1.6. Using Definition 1.1

$$\begin{aligned} F(s) &= \int_0^\infty (\sin 3t)e^{-st}\,dt \\ &= \lim_{T\to\infty}\int_0^T e^{-st}\sin 3t\,dt \end{aligned}$$

Integrating by parts,

$$\int e^{-st}\sin 3t\,dt = -\frac{1}{s}e^{-st}\sin 3t + \frac{3}{s}\int e^{-st}\cos 3t\,dt.$$

Perform another integration by parts on the integral on the right.

$$\begin{aligned} \int e^{-st}\sin 3t\,dt &= -\frac{1}{s}e^{-st}\sin 3t + \frac{3}{s}\left[-\frac{1}{s}e^{-st}\cos 3t - \frac{3}{s}\int e^{-st}\sin 3t\,dt\right] \\ &= -\frac{1}{s}e^{-st}\sin 3t - \frac{3}{s^2}e^{-st}\cos 3t - \frac{9}{s^2}\int e^{-st}\sin 3t\,dt \end{aligned}$$

Transferring the last integral to the left side of the equation,

$$\begin{aligned} \left(1+\frac{9}{s^2}\right)\int e^{-st}\sin 3t\,dt &= -\frac{1}{s}e^{-st}\sin 3t - \frac{3}{s^2}e^{-st}\cos 3t. \\ \int e^{-st}\sin 3t\,dt &= -\frac{se^{-st}}{s^2+9}\sin 3t - \frac{3}{s^2+9}e^{-st}\cos 3t. \end{aligned}$$

Thus,

$$\begin{aligned} F(s) &= \lim_{T\to 0} e^{-st}\left[-\frac{s}{s^2+9}\sin 3T - \frac{3}{s^2+9}\cos 3T\right]_0^T \\ &= \lim_{T\to 0}\left[-\frac{s}{s^2+9}\frac{\sin 3t}{e^{sT}} - \frac{3}{s^2+9}\frac{\cos 3t}{e^{sT}} + \frac{3}{s^2+9}\right] \\ &= \frac{3}{s^2+9}, \end{aligned}$$

provided $s > 0$.

1.10. Using Definition 1.1,

$$F(s) = \int_0^\infty (e^{-3t} \sin 2t)e^{-st}\,dt$$
$$= \lim_{T\to\infty} \int_0^T e^{-(s+3)t} \sin 2t\,dt.$$

Integrating by parts,

$$\int e^{-(s+3)t} \sin 2t\,dt = -\frac{1}{s+3} e^{-(s+3)t} \sin 2t + \frac{2}{s+3} \int e^{-(s+3)t} \cos 2t\,dt.$$

Integrating by parts a second time,

$$\int e^{-(s+3)t} \sin 2t\,dt$$
$$= -\frac{1}{s+3} e^{-(s+3)t} \sin 2t$$
$$+ \frac{2}{s+3} \left[-\frac{1}{s+3} e^{-(s+3)t} \cos 2t - \frac{2}{s+3} \int e^{-(s+3)t} \sin 2t\,dt \right]$$
$$= \frac{-e^{-(s+3)t} \sin 2t}{s+3} - \frac{2e^{-(s+3)t} \cos 2t}{(s+3)^2} - \frac{4}{(s+3)^2} \int e^{-(s+3)t} \sin 2t\,dt.$$

Transferring the last integral on the right to the left side of the equation,

$$\left(1 + \frac{4}{(s+3)^2}\right) \int e^{-(s+3)t} \sin 2t\,dt = \frac{-e^{-(s+3)t} \sin 2t}{s+3} - \frac{2e^{-(s+3)t} \cos 2t}{(s+3)^2}.$$

Therefore,

$$\int e^{-(s+3)t} \sin 2t\,dt = -\frac{s+3}{(s+3)^2+4} \cdot \frac{\sin 2t}{e^{(s+3)t}} - \frac{2}{(s+3)^2+4} \cdot \frac{\cos 2t}{e^{(s+3)t}}.$$

Hence,

$$F(s) = \lim_{T\to\infty} \left[-\frac{s+3}{(s+3)^2+4} \cdot \frac{\sin 2t}{e^{(s+3)t}} - \frac{2}{(s+3)^2+4} \cdot \frac{\cos 2t}{e^{(s+3)t}} \right]_0^T$$
$$= \lim_{T\to\infty} \left[-\frac{s+3}{(s+3)^2+4} \cdot \frac{\sin 2T}{e^{(s+3)T}} - \frac{2}{(s+3)^3+4} \cdot \frac{\cos 2T}{e^{(s+3)T}} + \frac{2}{(s+3)^2+4} \right]$$
$$= \frac{2}{(s+3)^2+4},$$

provided $s > -3$.

