

**Math53: Ordinary Differential Equations
Winter 2004**

Midterm II

February 25, 2004

Name: _____

I acknowledge the Honor Code. _____
sign here

Instructions

- (1) This exam is closed-book and closed-notes. No calculators.
- (2) You have 50 minutes to complete this exam.
- (3) Please write legibly. Circle or box your final answers.
- (4) Show your work. Correct answers will receive only partial credit.
- (5) Simplify your answers as much as possible.
- (6) If you continue your answer on the back of a page, indicate so on the front of the page.
- (7) If you are asked to sketch anything, your plot needs to show only the relevant qualitative behavior.
- (8) You may use the tables of Laplace Transforms provided on the back of this page. However, if you do use them, please state so in your solution.

$f(t)$	$F(s) = \{\mathcal{L}f\}(s)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
δ	1

$f(t)$	$F(s) = \{\mathcal{L}f\}(s)$
f'	$s \cdot F(s) - f(0)$
$t \cdot f(t)$	$-F'(s)$
$e^{at} f(t)$	$F(s-a)$
$H(t-a)f(t-a)$	$e^{-as}F(s)$

Laplace Transforms

1 (25pts)	
2 (15pts)	
3 (30pts)	
4 (30pts)	
Total (100pts)	

Problem 1 (25pts)

(a; 5pts) Show that if $Y=Y(s)$ is the Laplace Transform of the function $y=y(t)$, then the Laplace Transforms of y'' and of y'''' are given by

$$\{\mathcal{L}y''\}(s) = s^2Y(s) - sy(0) - y'(0), \quad \{\mathcal{L}y''''\}(s) = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0).$$

(b; 5pts) Show that if $y=y(t)$ is the solution to the initial value problem

$$y'''' + 2y'' + y = 9 \cos 2t, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -3, \quad y'''(0) = 0,$$

then the Laplace Transform $Y=Y(s)$ of y is given by

$$Y(s) = \frac{9s}{(s^2+4)(s^4+2s^2+1)} - \frac{3s}{s^4+2s^2+1}$$

(c; 15pts) Find the solution $y=y(t)$ to the initial value problem

$$y'''' + 2y'' + y = 9 \cos 2t, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -3, \quad y'''(0) = 0.$$

Problem 2 (15pts)

(a; 8pts) Find all values of the constant c such that the origin in the xy -plane is a spiral sink or source for the solutions of the linear system

$$\mathbf{y}' = \begin{pmatrix} 5 & c \\ -c & 1 \end{pmatrix} \mathbf{y}.$$

Specify whether the origin is a spiral source or a spiral sink for the values of c you find.

(b; 7pts) Sketch the phase-plane portraits, in the xy -plane, for the system of ODEs in (a) with the values of c you found. Clearly show all important qualitative information, including the direction of rotation. Each of your sketches should contain *at least two* solution curves. Explain your reasoning. *Hint:* You need to sketch two different phase-plane portraits. Do *not* solve the system.

Problem 3 (30pts)

(a; 20pts) Find the *general* solution $(x, y) = (x(t), y(t))$ to the system of ODEs

$$\begin{cases} x' = 4y - x \\ y' = x + 2y \end{cases}$$

(b; 8pts) Sketch the phase-plane portrait, in the xy -plane, for the system of ODEs in (a). Clearly show all important qualitative information. In particular, indicate the slope of every relevant line, the flow directions, and the asymptotic behavior of the solution curves. You may want to state some of this information next to the sketch. Sketch *at least two* solution curves in each region cut out by the half-line solutions.

(c; 2pts) Determine whether the origin is a stable, asymptotically stable, or an unstable equilibrium point. Explain why.

Problem 4 (30pts)

(a; 12pts) Let A be a square matrix. Write down the power-series definition of e^A and use it to show that

$$\text{if } B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad \text{then} \quad e^{tB} = \begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix} \quad \text{and} \quad e^{tC} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix}.$$

(b; 6pts) Use any approach you can justify to show that

$$\text{if } A = \begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix}, \quad \text{then} \quad e^{tA} = \begin{pmatrix} e^{3t} & 0 \\ 2t e^{3t} & e^{3t} \end{pmatrix}.$$

(c; 12pts) Find the solution $\mathbf{y} = \mathbf{y}(t)$ to the initial value problem

$$\mathbf{y}' = \begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix} \mathbf{y} - \begin{pmatrix} 3 \\ 6t \end{pmatrix}, \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$