The 2-D k-set Problem in Computational Geometry*

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1 Introduction

The k-set problem is one of the most challenging open problems in combinatorial geometry. The simplest variant of the problem is as follows: Let S be a set of n points in the plane. We assume they are in general position, i.e., no three of them are collinear and no parallel connecting lines. A *k-set* is a subset $S' \subseteq S$ such that $|S'| = k$ $(0 \leq k \leq n)$ and S' and $S \setminus S'$ can be separated by a straight line l. We denote $f_k(S)$ as the number of k-sets. The k-set problem thus asks for the asymptotic bound for $f_k(S)$. In the dual setting, given a set $\mathcal{L}$ of n lines in the plane in general position, denote by V_k the set of vertices of the arrangement $\mathcal{A}(\mathcal{L})$ that have exactly k lines passing below. The k-th level of the arrangement is defined to be the closure of the set of points that lie on the lines and have exactly k lines below them. So the set of vertices on the k-th level is $V_k \bigcup V_{k-1}$. Note that $f_k = |V_{k-1}|$, so the number of k-sets and the complexity of the k-th level are within a constant factor of each other. Figure 1 shows an illustration.

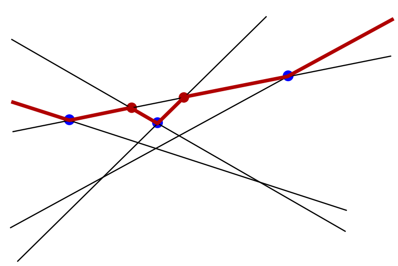

Figure 1: The third level in an arrangement of lines. The vertices of V_2 are shown by blue circles and the vertices of V_3 are shown by red circles.

The k-set problem was first studied in 1970's by Erdős *et al.* [21] and Lovász [33]. These papers gave an upper bound $O(n\sqrt{k})$ and a lower bound $\Omega(n \log k)$. The upper bound was improved slightly to $O(n\sqrt{k}/\log^* k)$ by Pach *et al.* [34]. No significant progress was made since then until in 1997 Dey [16] proved an upper bound of $O(nk^{1/3})$. The lower bound was later improved to $ne^{\Omega(\sqrt{\log k})}$ by Tóth [40].

In the dual setting, the problem can be generalized as follows: In the plane, we are given a collection Σ of n x-monotone curves, each being the graph of a continuous totally or partially defined function, we want to bound the complexity of the k-th level in the arrangement $\mathcal{A}(\Sigma)$. Agarwal *et al.* [1], combined with the result of Dey, gave a $O(nk^{1/3}\alpha(n/k))$ bound to the k-th level of n line segments, where $\alpha(n)$ is

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the inverse Ackermann function. The bounds for lines or line segments apply to the corresponding cases of pseudo-lines or extendible pseudo-segments as well [38, 1]. Two pseudo-lines (pseudo-segments) can intersect only once. Tamaki and Tokuyama [39] proved a bound $O(n^{23/12})$ on the complexity of a level in an arrangement of n pseudo-parabolas, which are graphs of total functions, each pair of which intersect at most twice. Their method is to cut the pseudo-parabolas into $O(n^{5/3})$ pseudo-segments. Applying Dey's result and the "k-sensitizing" technique by Agarwal *et al.* [1], this bound can be improved to $O(nk^{8/9})$. They also proved a lower bound $\Omega(n^{4/3})$ on the number of cuts to decompose pseudo-parabolas into pseudo-segments. Later Aronov and Sharir [8] improved the result by showing that circles can be cut into $O(n^{3/2+\epsilon})$ pseudo-segments. Chan [11] proved an $O(nk^{1/3}\alpha(n/k)\log^{2/3} k)$ bound on the k-th level of an arbitrary arrangement of n pseudo-segments. This result, also implies a improvement on the k-th level of pseudo-parabolas to $O(nk^{7/9}\log^{2/3} k)$. For an arrangement of n curves that are graphs of polynomial functions of an arbitrary fixed degree s, Chan [11] proved that the complexity of the k-th level is no more than $O(nk^{1-2/3^s})$.

The k-set problem can also be extended to high dimensions. In the primal plane, given a set of n points in $\mathbb{R}^d$ in general position. We want to bound the number of hyperplanes π passing through d of the points such that one of the halfspaces bounded by π contains exactly k points of S. For $d = 3$, Bárány *et al.* [9] proved an $O(n^{3-\gamma})$ bound, for any k and some absolute constant $\gamma > 0$. The bound was improved by Aronov *et al.* [7] and Eppstein [20] to $O(n^{8/3}\text{polylog}n)$, and then by Dey and Edelsbrunner [17] to $O(n^{8/3})$. Agarwal *et al.* [1] gave a k-sensitive bound $O(n(k+1)^{5/3})$. The best bound known, due to Sharir, Smorodinsky and Tardos, is $O(nk^{3/2})$ [37]. For $d > 3$, the best known upper bound is $O(n^{d-c_d})$, for some exponentially small but positive constant c_d [41]. For the number of k-sets in $\mathbb{R}^3$, a lower bound $\Omega(n^2 \log n)$ was proved in [9, 17]. Later Tóth [40] proved a lower bound $\Omega(nk^{d-2}e^{\Omega(\sqrt{\log k})})$ for k-set problem in d-dimensional space. Agarwal *et al.* [1] proved a bound $O(n^2k^{7/9}\alpha(n/k))$ on the complexity of the k-th level in an arrangement of n triangles in 3 dimensional space. This was later improved to $O((n-k)^{2/3}n^2)$ by Katoh and Tokuyama [31].

However, the bound on the k-set problem can be improved for some special point sets. Bárány and Steiger [10] proved a linear upper bound on the expected number of k-sets in a random planar point set. If the set of points in $\mathbb{R}^d$ is dense, i.e., the ratio of largest over smallest distance between any two points is at most $\delta n^{1/d}$, then the number of $\lfloor n/2 \rfloor$-sets can not exceed $O(n^{5/4}/\log^* n)$ for $d = 2$, $O(n^{7/3})$ for $d = 3$, and $O(n^{d-2/d})$ for $d \geq 4$ [18]. Alt *et al.* [6] proved that if all points lie on convex curves, the number of k-sets is linear.

