MATH 53 HOMEWORK 5 SOLUTIONS

Section 7.1

1.30.

$$-2\mathbf{v}_2 + 4\mathbf{v}_3 + 5\mathbf{x}_4 = -2\begin{pmatrix}0\\8\\6\end{pmatrix} + 4\begin{pmatrix}0\\-9\\7\end{pmatrix} + 5\begin{pmatrix}7\\-9\end{pmatrix}.$$

The dimensions do not allow these vectors to be added.

1.32.

$$\begin{aligned}\mathbf{v}_2 - 5\mathbf{v}_1 - 3\mathbf{v}_4 - 2\mathbf{v}_3 &= \begin{pmatrix}0\\8\\6\end{pmatrix} - 5\begin{pmatrix}10\\-5\\3\end{pmatrix} - 3\begin{pmatrix}-1\\3\\6\end{pmatrix} - 2\begin{pmatrix}0\\-9\\7\end{pmatrix}\\ &= \begin{pmatrix}0\\8\\6\end{pmatrix} + \begin{pmatrix}-50\\25\\-15\end{pmatrix} + \begin{pmatrix}3\\-9\\-18\end{pmatrix} + \begin{pmatrix}0\\18\\-14\end{pmatrix}\\ &= \begin{pmatrix}-47\\42\\-41\end{pmatrix}\end{aligned}$$

1.50.

$$\begin{aligned}x - 3y &= 5\\ -2x + 3y &= -2\end{aligned}$$

is equivalent to

$$\begin{pmatrix}1 & -3\\ -2 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\-2\end{pmatrix}.$$

Section 7.2

2.1. No. There are only three possible solution sets: a point, a line, or no solutions. A circle is none of these.

2.2. The set S consists of a circle of radius 2, centered at x_0. The set also includes the center at x_0. Again, this set is not a point or a line, so it cannot be the solution set of a system of linear equations in two unknowns.

2.4. Assuming that s is some fixed number, then $S = \left\{\begin{pmatrix}t\\s\end{pmatrix} : t > 0\right\}$ is a half line in the plane. This set is not a line or point. It is not infinitely long as we cannot move infinitely in two directions as we can on a line.

2.12. Solve for x_1,

$$x_1 = 2 - 2x_2 + 2x_3 - x_4.$$

Thus, solutions have the form

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}2 - 2x_2 + 2x_3 - x_4\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}2\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} + x_3\begin{pmatrix}2\\0\\1\\0\end{pmatrix} + x_4\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}.$$

The solution set has dimension 3 ($(2, 0, 0, 0)^T$ is just a translation).

2.14. Solve the third equation for x_3

$$x_3 = x_4.$$

Substitute this in the second equation and solve for x_2

$$\begin{aligned}x_2 - 3x_4 - x_4 &= 3\\ x_2 &= 3 + 4x_4.\end{aligned}$$

Substitute both of these in the first equation and solve for x_1

$$\begin{aligned}x_1 + 2(3 + 4x_4) - 2x_4 + x_4 &= 2\\ x_1 + 6 + 7x_4 &= 2\\ x_1 &= -4 - 7x_4.\end{aligned}$$

Thus, solutions have the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4-7x_4 \\ 3+4x_4 \\ x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -7 \\ 4 \\ 1 \\ 1 \end{pmatrix}.$$

This solution has dimension 1. Think of the set as a translated line in $\mathbf{R}^4$.

Section 7.3

3.12. Set up the augmented matrix.

$$\begin{pmatrix} -2 & 4 & 7 \\ 4 & -8 & 10 \end{pmatrix}$$

Multiply row 1 by 2 and add to row 2.

$$\begin{pmatrix} -2 & 4 & 7 \\ 0 & 0 & 24 \end{pmatrix}$$

The last row represents the equation $0x + 0y = 24$, which has no solutions. Therefore, the system has no solutions.