1.14. Using Definition 1.1,

$$F(s) = \int_0^\infty (e^{at} \sin \omega t)e^{-st}\, dt$$
$$= \lim_{T\to\infty} \int_0^T e^{-(s-a)t} \sin \omega t\, dt.$$

Integrating by parts,

$$\int e^{-(s-a)t} \sin \omega t\, dt = -\frac{e^{-(s-a)t} \sin \omega t}{s-a} + \frac{\omega}{s-a} \int e^{-(s-a)t} \cos \omega t\, dt.$$

Integrating the last integral on the right by parts,

$$\begin{aligned}\int e^{-(s-a)t} \sin \omega t\, dt &= -\frac{e^{-(s-a)t} \sin \omega t}{s-a} \\ &\quad + \frac{\omega}{s-a}\left[-\frac{e^{-(s-a)t} \cos \omega t}{s-a} - \frac{\omega}{s-a} \int e^{-(s-a)t} \sin \omega t\, dt\right] \\ &= -\frac{e^{-(s-a)t} \sin \omega t}{s-a} - \frac{\omega e^{-(s-a)t} \cos \omega t}{(s-a)^2} \\ &\quad - \frac{\omega^2}{(s-a)^2} \int e^{-(s-a)t} \sin \omega t\, dt.\end{aligned}$$

Transferring the last integral to the left side of the equation,

$$\left(1 + \frac{\omega^2}{(s-a)^2}\right) \int e^{-(s-a)t} \sin \omega t\, dt = -\frac{1}{s-a} \cdot \frac{\sin \omega t}{e^{(s-a)t}} - \frac{\omega}{(s-a)^2} \cdot \frac{\cos \omega t}{e^{(s-a)t}}$$
$$\int e^{-(s-a)t} \sin \omega t\, dt = -\frac{s-a}{(s-a)^2 + \omega^2} \cdot \frac{\sin \omega t}{e^{(s-a)t}} - \frac{\omega}{(s-a)^2 + \omega^2} \cdot \frac{\cos \omega t}{e^{(s-a)t}}.$$

Thus,

$$\begin{aligned}F(s) &= \lim_{T\to\infty}\left[-\frac{s-a}{(s-a)^2 + \omega^2} \cdot \frac{\sin \omega t}{e^{(s-a)t}} - \frac{\omega}{(s-a)^2 + \omega^2} \cdot \frac{\cos \omega t}{e^{(s-a)t}}\right]_0^T \\ &= \lim_{T\to\infty}\left[-\frac{s-a}{(s-a)^2 + \omega^2} \cdot \frac{\sin \omega T}{e^{(s-a)T}} - \frac{\omega}{(s-a)^2 + \omega^2} \cdot \frac{\cos \omega T}{e^{(s-a)T}}\right. \\ &\quad \left. + \frac{\omega}{(s-a)^2 + \omega^2}\right] \\ &= \frac{\omega}{(s-a)^2 + \omega^2},\end{aligned}$$

provided $s > a$.

2.6. Using Proposition 2.7 and Table 1,

$$\begin{aligned}\mathcal{L}(2 \sin 3t + 3 \cos 5t)(s) &= 2\mathcal{L}(\sin 3t)(s) + 3\mathcal{L}(\cos 5t)(s) \\ &= 2 \cdot \frac{3}{s^2 + 3^2} + 3 \cdot \frac{s}{s^2 + 5^2} \\ &= \frac{6}{s^2 + 9} + \frac{3s}{s^2 + 25},\end{aligned}$$

provided $s > 0$.

2.10. Using Proposition 2.7 and Table 1, if $y(t) = e^{2t}$, then

$$\begin{aligned} \mathcal{L}(y')(s) &= \mathcal{L}(2e^{2t})(s) \\ &= 2\mathcal{L}(e^{2t})(s) \\ &= 2 \cdot \frac{1}{s-2} \\ &= \frac{2}{s-2}, \end{aligned}$$

provided $s > 2$. On the other hand,

$$\begin{aligned} s\mathcal{L}(y)(s) - y(0) &= s\mathcal{L}(e^{2t}) - 1 \\ &= s \cdot \frac{1}{s-2} - \frac{s-2}{s-2} \\ &= \frac{2}{s-2}, \end{aligned}$$

provided $s > 2$.