Though the status of the k-set problem is miserable, the number of vertices of the first k-level, has a tight bound $\Omega(nk)$ [24, 5, 36]. Goodman and Pollack [24] were the first to prove a bound $2nk - 2k^2 - k$ for $2k \leq n$. With the same idea, Alon and Győri [5] improved the bound to nk. Peck [36] gave another simple proof of this result. For d-dimension, Clarkson and Shor [14] used a probabilistic analysis that yields a fairly sharp bound $\Omega(n^{\lfloor d/2 \rfloor}k^{\lceil d/2 \rceil})$.

Multiple techniques were developed to tackle the k-set problem. We will cover most of the techniques in the following part, e.g., allowable circular sequences of permutations, Lovász's Lemma, dissection graphs, concave chain decomposition, etc.

2 Allowable circular sequences

Allowable circular sequences were firstly studied by Goodman and Pollack [25, 24, 23]. We begin with the circular sequences of permutations for a point set.

Circular sequence of permutations For a set S of n points in the plane, a *circular sequence of permutations* was defined in the following way. Choose a directed line l which is not orthogonal to any direction determined by two points of S and project the points of S orthogonally onto l. Label the points of S by "$1, 2, \cdots, n$" so that their projections fall in increasing order on l. Now allow l to rotate counterclockwise. Whenever l passes through a direction orthogonal to the line determined by points i and j, the order of the points falling on l changes by having the indices i and j interchange. When l has rotated through 180 degrees the points fall on l in the reverse order "$n, n-1, \cdots, 1$". As l rotates counterclockwise, it defines an infinite sequence of permutations $P(S) = P_0, P_1, P_2, \cdots$, called circular sequences. This sequence is clearly periodic with period $2N = n(n-1)$, corresponding to a 360 degree rotation of l. It is determined by a half period, which corresponds to l rotating a half turn. It is easy to see the circular sequences of permutations have the following properties.

Property 1 *a) P_{k+1} differs from P_k by the interchange of two adjacent indices i and j, $i < j$. P_i and P_{i+N} are in reverse order;*

b) Each one of the N switches occurs exactly once in any subsequence of $N+1$ permutations.

Take $P'(S)$ to be a subsequence of $P(S)$ with length $2N+1$. A k-set of S is an initial k-segment of at least one permutation of $P'(S)$. In fact, for $2k < n$, the number of k-sets f_k is precisely the number of switches between position k and $k+1$, and of course the number of switches in position $n-k$.

($\leq k$)-sets Denote $g_{k,n} = \sum_{1 \leq i \leq k} f_i(S)$ to be the number of ($\leq k$)-sets of the point set S. Then $g_{k,n}$ can be shown to be at most nk [24, 5].

Theorem 2.1 *For $2k < n$, $g_{k,n} = \sum_{1 \leq i \leq k} f_i(S) \leq nk$.*

Proof. Let b be a fixed point in $\{1, 2, \cdots, n\}$, and $P'(S)$ be a subsequence of $P(S)$ with length $2N+1$. The total number of switches involving b is precisely $2n-2$ (twice with any other points). If b occurs in a switch in position $i \in (1, 2, \cdots, k)$ it also occurs in a switch in position $n-i$. If $i < j < n-i$, by continuity b occurs in at least two switches in position j. Thus any point occurs in at most $2n-2-2(n-2k-1) = 4k$ switches in position $\{1, 2, \cdots, k\} \bigcup \{n-k, \cdots, n-1\}$. The total number of switches in these positions is half of the sum of occurrences of points in such switches, i.e., $2nk$. The total number of switches in the first k positions is precisely half this quantity, i.e., nk. □

Allowable circular sequences As a generalization to the circular sequence of permutations for a set of points, an infinite sequence of permutations of the numbers $1, 2, \cdots, n$ is called an *allowable circular sequence* if it satisfies Properties 1. If an allowable circular sequence is induced by a point set then the sequence is said to be *realizable*. We call a subset A of $\{1, 2, \cdots, n\}$ an *allowable k-set* of an allowable circular sequence $\widetilde{P}$ of permutations of $1, 2, \cdots, n$, if A contains k numbers and there exists a permutation in $\widetilde{P}$ such that the k numbers of A occur at the leftmost k positions. The number of allowable k-sets is denoted as $\widetilde{f_k}$. Goodman and Pollack [23] showed that there exist allowable circular sequences which are not realizable. The distinction between point-sets and allowable circular sequences relates to the classical distinction between line arrangements and pseudoline arrangements, see, e.g., Gruenbaum [26].

Edelsbrunner and Welzl [19] proved the bound $O(n\sqrt{k})$ on the number of allowable k-sets for an allowable circular sequences, which also implies the bound on the number of k-sets for n points in the plane. Observe that the number of allowable k-sets is the number of switches at position k and $k+1$ among

$n(n-1)+1$ successive permutations of $\widetilde{P}$. Let $\widetilde{P}_0 = 1, 2, \cdots, n$ be the first permutation of $\widetilde{P}$ and let $Y = i, i+1, \cdots, j$ for $1 \leq i \leq j \leq n$ denote a subsequence of $\widetilde{P}_0$. Take $\widetilde{P}'$ to be a subsequence of $\widetilde{P}$ with length $n(n-1)+1$. The subsequences to the left and to the right of Y are denoted as X and Z. We have the following lemma:

Lemma 2.2 *At most* $\binom{y}{2} + \min\{n-y, 2k\}$ *of the switches in* $\widetilde{P}'$ *at position* k *and* $k+1$ *involve a number of* Y, y *is the number of numbers in* Y.

Proof. It is trivial to realize that at most $\binom{y}{2}$ of the relevant switches involve two numbers of Y. After those switches Y is totally reversed. It remains to derive a bound on those relevant switches which involve exactly one number of Y. To this end we perform certain transformations on $\widetilde{P}'$ such that the contribution of the numbers in Y does not change and such that no switch involving two numbers of either X or Z is performed before the last switch at position k involving a number of Y is completed.

The existence of the transformation can be proved in the following way. Let j denote the smallest positive integer such that the switch leading from the j-th permutation $\widetilde{P}_j$ in $\widetilde{P}'$ to the $(j+1)$-th permutation $\widetilde{P}_{j+1}$ involves two numbers i and $i+1$ of X. We delete $\widetilde{P}_{j+1}$ from $\widetilde{P}'$ and concatenate a permutation $\widetilde{P}^*_{j+1}$ at the end of $\widetilde{P}'$. In all permutations between $\widetilde{P}_j$ and $\widetilde{P}^*_{j+1}$, the numbers i and $i+1$ are replaced by each other. $\widetilde{P}^*_{j+1}$ is chosen to differ from its predecessor only by having the order of i and $i+1$ changed. Applying this local transformation repeatedly proves the existence of the transformation.