3.16. Set up the augmented matrix.

$$\begin{pmatrix} -3 & -6 & -3 & -3 \\ -6 & 4 & 1 & -8 \end{pmatrix}$$

Multiply row 1 by -2 and add to row 2.

$$\begin{pmatrix} -3 & -6 & -3 & -3 \\ 0 & 16 & 7 & -2 \end{pmatrix}$$

This gives

$$\begin{aligned} -3x - 6y - 3z &= -3, \\ 16y + 7z &= -2. \end{aligned}$$

Thus, x and y are pivot variables and z is free. Solve the second equation for y.

$$\begin{aligned} 16y &= -2 + 7z \\ y &= -\frac{1}{8} + \frac{7}{16}z \end{aligned}$$

Substitute this result in the first equation and solve for x.

$$\begin{aligned} -3x - 6\left(-\frac{1}{8} + \frac{7}{16}z\right) - 3z &= -3 \\ -3x + \frac{3}{4} - \frac{21}{8}z - \frac{24}{8}z &= -\frac{12}{4} \\ -3x &= -\frac{15}{4} + \frac{45}{8}z \\ x &= \frac{5}{4} - \frac{15}{8}z \end{aligned}$$

Thus,

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5/4 - 15/8z \\ -1/8 + 7/16z \\ z \end{pmatrix} = \begin{pmatrix} 5/4 \\ -1/8 \\ 0 \end{pmatrix} + z \begin{pmatrix} -15/8 \\ 7/16 \\ 1 \end{pmatrix},$$

where z is free.

3.20. Set up the augmented matrix and reduce.

$$\begin{pmatrix} -4 & 10 & -6 & -14 \\ 0 & -4 & 4 & 4 \\ 2 & -10 & 8 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This gives

$$\begin{aligned} x - z &= 1, \\ y - z &= -1. \end{aligned}$$

Thus, x and y are pivot variables and z is free. Solve each equation for its pivot variable.

$$x = 1 + z$$
$$y = -1 + z$$

Thus,

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+z \\ -1+z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where z is free.

Section 7.4

4.4 Inconsistent

4.10. The augmented matrix reduces to

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

4.16. The augmented matrix $[A\ I]$ reduces to

$$\begin{pmatrix} 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Matrix A is singular and has no inverse.

4.20. The augmented matrix $[A\ I]$ reduces to

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, matrix A is nonsingular and

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.24. The system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3$$

has more unknowns than equations. Therefore, there must be free variables. The system is either inconsistent or has an infinite solution set.

4.30. The coefficient matrix has row echelon form

$$\begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The presence of a zero on the diagonal of the row echelon form indicates that the coefficient matrix in singular. The system does not have a unique solution.

4.34. The system

$$x + 2y + 3x = 6$$
$$x + 2y + 3x = 10$$

represents two parallel, distinct planes. The system has no solutions.

Section 7.5

5.4. The nullspace consists of solutions of

$$\begin{pmatrix} 4 & 4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Set up the augmented matrix and reduce.

$$\begin{pmatrix} 4 & 4 & 0 \\ -2 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus, $x = 0$ and $y = 0$ and the nullspace consists of a single zero vector.

5.6. The nullspace consists of solutions of

$$\begin{pmatrix} -3 & 8 & -11 \\ -4 & 10 & -14 \\ -2 & 5 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We need only reduce the coefficient matrix if we mentally augment a zero vector in the fourth column.

$$\begin{pmatrix} -3 & 8 & -11 \\ -4 & 10 & -14 \\ -2 & 5 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, $x = y = z = 0$ and the nullspace consists of only the zero vector.

5.10. It is easy to see that $(-2, 3)^T$ is not a scalar multiple of $(2, -6)^T$. Thus, the vectors are independent. More formally, if

$$c_1\begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then the augmented matrix reduces as follows.

$$\begin{pmatrix} -2 & 2 & 0 \\ 3 & -6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus, $c_1 = c_2 = 0$ and the vectors are independent.

5.12. The vector $\mathbf{v}_1 = (-8, 9, -6)^T$ is not a scalar multiple of the vector $\mathbf{v}_2 = (-2, 0, 7)^T$. They are independent.

5.18. Place the vectors in a matrix and reduce.

$$\begin{pmatrix} -2 & 2 \\ 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Keep the columns that have nonzero pivots in the reduced matrix. Thus, a basis for span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is

$$B = \left\{ \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}.$$

The dimension of the span is 2.