3.4. Adjust as follows.

$$Y(s) = \frac{5s}{s^2+9} = 5 \cdot \frac{s}{s^2+9}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{5 \cdot \frac{s}{s^2+9}\right\} \\ &= 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} \\ &= 5\cos 3t \end{aligned}$$

3.6. Adjust as follows.

$$Y(s) = \frac{2}{3s^4} = \frac{1}{9} \cdot \frac{3!}{s^4}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{9} \cdot \frac{3!}{s^4}\right\} \\ &= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} \\ &= \frac{1}{9}t^3 \end{aligned}$$

3.16. Note the transform pairs.

$$\cos 2t \leftrightarrow \frac{s}{s^2+4}$$
$$\sin 2t \leftrightarrow \frac{2}{s^2+4}$$

By Proposition 2.12,

$$e^{-2t}\cos 2t \leftrightarrow \frac{s+2}{(s+2)^2+4}$$
$$e^{-2t}\sin 2t \leftrightarrow \frac{2}{(s+2)^2+4}.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4} - \frac{2}{(s+2)^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+4}\right\} \\ &= e^{-2t}\cos 2t - e^{-2t}\sin 2t \\ &= e^{-2t}(\cos 2t - \sin 2t). \end{aligned}$$

3.26. Find a partial fraction decomposition.

$$\frac{4s+15}{2s^2+3s} = \frac{A}{s} + \frac{B}{2s+3}$$
$$4s+15 = A(2s+3) + Bs$$

Now,

$$s = -3/2 \Rightarrow 9 = -3/2B \text{ or } B = -6$$
$$s = 0 \Rightarrow 15 = 3A \text{ or } A = 5.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{5}{s} - \frac{6}{2s+3}\right\} \\ &= 5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{6}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3/2}\right\} \\ &= 5 - 3e^{-(3/2)t}. \end{aligned}$$

3.30. Find a partial fraction decomposition.

$$\frac{7s^2+20s+53}{(s-1)(s^2+2s+5)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+2s+5}$$
$$7s^2+20s+53=A(s^2+2s+5)+(Bs+C)(s-1)$$
$$7s^2+20s+53=(A+B)s^2+(2A-B+C)s+(5A-C)$$

Thus,

$$\begin{aligned} A+B&=7\\ 2A-B+C&=20\\ 5A-C&=53, \end{aligned}$$

and $A=10$, $B=-3$, and $C=-3$. Thus,

$$\begin{aligned} y(t)&=\mathcal{L}^{-1}\left\{\frac{10}{s-1}+\frac{-3s-3}{s^2+2s+5}\right\}\\ &=10\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}-3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} \end{aligned}$$

Note the transform pair.

$$\cos 2t\leftrightarrow\frac{s}{s^2+4}$$

By Proposition 2.12,

$$e^{-t}\cos 2t\leftrightarrow\frac{s+1}{(s+1)^2+4}.$$

Thus,

$$y(t)=10e^{t}-3e^{-t}\cos 2t$$

4.2. Set $Y=\mathcal{L}(y)$. Then,

$$\begin{aligned} \mathcal{L}(y'+9y)&=\mathcal{L}(e^{-t})\\ \mathcal{L}(y')+9\mathcal{L}(y)&=\frac{1}{s+1}\\ s\cdot Y(s)-y(0)+9Y(s)&=\frac{1}{s+1}. \end{aligned}$$

But $y(0)=0$, so

$$\begin{aligned} (s+9)Y(s)&=\frac{1}{s+1}\\ Y(s)&=\frac{1}{(s+1)(s+9)} \end{aligned}$$

Find a partial fraction decomposition.

$$\frac{1}{(s+1)(s+9)}=\frac{A}{s+1}+\frac{B}{s+9}$$
$$1=A(s+9)+B(s+1)$$

Now,

$$s=-9\Rightarrow 1=-8B\text{ or }B=-\frac{1}{8}$$
$$s=-1\Rightarrow 1=8A\text{ or }A=\frac{1}{8}.$$

Thus,

$$Y(s)=\frac{1/8}{s+1}-\frac{1/8}{s+9},$$

and

$$\begin{aligned} y(t)&=\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+9}\right\}\\ y(t)&=\frac{1}{8}e^{-t}-\frac{1}{8}e^{-9t}. \end{aligned}$$

