we assume that the mass is displaced in a downward direction, $y(0) = -0.15\,\text{m}$ and $C_1 = -0.15$. Differentiate.

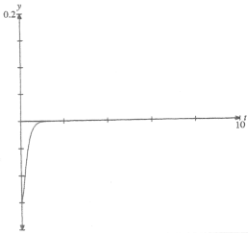
$$y'(t) = C_2 e^{-7t} - 7(C_1 + C_2 t)e^{-7t}$$

If the mass is released from rest, then $y'(0) = 0$ and

$$0 = C_2 - 7C_1.$$

Thus, $C_2 = -1.05$ and the solution is

$$y(t) = -(0.15 + 1.05t)e^{-7t}$$



4.5 Inhomogeneous Equations; the Method of Undetermined Coefficients

5.2. Let $y(t) = Ae^{-t}$. Then

$$y'(t) = -Ae^{-t}$$
$$y''(t) = Ae^{-t},$$

and $y'' + 6y' + 8y = -3e^{-t}$ becomes

$$Ae^{-t} + 6(-Ae^{-t}) + 8(Ae^{-t}) = -3e^{-t}$$
$$3A = -3$$
$$A = -1.$$

Thus, $y = -e^{-t}$ is a particular solution.

5.4. Let $y(t) = Ae^{2t}$. Then,

$$y'(t) = 2Ae^{2t}$$
$$y''(t) = 4Ae^{2t},$$

and $y'' + 3y' - 18y = 18e^{2t}$ becomes

$$4Ae^{2t} + 3(2Ae^{2t}) - 18(Ae^{2t}) = 18e^{2t}$$
$$-8A = 18$$
$$A = -\frac{9}{4}.$$

Thus, $y = -(9/4)e^{2t}$ is a particular solution.

5.8. Let $y_p = a\cos 3t + b\sin 3t$. Then

$$y_p' = -3a\sin 3t + 3b\cos 3t$$
$$y_p'' = -9a\cos 3t - 9b\sin 3t,$$

and the equation $y'' + 7y' + 10y = -4\sin 3t$ becomes

$$(a + 21b)\cos 3t + (-21a + b)\sin 3t = -4\sin 3t.$$

Thus,

$$a + 21b = 0$$
$$-21a + b = -4,$$

leading to $a = 42/221$ and $b = -2/221$ and the particular solution $y_p = (42/221)\cos 3t - (2/221)\sin 3t$.

5.12. Let $z = Ae^{i2t}$. Then

$$z' = 2iAe^{i2t}$$
$$z'' = (2i)^2Ae^{i2t},$$

and $z'' + 7z' + 6z = 3e^{i2t}$ leads to

$$(2i)^2Ae^{i2t} + 7(2i)Ae^{i2t} + 6Ae^{i2t} = 3e^{i2t}$$
$$\left((2i)^2 + 7(2i) + 6\right)A = 3$$
$$A = \frac{3}{2+14i}$$
$$A = \frac{3}{100} - \frac{21}{100}i.$$

Thus,

$$z = \left(\frac{3}{100} - \frac{21}{100}i\right)e^{i2t}$$
$$z = \left(\frac{3}{100} - \frac{21}{100}i\right)(\cos 2t + i\sin 2t)$$
$$z = \left(\frac{3}{100}\cos 2t + \frac{21}{100}\sin 2t\right) + i\left(-\frac{21}{100}\cos 2t + \frac{3}{100}\sin 2t\right)$$

is a solution of $z'' + 7z' + 6z = 3e^{i2t}$. The imaginary part,

$$y = -\frac{21}{100}\cos 2t + \frac{3}{100}\sin 2t$$

is a solution of $y'' + 7y' + 6y = 3\sin 2t$.