Thus, in the transformed sequence of permutations, no number of X (Z) is able to move to the left (right) before Y has completed its last contribution to the switches at position k and $k+1$. Then each one of the numbers in X and Z can only once be the partner of a number in Y when it is involved in a switch at position k and $k+1$. In addition, at most k numbers of X can move from position k to $k+1$ and at most k numbers of Z can move from position $k+1$ to k, before the last contribution of Y to the number of switches at these positions took place. This completes the argument. □

With the above lemma in mind, we have,

Theorem 2.3 $\widetilde{f}_k = \widetilde{f}_{n-k} = O(nk^{1/2})$ *for positive integers* n *and* k *with* $k \leq \lfloor n/2 \rfloor$.

Proof. We partition $\widetilde{P}_0$ into subsequences with $\lfloor k^{1/2} \rfloor$ or $\lfloor k^{1/2} \rfloor - 1$ numbers each. Due to the above lemma, the contributions to the number of switches at position k and $k+1$ of any one of those subsequences is at most $\binom{y}{2} + 2k < 3k$. There are at most $\lfloor n/(\lfloor k^{1/2} \rfloor - 1) \rfloor$ subsequences for $k \geq 4$, and at most n subsequences for $k < 4$. This proves the theorem. □

Corollary 2.4 *The number of* k*-sets for a set of* n *points is bounded by* $f_k = f_{n-k} = O(n\sqrt{k})$, *for* $k \leq \lfloor n/2 \rfloor$.

3 Lovász's lemma and dissection graphs

Lovász proved a lemma that implies the first nontrivial bound $O(n^{3/2})$ on the number of k-sets for n points. This lemma itself, is of great interests. Erdös, Lováz, Simmons and Straus [21] further extended the idea and improved the result on k-sets to $O(n\sqrt{k})$ and $\Omega(n \log n)$. Lovász's lemma has also been extended to high dimensions and the dual setting [1, 31].

Dissection graphs Denote by l an oriented line in the plane. For a set of points S in general position, let $N(l)$ be the set of those points of S which lie on its positive side, i.e., the open right hand side of l. l is called a *k-line* if and only if it contains two points of S and has $N(l) = k$. A segment joining two points of S is a *k-edge* if its line is a k-line. In particular, a $(n/2-1)$-line (edge) is called a *halving line (edge)*. Define a directed *k-graph* G_k of S whose edges are the k-edges, $0 \leq k \leq n-2$. Clearly $G_{n-k-2} = -G_k$, that is, the k-graph with all the edges reversed.

The k-graph can be constructed as follows. Let l be an oriented line containing no point of S and having $k+1$ points of S on its positive side. Translate l to its left until it meets a point p_1 of S. Call this line $l(0)$. Now rotate $l(0)$ counterclockwise by θ about p_1 into line $l(\theta)$ until it meets a second point p_2 of S at $l(\theta_1) = l_1$. Now rotate counterclockwise about p_2 until $l(\theta)$ meets a point p_3 of S at $l(\theta_2) = l_2$, etc. We thus get a sequence of points $p_1, p_2, \cdots, p_r$ of S with $p_{r+1} = p_1$, $p_{r+2} = p_2$ and a sequence of directed lines $l_1, l_2, \cdots, l_r, l_{r+1}$ with $l_{r+2} = l_1$.

Theorem 3.1 *The k-graph G_k consists of those vertices p_i and those edges $\overrightarrow{p_{i+1}p_i}$ for which the orientation $\overrightarrow{p_ip_{i+1}}$ is opposite to that of the line l_i.*

Proof. Clearly the number $N(\theta)$ of points on the positive side of $l(\theta)$ remains constant in any interval which does not contain one of the angles θ_i. If $\overrightarrow{p_ip_{i+1}}$ is in the direction of l_i then for small $\epsilon > 0$ we have $N(\theta_i - \epsilon) = N(\theta_i) = N(\theta_i + \epsilon)$, since the points p_i, p_{i+1} are either on or to the left of $l(\theta)$ for $\theta_i - \epsilon \leq \theta \leq \theta_i + \epsilon$. If $\overrightarrow{p_ip_{i+1}}$ is in the opposite direction of l_i then $N(\theta_i - \epsilon) - 1 = N(\theta_i) = N(\theta_i + \epsilon) - 1$, since one of p_i, p_{i+1} is to the right of $l(\theta)$ in $\theta_i - \epsilon \leq \theta \leq \theta_i + \epsilon$ except for $\theta = \theta_i$ when both are on l_i. Thus we have $N(\theta) = k+1$ for all $\theta \neq \theta_i$ and $N(\theta_i) = k+1$ or k depending on whether $\overrightarrow{p_ip_{i+1}}$ is in the direction l_i or not.

Finally we can see that all edges of G_k are included in the lines $l(\theta)$ since any line l' not included is parallel to a line $l(\theta')$ so that $N(l') - N(\theta') \neq 0$. If $\theta' = \theta_i$, and $\overrightarrow{p_ip_{i+1}}$ is in the direction opposite to l_i, this proves that $N(l') \neq k$. Otherwise $N(\theta') = k+1$ and l' passes through two points on the right of $l(\theta')$ so that $N(l') \leq k-1$. ∎

Lemma 3.2 (Antipodality Lemma) *If we order the oriented lines of the edges of G_k at a vertex v, in counterclockwise order, then between any two lines containing outgoing edges there is a line containing an incoming edge, and between any two lines containing an incoming edge there is a line containing an outgoing edge.*

Proof. Let l_1 and l_2 be successive oriented lines through v containing outgoing edges of G_k. Then as l rotates from l_1 to l_2 we have $k+1$ points of S on the positive side of l for l near to l_1 and k points of S on the positive side of l for l near to l_2. Since the number of points of S on the positive side of l increases by one each time l passes through a point p of S in the oriented angle $\angle(l_1, l_2)$ and decreases by one each time l passes through a point p of S in the opposite vertical angle $\angle(-l_2, -l_1)$, it follows that at some stage of the rotation the number of points on the positive side of l decreases from $k+1$ to k so that l contains an incoming edge $\overrightarrow{pv}$ of G_k. The argument for successive lines containing incoming edges is entirely analogous. ∎

Lemma 3.3 (Lovász Lemma) *Let l be a line containing no point of S, l divides S into two sets S_1, and S_2 with $|S_1| = m \leq n-m = |S_2|$. Then l intersects $m_0 = \min\{m, k+1\}$ edges of G_k going from S_1 to S_2 and m_0 edges of G_k from S_2 to S_1.*

Proof. Assume l is not parallel to any line connecting two points of S. Pick point $p_1 \in S_2$ and the directed line $l(0)$ of the family defined in Theorem 3.1 through p_1 with S_1 on the negative side of $l(0)$. As θ increases from 0 to π, the number $N(\theta, S_1)$ of points of S_1 on the positive side of $l(\theta)$ increases from 0 to m_0. This increase is monotonic if we ignore the values $\theta = \theta_i$.