4.6. Let $\mathcal{L}(y) = Y(s)$. Then,

$$\begin{aligned} \mathcal{L}(y' + 8y) &= \mathcal{L}(t^2) \\ \mathcal{L}(y') + 8\mathcal{L}(y) &= \frac{2!}{s^3} \\ s \cdot Y(s) - y(0) + 8Y(s) &= \frac{2}{s^3}. \end{aligned}$$

But, $y(0) = -1$.

$$\begin{aligned} (s+8)Y(s) + 1 &= \frac{2}{s^3} \\ Y(s) &= \frac{2}{s^3(s+8)} - \frac{1}{s+8} \end{aligned}$$

Find a partial fraction decomposition.

$$\begin{aligned} \frac{2}{s^3(s+8)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+8} \\ 2 &= As^2(s+8) + Bs(s+8) + C(s+8) + Ds^3 \\ 2 &= (A+D)s^3 + (8A+B)s^2 + (8B+C)s + 8C \end{aligned}$$

Then,

$$\begin{aligned} A + D &= 0 \\ 8A + B &= 0 \\ 8B + C &= 0 \\ 8C &= 2, \end{aligned}$$

and $A = 1/256$, $B = -1/32$, and $C = 1/4$, and $D = -1/256$. Thus,

$$\begin{aligned} Y(s) &= \frac{-1}{s+8} + \frac{1/256}{s} - \frac{1/32}{s^2} + \frac{1/4}{s^3} - \frac{1/256}{s+8} \\ Y(s) &= \frac{1}{256}\left(\frac{1}{s}\right) - \frac{1}{32}\left(\frac{1}{s^2}\right) + \frac{1}{4} \cdot \frac{1}{2}\left(\frac{2!}{s^3}\right) + \left(\frac{-257}{256}\right)\left(\frac{1}{s+8}\right). \end{aligned}$$

Therefore,

$$y(t) = \frac{1}{256} - \frac{1}{32}t + \frac{1}{8}t^2 - \frac{257}{256}e^{-8t}.$$

4.10. Let $\mathcal{L}(y) = Y(s)$. Then,

$$\begin{aligned}\mathcal{L}(y' - 4y) &= \mathcal{L}(e^{-2t}t^2)\\ \mathcal{L}(y') - 4\mathcal{L}(y) &= \frac{2!}{(s+2)^3}\\ s \cdot Y(s) - y(0) - 4Y(s) &= \frac{2}{(s+2)^3}.\end{aligned}$$

But $y(0) = 1$, so

$$\begin{aligned}(s-4)Y(s) - 1 &= \frac{2}{(s+2)^3}\\ Y(s) &= \frac{1}{s-4} + \frac{2}{(s-4)(s+2)^3}.\end{aligned}$$

Find a partial fraction decomposition.

$$\begin{aligned}\frac{2}{(s-4)(s+2)^3} &= \frac{A}{s-4} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}\\ 2 &= A(s+2)^3 + B(s-4)(s+2)^2 + C(s-4)(s+2) + D(s-4)\\ 2 &= (A+B)s^3 + (6A+C)s^2 + (12A - 12B - 2C + D)s\\ &\quad + (8A - 16B - 8C - 4D)\end{aligned}$$

Thus,

$$\begin{aligned}A + B &= 0\\ 6A + C &= 0\\ 12A - 12B - 2C + D &= 0\\ 8A - 16B - 8C - 4D &= 2,\end{aligned}$$

and $A = 1/108$, $B = -1/108$, $C = -1/18$, and $D = -1/3$. Thus,

$$\begin{aligned}Y(s) &= \frac{1}{s-4} + \frac{1/108}{s-4} - \frac{1/108}{s+2} - \frac{1/18}{(s+2)^2} - \frac{1/3}{(s+2)^3}\\ &= \frac{109}{108}\left(\frac{1}{s-4}\right) - \frac{1}{108}\left(\frac{1}{s+2}\right) - \frac{1}{18}\left(\frac{1}{(s+2)^2}\right) - \frac{1}{3}\cdot\frac{1}{2}\left(\frac{2!}{(s+2)^2}\right)\end{aligned}$$

Therefore,

$$y(t) = \frac{109}{108}e^{4t} - \frac{1}{108}e^{-2t} - \frac{1}{18}te^{-2t} - \frac{1}{6}t^2e^{-2t}.$$