5.18. The homogeneous equation $y'' + 3y' + 2y = 0$ has characteristic equation $\lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$ with zeros $\lambda_1 = -1$ and $\lambda_2 = -2$. This leads to the homogeneous solution

$$y_h = C_1e^{-t} + C_2e^{-2t}.$$

The particular solution $y_p = Ae^{-4t}$ has derivatives $y_p' = -4Ae^{-4t}$ and $y_p'' = 16Ae^{-4t}$, which when substituted into the equation $y'' + 3y' + 2y = 3e^{-4t}$ provides

$$16Ae^{-4t} + 3(-4Ae^{-4t}) + 2Ae^{-4t} = 3e^{-4t}$$
$$6A = 3$$
$$A = \frac{1}{2}.$$

Thus, a particular solution is $y_p = (1/2)e^{-4t}$. This leads to the general solution

$$y = C_1e^{-t} + C_2e^{-2t} + \frac{1}{2}e^{-4t}.$$

The initial condition $y(0) = 1$ provides

$$1 = C_1 + C_2 + \frac{1}{2}.$$

Differentiating,

$$y' = -C_1e^{-t} - 2C_2e^{-2t} - 2e^{-4t}.$$

The initial condition $y'(0) = 0$ provides

$$0 = -C_1 - 2C_2 - 2.$$

This system has solution $C_1 = 3$ and $C_2 = -5/2$, leading to the solution

$$y = 3e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-4t}.$$

5.20. The homogeneous equation $y'' + 2y' + 2y = 0$ has characteristic equation $\lambda^2 + 2\lambda + 2 = 0$ with zeros $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. This leads to the homogeneous solution

$$y_h = e^{-t}(C_1 \cos t + C_2 \sin t).$$

The particular solution $z = Ae^{i2t}$ has derivatives

$$z' = (2i)Ae^{i2t}$$
$$z'' = (2i)^2 Ae^{i2t}$$

which, when inserted in the complex equation $z'' + 2z' + 2z = 2e^{i2t}$, gives

$$\left((2i)^2 + 2(2i) + 2\right) Ae^{i2t} = 2e^{i2t}$$
$$A = \frac{2}{-2+4i}$$
$$A = -\frac{1}{5} - \frac{2}{5}i.$$

This gives the particular solution

$$z = \left(-\frac{1}{5} - \frac{2}{5}i\right) e^{i2t}$$
$$z = \left(-\frac{1}{5} - \frac{2}{5}i\right) (\cos 2t + i \sin 2t).$$

The real part of this solution,

$$y_p = -\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t,$$

is a particular solution of $y'' + 2y' + 2y = 2\cos 2t$. Thus, the general solution is

$$y = e^{-t}(C_1 \cos t + C_2 \sin t) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t.$$

The initial condition $y(0) = -2$ gives $-2 = C_1 - 1/5$. Differentiating,

$$y' = e^{-t}(-C_1 \sin t + C_2 \cos t) - e^{-t}(C_1 \cos t + C_2 \sin t) + \frac{2}{5}\sin 2t + \frac{4}{5}\cos 2t.$$

The initial condition $y'(0) = 0$ provides

$$0 = C_2 - C_1 + \frac{4}{5}.$$

This system has solution $C_1 = -9/5$ and $C_2 = -13/5$. Therefore, the solution is

$$y = e^{-t}\left(-\frac{9}{5}\cos t - \frac{13}{5}\sin t\right) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t.$$

5.26. The homogeneous equation $z'' + 4z = 0$ has characteristic equation $\lambda^2 + 4 = 0$ and zeros $\pm 2i$. Thus, the homogeneous solution is

$$z_h = C_1 e^{2it} + C_2 e^{-2it}.$$

The forcing term of $z'' + 4z = 4e^{i2t}$ is also a solution of the homogeneous equation, so multiply by a factor of t and try $z_p = Ate^{i2t}$. The particular solution has derivatives

$$z_p' = Ae^{i2t}(1 + 2it)$$
$$z_p'' = 4Ae^{i2t}(i - t).$$

If these are substituted in $z'' + 4z = 4e^{i2t}$, then

$$4Ae^{i2t}(i - t) + 4Ate^{i2t} = 4e^{i2t}$$
$$(4(i - t) + 4t)\, A = 4$$
$$A = \frac{1}{i} = -i.$$

Hence,

$$
\begin{aligned}
z_p &= -ite^{i2t} \\
&= it(\cos 2t + i \sin 2t) \\
&= t \sin 2t - it \cos 2t.
\end{aligned}
$$

The real part of this solution is a particular solution of $y'' + 4y = 4\cos 2t$.

$$y_p = t \sin 2t$$