Note $N(\theta, S_1)$ is constant in any interval which does not contain a θ_i. If both points p_i, p_{i+1} of $l(\theta_i)$ are in S_2, or $\overrightarrow{p_i p_{i+1}}$ is in the direction of $l(\theta_i)$, then $N(\theta_i - \epsilon, S_1) = N(\theta_i, S_1) = N(\theta_i + \epsilon, S_1)$ for small $\epsilon > 0$. If p_i, p_{i+1} are both in S_1 and $\overrightarrow{p_i p_{i+1}}$ is in the opposite direction of $l(\theta_i)$, we have $N(\theta_i - \epsilon, S_1) = N(\theta_i + \epsilon, S_1)$ for small $\epsilon > 0$. Finally if p_i and p_{i+1} are in the opposite sides of l and $\overrightarrow{p_i p_{i+1}}$ is in the opposite direction of $l(\theta_i)$ for $0 < \theta_i < \pi$, then $P_i \in S_1$, $P_{i+1} \in S_2$ and $N(\theta_i + \epsilon, S_1) = N(\theta_i, S_1) + 1 = N(\theta_i - \epsilon, S_1) + 1$. Thus we have shown that $N(\theta, S_1)$ increases by one in the interval $0 < \theta_i < \pi$ when l is intersected by a k-edge $\overrightarrow{p_{i+1} p_i}$ of G_k going from S_2 to S_1. As θ increases from π to 2π, the number $N(\theta, S_1)$ decreases from m_0 to 0, similarly we can prove that l intersects m_0 k-edges going from S_1 to S_2. □

k-set problem Lovász's lemma immediately implies a $O(n\sqrt{k})$ bound on the k-set problem. Observe that the number of k-sets of S equals to the number of edges in the $(k-1)$-graphs. We only need to prove the bound $O(n\sqrt{k})$ on the number of k-edges. Assume $e_1, e_2, \cdots, e_{n-1}$ be parallel lines cutting the plane into strips, each containing one point of S. According to Theorem 3.3, the total number of intersections of these lines with the edges of G_k is $2(2 + 4 + \cdots + 2k) + 2(n - 1 - 2k)(k + 1) = O(nk)$. Now divide the edges of G_k into two classes according to whether they intersect at least $\sqrt{k}$ of the parallel lines or not. The first class contains no more than $O(n\sqrt{k})$ k-edges. The second class contains pairs of points of S separated by fewer than $\sqrt{k}$ of the parallel lines, which is $2(n - 2 + n - 3 + \cdots + n - \sqrt{k}) = O(n\sqrt{k})$. Thus the number of edges in G_k is less than $O(n\sqrt{k})$.

Peck's proof on ($\leq k$)-sets Peck [36] gave a different but very simple proof to the bound on ($\leq k$)-sets, using the same idea with the dissection graph. Define a graph G' on the points S whose edge set is the union of the edges in j-graphs, $0 \leq j < k$. Then the claim is that no point can have degree above $2k$, which proves the bound on the ($\leq k$)-sets.

Consider any point p in S. Look at the set $\mathcal{L}(p)$ containing all the ($< k$)-lines through p in G'. Associate to each line with the normal direction that points to the side of it which has $k - 1$ or fewer points. Any two such lines determine four quadrants and there is exactly one quadrant that the normals of both lines point away from, which we will call a good quadrant. Choose two such lines whose good quadrant contains no other lines. (Such a quadrant exists if there are only two lines; and of the two regions obtained when a new line splits a good empty quadrant one will still be good and empty, so that such two lines exist by induction.) There are at most $2(k - 1)$ points in the interior of the three quadrants other than the good one by the definition. Then there are at most $2(k - 1)$ other edges or lines emanating from our point, since each of these must contain a point in these quadrants.

Lower bound With dissection graphs, Erdös *et al.* [21] provided the first lower bound construction for the k-set problem. Let $e_{n,k}$ denote the maximum number of edges of a graph $G_k(S)$ where S contains n points. They proved that $e_{n,k} = \Omega(n \log n)$. We begin with a simple lemma.

Lemma 3.4 *A point p of S is a vertex of $G_k(S)$, $k \leq (n - 2)/2$, if and only if there exists a directed line through p whose positive side contains no more than k points of S.*

Proof. The necessity is obvious since any line of an edge in G_k with vertex p has the property. The sufficiency follows from the fact that, as a directed line l is rotated around p, the number of points on its positive side ranges through all values from the minimum m to the maximum M. We have $m \leq k$ and

$M = n - m - 1 \geq n - k - 1 \geq k + 1$. Thus there must be an instant at which the number of points on the positive side of l changes from k to $k + 1$. $\square$

Lemma 3.5 *If* $k \neq (n-2)/2$ *then* $e_{2n,2k+1} \geq 2e_{n,k} + n$.

Proof. We first show that a $G_k(S)$ with a maximal number of edges must have n vertices. Assume there is a point $p \in S$ which is not a vertex of G_k. From Lemma 3.4, p is on the negative side of all the edges of G_k. If we move p across the line of an edge of G_k then that edge is removed from the graph but there will be at least two new edges incident to p, contrary to the maximality assumption for G_k.

Now associate an outgoing edge e_p to each vertex p of G_k and construct a set S' with $2n$ points by splitting each point p into two points p' and p'' at a small distance ϵ from p, with $\overrightarrow{p'p}$ and $\overrightarrow{pp''}$ in the direction of e_p. Consider $G_{2k+1}(S')$. First for each $p \in S$ we get $p'p''$ as an edge of G_{2k+1} sine each point on the positive side of e_p has become two points; and, if $e_p = \overrightarrow{pq}$, then exactly one of the two points q', q'' is on the positive side of e_p. In addition, both $\overrightarrow{p''q'}$ and $\overrightarrow{p''q''}$ are edges of G_{2k+1}. See Figure 2 (i).