4.20. Let $\mathcal{L}(y) = Y(s)$. Then,

$$\begin{aligned}\mathcal{L}(y'' - 2y' - 3y) &= \mathcal{L}(e^{4t})\\ \mathcal{L}(y'') - 2\mathcal{L}(y') - 3\mathcal{L}(y) &= \frac{1}{s-4}\\ s^2Y(s) - s\cdot y(0) - y'(0) - 2(s\cdot Y(s) - y(0)) - 3Y(s) &= \frac{1}{s-4}\end{aligned}$$

But, $y(0) = 1$ and $y'(0) = -1$.

$$\begin{aligned}(s^2 - 2s - 3)Y(s) - s + 1 + 2 &= \frac{1}{s-4}\\ Y(s) &= \frac{s-3}{(s+1)(s-3)} + \frac{1}{(s-4)(s+1)(s-3)}\end{aligned}$$

The terms on the right have decompositions

$$\begin{aligned}\frac{s-3}{(s+1)(s-3)} &= \frac{1}{s+1}\\ \frac{1}{(s-4)(s+1)(s-3)} &= \frac{1/5}{s-4} + \frac{1/20}{s+1} + \frac{-1/4}{s-3}\\ Y(s) &= \frac{21/20}{s+1} - \frac{1/4}{s-3} + \frac{1/5}{s-4}\end{aligned}$$

and therefore

$$y(t) = \frac{21}{20}e^{-t} - \frac{1}{4}e^{3t} + \frac{1}{5}e^{4t}.$$

4.27. (a) Let $y = e^{rt}$. Then

$$y'' - 4y' + 3y = 0$$
$$(r^2 - 4r + 3)e^{rt} = 0$$
$$r^2 - 4r + 3 = 0$$
$$(r - 3)(r - 1) = 0$$

Thus, $r = 3$ and $r = 1$ lead to independent solutions $y = e^{3t}$ and $y = e^{t}$. The general solution is

$$y = C_1e^{3t} + C_2e^{t}.$$

The initial condition $y(0) = 1$ gives us

$$1 = C_1 + C_2.$$

Differentiating,

$$y' = 3C_1e^{3t} + C_2e^{t}.$$

The initial condition $y'(0) = -1$ gives

$$-1 = 3C_1 + C_2.$$

Therefore, $C_1 = -1$ and $C_2 = 2$ and the solution is $y = -e^{3t} + 2e^{t}$.

(b) Let $\mathcal{L}(y) = Y$. Then

$$\mathcal{L}(y'' - 4y' + 3y) = \mathcal{L}(0)$$
$$s^2Y(s) - s \cdot y(0) - y'(0) - 4(s \cdot Y(s) - y(0)) + 3Y(s) = 0.$$

The initial conditions $y(0) = 1$ and $y'(0) = -1$ give

$$(s^2 - 4s + 3)Y(s) - s + 1 + 4 = 0$$
$$Y(s) = \frac{s - 5}{s^2 - 4s + 3}.$$

This has decomposition

$$Y(s) = \frac{2}{s - 1} - \frac{1}{s - 3}.$$

Thus, $y(t) = 2e^{t} - e^{3t}$.

5.2. Since $f(t) = e^{2t}$ has transform

$$F(s) = 1/(s - 2),$$

$g(t) = H(t - 1)e^{2(t-1)}$ has transform

$$G(s) = e^{-s}F(s)$$
$$= e^{-s}\frac{1}{s - 2}$$
$$= \frac{e^{-s}}{s - 2}.$$

5.6. Since $-t = -(t - 2) - 2$,

$$H(t - 2)e^{-t} = H(t - 2)e^{-(t-2)-2} = H(t - 2)e^{-(t-2)}e^{-2}.$$

Thus $f(t) = e^{-2}e^{-t}$ has transform

$$F(s) = \frac{e^{-2}}{s + 1}$$

so $g(t) = H(t - 2)e^{-(t-2)}e^{-2}$ has transform

$$G(s) = e^{-2s}F(s)$$
$$= e^{-2s}\frac{e^{-2}}{s + 1}$$
$$= \frac{e^{-2(s+1)}}{s + 1}.$$

