Math53: Ordinary Differential Equations
Winter 2004

Midterm II Practice Tests

Practice Test A

Problem 1

(a) Show that if $Y = Y(s)$ is the Laplace Transform of the function $y = y(t)$, then the Laplace Transforms of y'' and of y''' are given by

$$\{\mathcal{L}y''\}(s) = s^2Y(s) - sy(0) - y'(0), \qquad \{\mathcal{L}y'''\}(s) = s^3Y(s) - s^2y(0) - sy'(0) - y''(0).$$

(b) Find the solution to the initial value problem

$$y''' + 3y'' + 3y' + y = e^t, \qquad y(0) = 0, \;\; y'(0) = 0, \;\; y''(0) = 1.$$

Problem 2

(a) For what values of the constant c, the origin in the xy-plane is a spiral sink or source for the solutions of the linear system

$$\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ -1 & c \end{pmatrix} \mathbf{y}.$$

(b) Sketch the phase-plane portraits, in the xy-plane, for the system of ODEs in (a) with the values of c you found. Clearly show all important qualitative information, including the direction of rotation.

Problem 3

Find the general solution to the system of linear ODEs

$$\begin{cases} x' = y - 3x \\ y' = -x - y \end{cases} \qquad \mathbf{y} = \mathbf{y}(t).$$

Sketch the phase-plane portrait of solution curves. Determine whether the origin is a stable, asymptotically stable, or unstable equilibrium.

Problem 4

(a) Let A be a square matrix. Write down the power-series definition of e^A.

(b) Compute e^{tA} for $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$.

(c) Find the solution to the initial value problem

$$\mathbf{y}' = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 6t \\ 2 \end{pmatrix}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Practice Test B

Problem 1

(a) Show that if $Y = Y(s)$ is the Laplace Transform of the function $y = y(t)$, then the Laplace Transforms of y'' and of y'''' are given by

$$\{\mathcal{L}y''\}(s) = s^2Y(s) - sy(0) - y'(0), \qquad \{\mathcal{L}y''''\}(s) = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0).$$

(b) Find the solution to the initial value problem

$$16y'''' + 8y'' + y = 4\sin(3t/2), \qquad y(0) = 0, \;\; y'(0) = 0, \;\; y''(0) = 0, \;\; y'''(0) = -3.$$

Problem 2

Find the inverse Laplace Transform of the function

$$Y(s) = \frac{1}{s} \cdot \frac{1}{s^2+1}$$

Problem 3

Find the general solution, either real or complex, to the system of linear ODEs

$$\mathbf{y}' = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

Sketch the phase-plane portrait of solution curves. Determine whether the origin is a stable, asymptotically stable, or unstable equilibrium.

Problem 4

(a) Find the general solution to the system of linear ODEs

$$\mathbf{y}' = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

(b) Find the solution to the initial value problem

$$\mathbf{y}' = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 3t^2 \\ 2t \end{pmatrix}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Remarks: For Problems A1, B1, and B2, you should use the two tables of Laplace Transforms in *Unit 3 Summary*. They will appear on the back of the cover page on the second midterm. The answer for A2 is $c \in (-3, -1)$ for sink and $c \in (-1, 5)$ for source. If $c = -1$, the real part of the eigenvalues is zero, and the solution curves are ellipses. For A3, B3, and B4a, see *PS4 Solutions* for 9.2:44, 9.2:24, and 9.2:40. The solution in B4b is obtained using (a) and Theorem 8.11 on p532. A4 was discussed in class last Friday. In A1,3,4 and B1,3,4, you can also check your answers by plugging your solutions back into the corresponding equations.