Finally, if $\overrightarrow{pq}$ is an edge of G_k other than e_p, see Figure 2 (ii), then the edge which joins the point p' or p'' on the negative side of $\overrightarrow{pq}$ to the point q' or q'' on the positive side of $\overrightarrow{pq}$ as well as the edge which joins the point p' or p'' on the positive side of $\overrightarrow{pq}$ to the point q' or q'' on the negative side of $\overrightarrow{pq}$ are edges of G_{2k+1}. Thus in this splitting process each edge of G_k yields two edges of G_{2k+1} and the n edges e_p of G_k yield an additional edge $\overrightarrow{p'p''}$. Thus $e_{2n,2k+1} \geq e(G_{2k+1}) = 2e_{n,k} + n$. $\square$

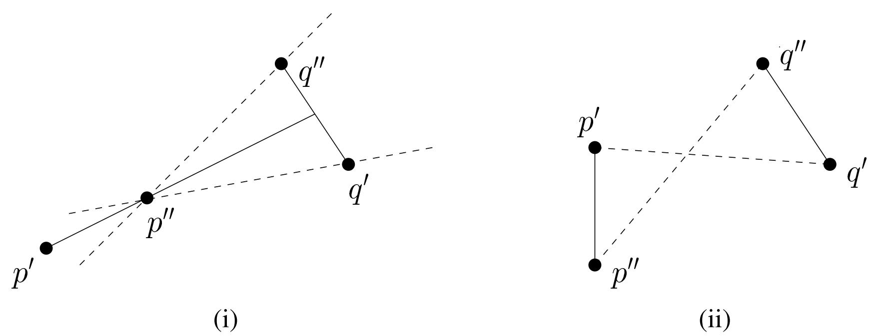


Figure 2: (i) The edge from p to q is e_p; (ii) The edge from p to q is not e_p.

It is a little troublesome to prove the similar theorem for the case $k = (n-2)/2$. But it can be done [21]. This implies the lower bound $\Omega(n \log n)$. Tóth's bound $ne^{\Omega(\sqrt{\log k})}$ [40] follows the same inductive idea of duplicating one point to multiple points.

4 Potential argument

Gusfield [27] proves the bound $O(e\sqrt{n})$ on the number of changes of the minimum spanning tree on a graph with n nodes and e edges, if the weights of the edges are linear function in a parameter t. This problem is actually congruent to the k-set problem and Gusfield's proof gives a $O(n\sqrt{k})$ bound on the number of k-sets of n points. We describe the analysis in the setting of the k-set problem, as Agarwal *et al.* did in [1]. Let the lines in $\mathcal{L}$ be $l_1, l_2, \cdots, l_n$, sorted in the order of decreasing slope, and we want to bound the complexity of the k-th level. For any $\alpha \in \mathbb{R}$, define the level of a line $l \in \mathcal{L}$ at α is j if exactly j lines of $\mathcal{L}$ intersect the vertical line $x = \alpha$ below l. Define the *potential function* as the following:

$$\Phi(x) = \sum\{j \mid \text{the level of } l_j \text{ at } x \text{ is less than } k\}.$$

Note that the potential function is upper bounded by $O(nk)$ for any $x \in \mathbb{R}$. When x goes from $-\infty$ to ∞, $\Phi(x)$ is monotonically increasing. The value of $\Phi(x)$ only changes at the abscissa v_x of a vertex $v \in V_{k-1}$. Assume v_x is the intersection of l_i and l_j, with $j \geq i$. Then the increase of $\Phi(x)$ is $j - i$. Therefore we have $\sum_{v \in V_{k-1}}(\Phi(v_x + \epsilon) - \Phi(v_x - \epsilon)) = O(nk)$, for a sufficiently small $\epsilon > 0$.

Note that the number of vertices v at which $\Phi(v_x + \epsilon) - \Phi(v_x - \epsilon) > \sqrt{k}$ is no more than $O(n\sqrt{k})$. The number of vertices v with $\Phi(v_x + \epsilon) - \Phi(v_x - \epsilon) \leq \sqrt{k}$ is bounded by $(n-1) + (n-2) + \cdots + (n - \sqrt{k} + 1) < n\sqrt{k}$. Therefore the total number of vertices in V_{k-1} is at most $O(n\sqrt{k})$.

Note that this argument can be extended to the cases of pseudo-lines, line segments, and pseudo-segments [1], combined with the concave chain data structure in the next section.

5 Concave chains

The concave chains of the line arrangements were first studied by Halperin and Sharir [29, 28, 30]. Later Agarwal *et al.* [1] applied concave chains on the k-set problem. This later led to Dey's result [16].

A *concave chain* is an unbounded x-monotone concave polygonal curve contained in the union of the lines of $\mathcal{L}$. The k-th level can be associated with a set of k concave chains $c_1, c_2, \cdots, c_k$, which start at $x = -\infty$ along the k-th lowest line of the arrangement. When a chain c_i reaches the k-th level, it reaches a vertex $v \in V_{k-1}$. Then c_i continues to the right along the other line incident to v. The chain bends to the right only at vertices of V_{k-1}. See Figure 3 (i) for an example. It is easy to see that the concave chains satisfy the following properties:

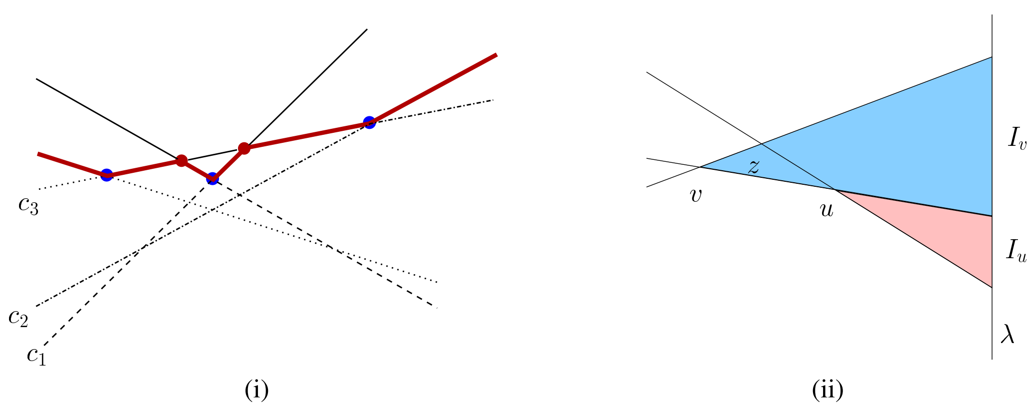


Figure 3: (i) The concave chains associated with the third level; the level itself is drawn in red, and the dashed lines denote the concave chains c_1, c_2, c_3; (ii) Each common tangent is charged for only once.

a) The number of vertices on a concave chain is at most $n - 1$.

b) The union of the chains is the closure of the portion of the union of the lines that lies below the k-th level. Except for the vertices of V_{k-1}, the union of the chains lies strictly below the k-th level.

c) The chains are vertex-disjoint and have non-overlapping edges, but they generally cross each other.

d) All the vertices of the chains lie on the upper envelope of the chains. Each chain, except for its vertices, lies fully below the k-th level, so any vertex of any chain lies above all the chains that are not incident to it.