6.4. By Theorem 6.10, the unit impulse response of

$$y'' + 4y = \delta(t)$$

is $E(s) = 1/P(s)$, where $P(s) = s^2 + 4s$ is the characteristic polynomial. Thus,

$$E(s) = \frac{1}{s^2 + 4s}.$$

6.8. (a)

$$\mathcal{L}\{\delta(t)\}(s) = 1$$

$$x'' = 2\delta(t)$$

$$\mathcal{L}\{x''\} = 2\mathcal{L}\{\delta(t)\}$$

$$s^2X(s) - sx(0) - x'(0) = 2$$

But $x(0) = x'(0) = 0$, so

$$s^2X(s) = 2$$

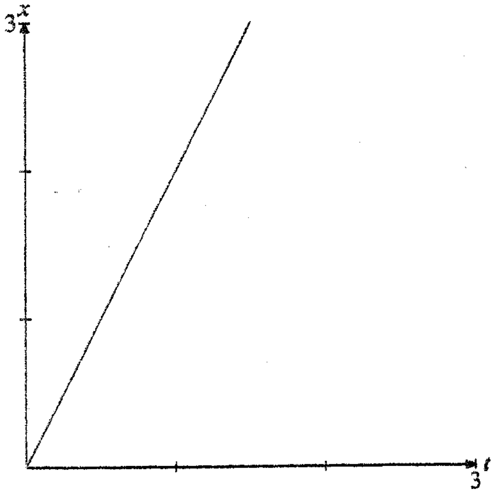
$$X(s) = \frac{2}{s^2}.$$

Thus,

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\},$$

and

$$x(t) = 2tH(t).$$



(b) From Exercise 1

$$\mathcal{L}\{\delta_p^\epsilon(t)\} = e^{-sp}\frac{1-e^{-s\epsilon}}{s\epsilon},$$

so,

$$\mathcal{L}\{\delta_0^\epsilon(t)\} = \frac{1-e^{-s\epsilon}}{s\epsilon}.$$

Thus,

$$\begin{aligned} x'' &= 2\delta_0^\epsilon(t) \\ \mathcal{L}\{x''\} &= 2\mathcal{L}\{\delta_0^\epsilon(t)\} \\ s^2X(s) - sx(0) - x'(0) &= \frac{2(1-e^{-s\epsilon})}{s\epsilon} \end{aligned}$$

But $x(0) = x'(0) = 0$.

$$\begin{aligned} s^2X(s) &= \frac{2(1-e^{-s\epsilon})}{s\epsilon} \\ X(s) &= \frac{2(1-e^{-s\epsilon})}{s^3\epsilon} \\ X(s) &= \frac{2}{s^3\epsilon} - \frac{2e^{-s\epsilon}}{s^3\epsilon}. \end{aligned}$$

Now, work backwards from the solution $x(t)$ and confirm it is consistent with $X(s)$.

$$x_\epsilon(t) = \begin{cases} t^2/\epsilon, & \text{if } 0 \le t < \epsilon, \\ 2t - \epsilon, & \text{if } t \ge \epsilon \end{cases}$$

Thus,

$$x_\epsilon(t) = \frac{1}{\epsilon}t^2 H_{0\epsilon} + (2t-\epsilon)H_\epsilon$$

$$x_\epsilon(t) = \frac{1}{\epsilon}t^2(H(t) - H(t-\epsilon)) + (2t-\epsilon)H(t-\epsilon)$$

$$x_\epsilon(t) = \frac{1}{\epsilon}t^2 H(t) - \frac{1}{\epsilon}t^2 H(t-\epsilon) + (2t-\epsilon)H(t-\epsilon)$$

$$x_\epsilon(t) = \frac{1}{\epsilon}t^2 H(t) - \frac{1}{\epsilon}[(t-\epsilon)^2 + 2\epsilon(t-\epsilon) + \epsilon^2]H(t-\epsilon) + (2(t-\epsilon)+\epsilon)H(t-\epsilon).$$

Therefore,

$$X(s) = \frac{2}{\epsilon s^3} - \frac{1}{\epsilon}\cdot e^{-\epsilon s}\left[\frac{2}{s^3} + \frac{2\epsilon}{s^2} + \frac{\epsilon^2}{s}\right] + e^{-\epsilon s}\left(\frac{2}{s^2} + \frac{\epsilon}{s}\right)$$

$$X(s) = \frac{2}{\epsilon s^3} - \frac{2e^{-\epsilon s}}{\epsilon s^3}$$

(c) As long as $t \geq 0$, the solution in part (b) will approach the solution in part (a).