5.22. Place the vectors in a matrix and reduce.

$$\begin{pmatrix} -8 & -2 & 8 \\ 9 & 0 & -18 \\ -6 & 7 & 40 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Keep the vectors that have nonzero pivots in the reduced matrix. Thus, a basis for span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is

$$B = \left\{ \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix} \right\}.$$

The dimension is 2.

5.28. The nullspace contains solutions of

$$\begin{pmatrix} 4 & 4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The reduced form of the coefficient matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the nullspace contains only the zero vector and a basis is

$$B = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

5.34. The equation

$$-3x + 5y = 2$$

has particular solution $\mathbf{x}_p = (-2/3, 0)^T$. We found a basis for the nullspace in Exercise 26.

$$B = \left\{ \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} \right\}$$

Thus all solutions of $-3x + 5y = 2$ are given by

$$\mathbf{x} = \begin{pmatrix} -2/3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5/3 \\ 1 \end{pmatrix},$$

where $t \in \mathbf{R}$.

Section 7.6

6.8. Add 4/5 times row 1 to row 2; add $-4/5$ times row 1 to row 3.

$$\begin{vmatrix} 5 & 6 & 4 \\ -4 & -9 & -8 \\ 4 & 6 & 5 \end{vmatrix} = \begin{vmatrix} 5 & 6 & 4 \\ 0 & -21/5 & -24/5 \\ 0 & 6/5 & 9/5 \end{vmatrix}$$

Add 2/7 times row 2 to row 3.

$$\begin{aligned} &= \begin{vmatrix} 5 & 6 & 4 \\ 0 & -21/5 & -24/5 \\ 0 & 0 & 3/7 \end{vmatrix} \\ &= 5\left(-\frac{21}{5}\right)\left(\frac{3}{7}\right) \\ &= -9 \end{aligned}$$

6.14. (a) Suppose matrix $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$ is $n \times n$, where $\mathbf{a}_j$ denotes the jth column. Assume that column j is a linear combination of the columns that proceed it,

$$\mathbf{a}_j = c_1\mathbf{a}_1 + \cdots + c_{j-1}\mathbf{a}_{j-1}.$$

Thus,

$$\mathbf{a}_j - c_1\mathbf{a}_1 - \cdots - c_{j-1}\mathbf{a}_{j-1} = \mathbf{0}.$$

Thus, if we perform the following

(1) Add $-c_1$ times column 1 and add to column j.
(2) Add $-c_2$ times column 1 and add to column j.

$\vdots$

($j-1$) Add $-c_{j-1}$ times column 1 to column j.

Then,

$$\begin{aligned} &|\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_j, \dots, \mathbf{a}_n|, \\ &= |\mathbf{a}_1, \mathbf{a}_2, \dots, (\mathbf{a}_j - c_1\mathbf{a}_1 - \cdots - c_{j-1}\mathbf{a}_{j-1}), \dots, \mathbf{a}_n|, \\ &= |\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{0}, \dots, \mathbf{a}_n|, \\ &= 0, \end{aligned}$$

because there is a zero column.

If the jth row of A is a linear combination of its preceding rows, then the jth column of A^T is a linear combination of its preceding columns and has zero determinant. Thus,

$$|A| = |A^T| = 0.$$

(b) The first matrix has zero determinant because its third row is the sum of its first two rows. The second matrix has zero determinant because its third column is the sum of its first two columns. Matrix three has zero determinant because its third row equals twice its first row added to its second row. The fourth matrix has zero determinant because its third column is the sum of twice its first column and three times its second column.

6.28. To take advantage of the zero we could expand by the first row or the second column. Let's choose the second column.

$$\det A = -1 \cdot \det \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = 1 - 1 = 0.$$

Since $\det A = 0$, null(A) is nontrivial, and the column vectors of A are linearly dependent. null(A) is the subspace with basis $(1, 1, -1)^T$.

6.34. To have a nontrivial nullspace, the determinant must equal zero.

$$0 = \begin{vmatrix} 2-x & 1 \\ 0 & -1-x \end{vmatrix}$$

$$0 = (2-x)(-1-x)$$

$$x = 2, 1$$