Note that the crossings of the k concave chains covers all the vertices of level at most $k-2$. Since one line can intersect a concave chain at most twice, it immediately implies that the complexity of the ($\leq k$)-levels is bounded by $O(nk)$. In addition, the overall number of vertices of k concave chains in an arrangement of n lines in the plane, is bounded by $O(n\sqrt{k})$. Gusfield's potential method can be easily extended to prove the above result. Halperin and Sharir, independently, proved this using extremal graph theory [28, 30].

Agarwal *et al.* [1] gave a simple proof of the Lovász's lemma in the dual setting, using the concave chain structure. Define a *double wedge* W_v formed at a vertex v by two lines l_1, l_2 meeting at v to be the closed region between the upper and lower envelopes of $\mathcal{A}(l_1 \bigcup l_2)$. Then the planar dual Lovász's lemma says:

Lemma 5.1 (Dual Lovász's Lemma) *For any point $z \in \mathbb{R}^2$ not lying on any line of $\mathcal{L}$, the number of double wedges W_v, for $v \in V_{k-1}$, that contain z is at most $2\min\{k, j\} \leq 2k$, where j is the number of lines of $\mathcal{L}$ that pass below z.*

Proof. Let c be one of the concave chains, consider the set of double wedges W_v, over all vertices v of c. The concavity of c implies that z can lie in at most two of these double wedges (at most one right wedge and one left wedge), see Figure 3 (ii). Since we have k concave chains, z can lie in at most $2k$ double wedges W_z, for $v \in V_{k-1}$. More over, if z lies above exactly $j < k$ lines of $\mathcal{L}$, then it lies only above j concave chains, and can therefore only belong to double wedges corresponding to vertices of these chains. This proves the lemma. □

Also with the dual setting, the complexity of the k-level can be easily bounded by $O(n\sqrt{k})$. Fix a vertical line λ, and intersect each W_v, for $v \in V_{k-1}$, with λ, to obtain a set of $|V_{k-1}|$ intervals on λ, having a total of n endpoints (intersections of the lines in $\mathcal{L}$ with λ), see Figure 3 (ii). It follows from a counting argument [12, 33] that λ must contain a point that lies in at least $|V_{k-1}|^2/4n^2$ intervals. Since this number cannot be more than $2k$ due to Lemma 5.1, we obtain that $|V_{k-1}| \leq O(n\sqrt{k})$.

6 Dey's bound

Dey's proof [16] is based on the concave chain structure, dissection graphs and the crossing number of geometric graphs. The proof can be done in both primal [16] and dual plane [15]. We begin from the proof in the dual plane since it follows simply from the concave chain structure described above.

Dual plane Consider the $(k-1)$-graph $G_{k-1} = (S, E)$ in the primal plane. Let e_{pq} and e_{rs} be two vertex disjoint edges in E that cross. The edge e_{pq} is mapped to the double wedge formed at the vertex u where the lines p^* and q^* meet. Similarly e_{rs} is mapped to the double wedge at the vertex v where r^* and s^* meet. If edges e_{pq} and e_{rs} crosses, in the dual plane, the line passing through u and v lies in the double wedges of both vertices. This means that the edge connecting u and v is tangent to the concave chains that contain u and v respectively. See Figure 4. This observation is crucial for counting the crossings between $(k-1)$-edges of G_{k-1}. A *common tangent* between two chains c_i and c_j is a line segment that connects a vertex of c_i with a vertex of c_j, and whose supporting line is a tangent to both chains. Let x_{ij} denote the number of crossings between the chains c_i and c_j. We have the lemma:

Lemma 6.1 *The number of common tangents between any two chains c_i and c_j is at most x_{ij}.*

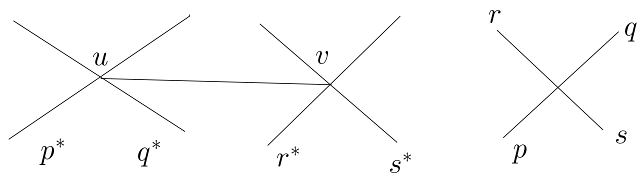
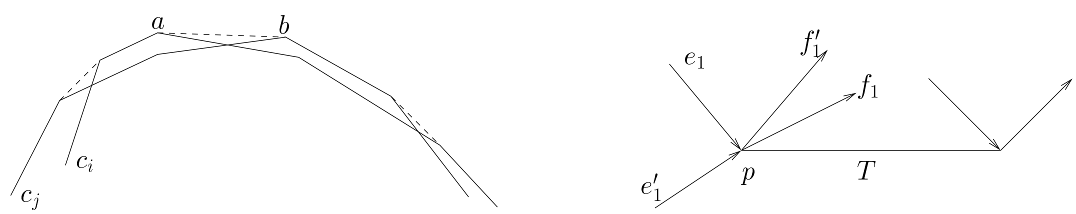


Figure 4: A pair of crossing $(k-1)$-set edges and their duals.

Proof. Note that all common tangents between c_i and c_j appear on the upper hull of the vertices of c_i and c_j together. Consider a common tangent connecting a vertex a on c_i with a vertex b on c_j. Due to the properties of the concave chains, we know that c_i lies below b and c_j lies below a. So there must be a crossing of the chain c_i and c_j somewhere, we charge the crossing for the common tangent $\overline{ab}$. The number of common tangents between c_i and c_j is therefore at most x_{ij}. See Figure 5 (i). □



(i) (ii)

Figure 5: (i) Common tangents between two concave chains are shown in dashed lines. A crossing between two chains corresponds to the common tangent above; (ii) Each common tangent is charged for only once.