6.9. (a) If

$$x'' + 2x' + 2x = \delta(t), \qquad x(0) = x'(0) = 0,$$

then, by Theorem 6.10, the unit impulse response is

$$X(s) = \frac{1}{P(s)} = \frac{1}{s^2 + 2s + 2}.$$

Completing the square,

$$X(s) = \frac{1}{(s+1)^2 + 1}.$$

Note the transform pair

$$\sin t \Longleftrightarrow \frac{1}{s^2+1}.$$

Thus, by Proposition 2.12,

$$x(t) = e^{-t}\sin t.$$

(b) The Laplace transform of the right-hand side, δ_0^ϵ, is

$$\begin{aligned}
\mathcal{L}\{\delta_0^\epsilon(t)\}(s) &= \mathcal{L}\left\{\frac{1}{\epsilon}(H(t) - H(t-\epsilon))\right\}(s) \\
&= \frac{1}{\epsilon}[\mathcal{L}\{H(t)\}(s) - \mathcal{L}\{H(t-\epsilon)\}(s)] \\
&= \frac{1}{\epsilon}\left[\frac{1}{s} - \frac{e^{-\epsilon s}}{s}\right] \\
&= \frac{1-e^{\epsilon s}}{\epsilon s}.
\end{aligned}$$

Thus, if

$$x'' + 2x' + 2x = \delta_0^\epsilon(t), \qquad x(0) = x'(0) = 0,$$

then

$$\begin{aligned}
\mathcal{L}(x'' + 2x' + 2x)(s) &= \mathcal{L}\{\delta_0^\epsilon(t)\}(s) \\
(s^2 + 2s + 2)X(s) &= \frac{1-e^{-\epsilon s}}{\epsilon s} \\
X(s) &= \frac{1-e^{-\epsilon s}}{\epsilon s(s^2+2s+2)}.
\end{aligned}$$

A partial fraction decomposition gives

$$\begin{aligned}\frac{1}{s(s^2+2s+2)} &= \frac{1/2}{s} + \frac{-(1/2)s-1}{s^2+2s+2}\\ &= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s+2}{(s+1)^2+1}\\ &= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\left[\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right].\end{aligned}$$

Thus,

$$\begin{aligned}x_\epsilon(t) &= \frac{1}{\epsilon}\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}(t) - \frac{1}{\epsilon}\mathcal{L}^{-1}\left\{e^{-\epsilon s}\cdot\frac{1}{s(s^2+2s+2)}\right\}(t)\\ &= \frac{1}{\epsilon}\left[\frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t)\right]\\ &\quad - \frac{1}{\epsilon}H(t-\epsilon)\left[\frac{1}{2} - \frac{1}{2}e^{-(t-\epsilon)}(\cos(t-\epsilon) + \sin(t-\epsilon))\right].\end{aligned}$$

Thus,

$$x_\epsilon(t) = \frac{1}{\epsilon}\begin{cases}\frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t), & \text{if } 0 \le t < \epsilon,\\ -\frac{1}{2}e^{-t}(\cos t + \sin t) & \\ \qquad + \frac{1}{2}e^{-(t-\epsilon)}(\cos(t-\epsilon) + \sin(t-\epsilon)), & \text{if } t \ge \epsilon.\end{cases}$$

(c) As $\epsilon \to 0$, the first interval $(0 \le t < \epsilon)$ in the piecewise definition in part (b) melts away. Let's concentrate on the second piece. By l'Hôpital's rule,

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{-\frac{1}{2}e^{-t}(\cos t + \sin t) + \frac{1}{2}e^{-(t-\epsilon)}(\cos(t-\epsilon) + \sin(t-\epsilon))}{\epsilon}\\ &\quad = \lim_{\epsilon\to 0}\frac{e^{-(t-\epsilon)}\sin(t-\epsilon)}{1}\\ &\quad = e^{-t}\sin t.\end{aligned}$$

7.6. If $f(t) = t - 1$ and $g(t) = t - 2$, then

$$\begin{aligned}f * g(t) &= \int_0^t f(u)g(t-u)\,du\\ &= \int_0^t (u-1)(t-u-2)\,du\\ &= \int_0^t (ut - u^2 - u - t + 2)\,du\\ &= \frac{u^2 t}{2} - \frac{u^3}{3} - \frac{u^2}{2} - tu + 2u\Big|_0^t\\ &= \frac{t^3}{2} - \frac{t^3}{3} - \frac{t^2}{2} - t^2 + 2t\\ &= \frac{t^3}{6} - \frac{3t^2}{2} + 2t\end{aligned}$$