Since a crossing between edges in G_{k-1} corresponds to a common tangent between two concave chains. Due to Lemma 6.1, the number of crossings between edges in G_{k-1} is no more than the number of crossings among the concave chains. Since the concave chains only intersect at the vertices with level $\leq k$, whose number is bounded to be $O(nk)$ [24, 5, 36]. Note that in the graph G there are exactly $|V_{k-1}|$ edges. Note the result on geometric graphs proved by Ajtai, Chvátal, Newborn, Szemerédi [4] and Leighton [32]:

Lemma 6.2 (Crossing Lemma) *Let G be a geometric graph with n vertices and $m \geq 4n$ edges. Then there are $\Omega(m^3/n^2)$ pairs of edges in G whose relative interiors cross.*

Proof. [1] Consider a planar embedding of a graph with n vertices, e edges, and c pairs of crossing edges. Euler's formula implies that $c > e - 3n$. Take a random subset of the vertices, each vertex with probability p. The expected number of vertices, edges, and crossings in the induced subgraph are at least pn, p^2e, and p^4c, respectively. Thus, $p^4c > p^2e - 3pn$, which implies $c > e/p^2 - 3n/p^3$. Taking $p = 4n/e$ gives us $c > e^3/64n^2$. □

[1] According to János Pach, this probabilistic proof has been independently rediscovered several times, by Lovász, Matousék, Füredi, Alon, Seidel, and many others, but was not published since it is essentially equivalent to the original counting argument of Ajtai *et al.*. The proof was first published by Székely. Using a more complicated probabilistic argument, Pach and Tóth [35] improved the constant from $1/64$ to $4/135$, which is currently the best known.

Assume $|V_{k-1}| > 4n$, then apply the above lemma, there are at least $c \cdot t^3/n^2$ crossings among $t > 4n$ edges connecting pairs of points in a set of n points in plane, where c is some constant, we have $|V_{k-1}|^3/n^2 < c_1 \cdot nk$ for some constant c_1. This immediately proves the following theorem.

Theorem 6.3 *There are $O(nk^{1/3})$ k-sets for a set of n points in $\mathbb{R}^2$.*

Primal plane The proof can also be done in the primal plane [16]. Consider the k-graph defined in section 3. We only keep the k-edges that orient from left to right, without loss of generality, we assume the number of edges is at least half the number of the k-edges. Otherwise we can just keep the edges that orient from right to left. We call this graph G'_k. Note that Lemma 3.2 remains true for the graph G'_k. Denote $s(e)$ as the slope of the edge e. We define a relation R on the edge set, such that for an incoming edge e and an outgoing edge f incident to the same vertex, eRf if and only if $s(e) < s(f)$ and there doesn't exist any outgoing edge f' with $s(e) < s(f') < s(f)$. Lemma 3.2 simply implies that the there is no edge f such that eRf and gRf where $e \neq g$.

Let R^* denote the reflective, symmetric, and transitive closure of R. The equivalence relation R^* partitions the edge set of G'_k into a set of convex chains, each of them is composed of non-overlapping directed edges going from left to right. In addition, those chains have the following properties.

Lemma 6.4 *Let $c_1, c_2, \cdots, c_r$ be the convex chains obtained by partition of the edge set with R^*. Each c_i has a unique endpoint which is one of the $k+1$ leftmost points of P.*

Proof. Lemma 3.2 implies that two chains cannot share a common starting point. Assume c_i starts from the m-th leftmost point p_m, $m > k+1$. f is the first edge of the chain c_i. Rotate f clockwise around p_m. At the beginning of this rotation there are exactly k points on its right. When the line is rotated up to the vertical position, it has exactly $m-1 > k$ points on its right. This means that its supporting line has gained at least one point on its right during the rotation. So there exists a point x such that the oriented edge $\overrightarrow{xp_m}$ must be a k-edge. Furthermore $s(\overrightarrow{xp_m}) < s(f)$, this leads to a contradiction with the fact that f is the leftmost edge of c_i. □

The way we bound the number of crossings between the convex chains is to charge each crossing between c_i and c_j to the common tangent of c_i and c_j directly below. It is easy to see that two crossings between c_i and c_j cannot charge the same common tangent. In addition, even two crossings from different pairs of chains can not charge the same tangent.

Lemma 6.5 *Each common tangent is charged only once for crossings over all pairs of chains.*

Proof. Assume a tangent T is charged by two pairs of chains, c_1, c_2 and c'_1, c'_2. Let p be an endpoint of T which is not an endpoint of one chain, say c_1. See Figure 5 (ii). p must exist, otherwise Lemma 6.4 is violated. Let e_1 an f_1 be the incoming and outgoing edge of c_1 incident with p. Consider the outgoing edge f'_1 of c'_1 incident with p. Since T is tangent to c_1 and c'_1, we have either $s(e_1) < s(f'_1) < s(f_1)$ or $s(e_1) < s(f_1) < s(f'_1)$. The first possibility contradicts with the fact that $e_1 R f_1$. The second possibility implies an incoming edge e'_1 of c'_1 such that $s(f_1) < s(e'_1) < s(f'_1)$ due to Lemma 3.2. This implies that T can't be tangent to c'_1. □

Therefore, each vertex of G'_k occurs at most once as the left endpoint of a tangent to each convex chain not containing p. Since there are at most $k+1$ such chains, there are at most $n(k+1)$ common tangents. Applying the crossing lemma 6.2, we can get the bound $O(nk^{1/3})$ for k-set problem.

7 k-sensitive bounds

Agarwal *et al.* [1] proposed a general method for proving the k-sensitive bounds on the complexity of k-level in an arrangement of curves or surfaces, given a k-insensitive bound. Basically, we take a random sample of about n/k of the curves/surfaces, compute their lower envelope, and construct the vertical decomposition of the region below the envelope. Within each cell τ of the decomposition, the k-th level of the whole arrangement coincides with the k-th level of the sub-arrangement formed by the curves/surfaces that cross τ. The number of these surfaces, is $O(k)$, following the standard probabilistic analysis of Clarkson and Shor [14]. Then we apply the k-insensitive bound on the complexity of the level within each τ, and multiply the bound by the number of cells of the decomposition. By doing this way, given a k-insensitive bound $O(n^{\alpha})$ for the complexity of the k-th level in an arrangement of n lines, we can get a k-sensitive bound of $O(nk^{\alpha-1})$. For line segments, since the lower envelope of n segments is bounded by $O(n\alpha(n))$ [3], we can get k-sensitive bound of $O(nk^{\alpha-1}\alpha(n/k))$. This method can be extended to high dimensions and is used to get k-sensitive bounds of the complexity of k-level in arrangements of line segments, planes, and triangles [1, 37].