7.12. If $f(t) = \cos t$ and $g(t) = t^2$, then

$$f * g(t) = \int_0^t f(u)g(t-u)\,du = \int_0^t (\cos u)(t-u)^2\,du.$$

Integrating by parts,

$$\int (\cos u)(t-u)^2\,du = (\sin u)(t-u)^2 + 2\int (\sin u)(t-u)\,du.$$

Again,

$$\begin{aligned} &= (\sin u)(t-u)^2 + 2\left[(-\cos u)(t-u) - \int \cos u\,du\right], \\ &= (\sin u)(t-u)^2 - 2(\cos u)(t-u) - 2\sin u. \end{aligned}$$

Thus,

$$\begin{aligned} f * g(t) &= \left[(\sin u)(t-u)^2 - 2(\cos u)(t-u) - 2\sin u\right]_0^t \\ &= -2\sin t + 2t. \end{aligned}$$

However,

$$\begin{aligned} \mathcal{L}\{f * g(t)\}(s) &= \mathcal{L}\{-2\sin t + 2t\}(s), \\ &= \mathcal{L}\{-2\sin t\}(s) + \mathcal{L}\{2t\}(s), \\ &= \frac{-2}{s^2+1} + \frac{2}{s^2}, \\ &= \frac{2}{s^2(s^2+1)}. \end{aligned}$$

Alternatively, we have the transform pairs

$$\begin{aligned} f(t) = \cos t &\Longleftrightarrow F(s) = \frac{s}{s^2+1}, \\ g(t) = t^2 &\Longleftrightarrow G(s) = \frac{2}{s^3}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f * g(t)\}(s) &= F(s)G(s), \\ &= \left(\frac{s}{s^2+1}\right)\left(\frac{2}{s^3}\right), \\ &= \frac{2}{s^2(s^2+1)}. \end{aligned}$$

7.18. Note the expression

$$\frac{1}{s^2-3s} = \frac{1}{s(s-3)} = \frac{1}{s}\cdot\frac{1}{s-3},$$
$$= F(s)G(s).$$

We have the transform pairs

$$F(s) = \frac{1}{s} \Longleftrightarrow f(t) = 1,$$
$$G(s) = \frac{1}{s-3} \Longleftrightarrow g(t) = e^{3t}.$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2-3s}\right\}(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t),\\
&= f * g(t),\\
&= \int_0^t f(u)g(t-u)\,du,\\
&= \int_0^t e^{3(t-u)}\,du,\\
&= -\frac{1}{3}e^{3(t-u)}\Big|_0^t,\\
&= -\frac{1}{3}+\frac{1}{3}e^{3t},\\
&= -\frac{1}{3}\left(1-e^{3t}\right).
\end{aligned}$$

7.22. The expression

$$\frac{1}{(s+1)(s^2+4)} = \frac{1}{s+1}\cdot\frac{1}{s^2+4},$$
$$= F(s)G(s).$$

We have transform pairs

$$F(s) = \frac{1}{s+1} \Longleftrightarrow f(t) = e^{-t},$$
$$G(s) = \frac{1}{2}\cdot\frac{2}{s^2+4} \Longleftrightarrow g(t) = \frac{1}{2}\sin 2t.$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\} &= \mathcal{L}^{-1}\{F(s)G(s)\}(t),\\
&= f * g(t),\\
&= \int_0^t f(u)g(t-u)\,du,\\
&= \frac{1}{2}\int_0^t e^{-u}\sin 2(t-u)\,du.
\end{aligned}$$

Integration gives

$$\int e^{-u}\sin(2t-2u)\,du =$$
$$-e^{-u}\sin(2t-2u) + 2e^{-u}\cos(2t-2u) - 4\int e^{-u}\sin(2t-2u)\,du.$$

Thus,

$$\int e^{-u}\sin(2t-u)\,du = -\frac{1}{5}e^{-u}\sin(2t-2u) + \frac{2}{5}e^{-u}\cos(2t-2u),$$

and

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\}(t) &= \frac{1}{2}\left[-\frac{1}{5}e^{-u}\sin(2t-2u) + \frac{2}{5}e^{-u}\cos(2t-2u)\right]_0^t,\\
&= \frac{1}{5}e^{-t} + \frac{1}{10}\sin 2t - \frac{1}{5}\cos 2t.
\end{aligned}$$