8 Some extensions

Complexity of j consecutive levels Dey [16] applied the bound on the complexity of k-th level in determining the number of j consecutive levels, that is, the number of vertices on those j levels, $|\bigcup_{k\leq i\leq k-j+1} V_i|$. Consider the concave chains associated with each i-th level, for $k \leq i \leq k-j+1$. We are interested in counting the number of vertices of all concave chains. In the primal we consider the graph G containing all i-edges where $k \leq i \leq k-j+1$. The number of crossings among the edges of G is again determined by the number of common tangents among all pairs of concave chains. A common tangent between two chains c_1, c_2 is charged to the vertex v where c_1 and c_2 cross below the common tangent. The vertex v must lie below the k-th level. Observe that v may be charged for more than one pair of concave chains, however, it can't be charged by two pairs of chains coming from the same level. So a vertex can be charged for at most $O(j^2)$ times, which gives a $O(nkj^2)$ bound on the number of common tangents. Now applying the crossing lemma 6.2, we obtain that the number of edges in G is at most $O(nk^{1/3}j^{2/3})$.

Line segments Dey's technique [16] also applies to computing the complexity of the k-th level for line segments. Let $\mathcal{L}$ be a set of n line segments in $\mathbb{R}^2$ and $\mathcal{A}(\mathcal{L})$ denote the arrangement. The k-th level in $\mathcal{A}(\mathcal{L})$ is defined as the closure of all points on line segments, which have exactly k line segments strictly below them. Notice that the k-th level may not be continuous. However, each discontinuity can be charged to an endpoint of the line segments. So the number of such discontinuities is at most $2n$. Agarwal *et al.* [1] extended the concave chain structure for the case of line segments. A new chain is started (i) at the left endpoint of any segment if that point lies below the k-th level, and (ii) at any point of discontinuity of the level, when the level jumps up from a segment s_i to s_j. As x increases, each chain c follows the segment that it lies on, except the following cases. If c reaches the right endpoint of the segment, then c terminates there. If c reaches a discontinuity of the k-th level, where the level jumps down, in which case c is terminated at that point. If c reaches a vertex $v \in V_{k-1}$, in which case c bends to the right, and continues along the other segment incident to v. There are at most $2n$ concave chains which have the same properties as the concave chains in line arrangements. We extend the leftmost and rightmost edges of the concave chains to $-\infty$ and ∞ respectively. With this modification, the concave chains remain non-overlapping except possibly at their leftmost and rightmost infinite edges. Note that a vertex can be charged for multiple times by the pairs of chains whose infinite edges cross at v. But these charges accumulated over all pairs are at most $O(n^2)$. All other crossings between the concave chains are charged only once over all pairs. So the number of common tangents is at most $O(n^2)$. By performing the same analysis, we can get a bound $O(n^{4/3})$ on the k-level of the n line segments. Using the technique by Agarwal *et al.* [1], we can in addition get the k-sensitive bound $O(nk^{1/3}\alpha(n/k))$.

3-D k-set problem Lovász Lemma can also be extended to high dimensions [9, 1]. The extension of a k-edge is a k-facet. The extension of a double wedge is a corridor, i.e., the closed region lying between the upper and lower envelopes of the hyperplanes passing through a vertex of the arrangement. In d-dimensional case, both the number of k-facets intersecting a line l and the number of corridors containing a $(d-2)$-flat are bounded by $O(k^{d-1})$.

The Lovász Lemma for 3-space says any line intersects at most $O(k^2)$ k-triangles. This bound, combined with a crossing lemma for triangles, which says for any collection of t triangles spanned by the points of S there exists a line that crosses at least $\Omega(t^3/n^6)$ triangles [17], gives a bound $O(n^{8/3})$ on the number of k-triangles. Apply the technique of Agarwal *et al.*, we can get a k-sensitive bound $O(nk^{5/3})$ [17, 1]. Recently Sharir, Smorodinsky and Tardos [37] proved a bound $\Omega(n^{5/2})$ on the number of k-triangles spanned on n points, by exploring the antipodal property of the k-triangles. This immediately proves a bound $O(nk^{3/2})$ on the number of k-sets in 3-D.

Triangles in 3-D Agarwal *et al.* [1] proved a bound $O(n^2k^{7/9}\alpha(n/k))$ on the complexity of k-level in an arrangement of n triangles $T = \{\triangle_i\}$ in 3-D. Katoh and Tokuyama [31] improved it to $O((n-k)^{2/3}n^2)$. They showed an analogue of the concave chain structure in 3-dimension and proved a variant of the Lovász lemma, that is, there are $k(k+1)/2$ local maxima in the k-level of an arrangement of concave surfaces. The bound on the complexity of the k-level is then obtained by applying the crossing lemma for triangles in 3-space [17].

Other applications The k-set problem has more applications like convex polygons and matroid optimization [16], parametric minimum spanning tree [27], half-range search [13] etc.

9 Future work

The 2-D k-set problem remains open and challenging due to the big gap between the upper bound $O(nk^{1/3})$ and lower bound $O(ne^{\sqrt{\log n}})$. Here I include some of the possible directions of future research.

1 Motivated by the elegant proof by Dey on the upper bound case, we hope to get some ideas from extremal graph theory.

2 If we look at the lower bound construction by Erdös *et al.* [21] and Tóth [40], they follow the same idea of duplicating one point into a set of points that are very near to the original point, which generate more halving lines. In [21], they duplicate one point into two, each halving line is then duplicated into two, plus another n newly generated halving lines. In [40], one point is duplicated into a points, but with another 2 points, one can generate $2a$ halving lines. However, some additional points should be added to make it balanced. Assume we can place the a points properly so that between two set of a points, there are b halving lines, then we can generate a set of n points with $\Omega(n^{\log_a b}) = \Omega(n^{1+\epsilon})$ halving lines.

3 Instead of computing the exact k-level of an arrangement, whose complexity is not tightly bounded, we can use the approximate k-level [2, 22]. Define an ϵ-approximate k-level to be a x-monotone curve that is bounded by the $(k + \epsilon n)$-level and the $(k - \epsilon n)$-level. For arrangement of lines, we can find an ϵ-approximate k-level whose complexity is bounded by $O(1/\epsilon^2)$. Cases like arrangements of line segments, pseudo-lines and pseudo-algebraic arcs were also studied [22].

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