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Laguerre Minimal Surfaces, Isotropic Geometry and Linear Elasticity

Abstract Laguerre minimal (L-minimal) surfaces are the minimizers of the energy $\int (H^2 - K)/K dA$. They are a Laguerre geometric counterpart of Willmore surfaces, the minimizers of $\int (H^2 - K) dA$, which are known to be an entity of Möbius sphere geometry. The present paper provides a new and simple approach to L-minimal surfaces by showing that they appear as graphs of biharmonic functions in the isotropic model of Laguerre geometry. Therefore, L-minimal surfaces are equivalent to Airy stress surfaces of linear elasticity. In particular, there is a close relation between L-minimal surfaces of the spherical type, isotropic minimal surfaces (graphs of harmonic functions), and Euclidean minimal surfaces. This relation exhibits connections to geometrical optics. In this paper we also address and illustrate the computation of L-minimal surfaces via thin plate splines and numerical solutions of biharmonic equations. Finally, metric duality in isotropic space is used to derive an isotropic counterpart to L-minimal surfaces and certain Lie transforms of L-minimal surfaces in Euclidean space. The latter surfaces possess an optical interpretation as anticaustics of graph surfaces of biharmonic functions.

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Keywords differential geometry, Laguerre geometry, Laguerre minimal surface, isotropic geometry, linear elasticity, Airy stress function, biharmonic function, thin plate spline, geometrical optics.

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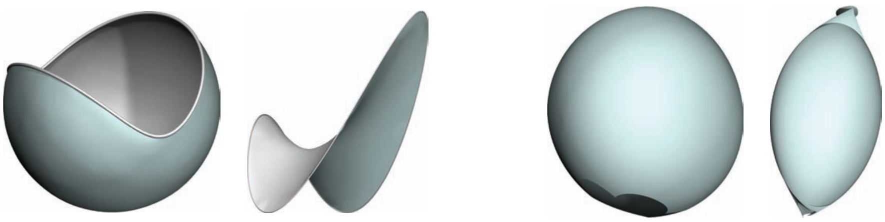

Fig. 1 Examples of Laguerre minimal surfaces created by solving the biharmonic boundary value problem (left), and by thin plate spline interpolation (right).

1 Introduction

The motivation for the present study comes from research on discrete freeform surfaces for architecture [G*02, Sch03, BP06, CW06, LPW*06, PBCW07, PLW*07]. The design and construction process favors certain discrete surface representations such as special quadrilateral meshes with planar faces. The most important classes of such polyhedral surfaces, namely conical meshes [LPW*06, PW07] and meshes with edge offsets [PLW*07] are actually objects of Laguerre sphere geometry. Aesthetics plays a crucial role in this application and therefore minimizers of geometric energies are of great interest. In view of these facts it is most natural to ask for minimizers of geometric energies which are invariant under Laguerre transformations. The simplest energy of this type has been introduced by Blaschke [Bla24, Bla25, Bla29]. Using mean curvature H, Gaussian curvature K, and the surface area element dA of a surface Φ in Euclidean 3-space E^3, this energy can be expressed as the surface integral:

$$\Omega = \int_{\Phi} (H^2 - K)/K dA. \tag{1}$$

Though the quantities H, K, A used for the definition are not objects of Laguerre geometry, the functional Ω and its local minimizers, known as *Laguerre-minimal surfaces* (L-minimal surfaces), are invariant under Laguerre transformations.

Our research goal is to find computational approaches for discrete L-minimal surfaces as a generalization of recent work on discrete Euclidean minimal surfaces [BP96, BHS06, PLW*07, WP07]. The present paper is to be considered as a first step towards reaching this goal. It links smooth L-minimal surfaces with well-known functions and surfaces of computational mathematics, namely graphs of biharmonic functions. These are also known as Airy stress surfaces of planar elastic systems [Str62]. The frequently used thin plate splines [Duc77] are biharmonic radial basis functions and thus also provide methods for the computation of L-minimal surfaces. The key for obtaining this new relation between L-minimal surfaces and biharmonic functions is provided by a special model of Laguerre geometry, which is formulated with help of so-called isotropic geometry. Before we briefly describe in Section 2 the classical geometric fundamentals such as the relation between Laguerre geometry and isotropic geometry, we sketch the state-of-the-art.

1.1 Previous work

Differential sphere geometry. Differential geometry in the three classical sphere geometries of Möbius, Laguerre and Lie, respectively, is the subject of Blaschke's third volume on differential geometry [Bla29]. For a more modern treatment we refer to Cecil [Cec92]. Here we focus on contributions to Laguerre-minimal surfaces. Many of them are found in the work of Blaschke [Bla24, Bla25, Bla29] and in papers by his student König [Kön26, Kön28]. Recently, this topic found again the interest of differential geometers. Musso and Nicolodi derived all L-minimal surfaces which are envelopes of a one-parameter family of spheres [MN95] and studied general L-minimal surfaces by the method of moving frames [MN96]. The stability of L-minimal surfaces has been analyzed by Palmer [Pal99]; he also showed that these surfaces are indeed local minimizers of (1). Willmore surfaces, the minimizers of $E = \int (H^2 - K)dA$ are a Möbius-geometric counterpart to L-minimal surfaces [Bla29]. These surfaces and gradient flows of E have found various applications in geometric computing; we refer to the discrete Willmore flow of Bobenko and Schröder [BS05] and the references therein. In fact, the present paper shows that L-minimal surfaces are essentially the isotropic analogues of Willmore surfaces (Sections 3.2 and 3.3).

Laguerre geometry is the geometry of oriented planes and spheres in Euclidean 3-space [Bla29, Cec92]. Its recent application in the study of discrete surfaces with applications to multi-layer freeform structures for architecture [BS07, LPW*06, PLW*07, WP07] is one of the motivations for the present work. Further applications of Laguerre geometry in geometric computing concern rational offsets and developable surfaces [PP98a, PP98b, Pet04, PL02].

Isotropic geometry has been systematically developed by Strubecker [Str41, Str42a, Str42b] in the 1940s. It is based on a simple semi-Riemannian metric and briefly surveyed in Section 2.1. There are numerous purely geometric papers on isotropic geometry, most of which are covered in the monograph by Sachs [Sac90]. Isotropic geometry naturally appears when properties of functions shall be geometrically visualized and interpreted at hand of their graph surfaces [PO94]. In particular, this holds for the visualization of stress properties in planar elastic systems at hand of their Airy surfaces [Str62], a topic addressed in Section 3.3. A remarkable application of isotropic geometry in Image Processing has been presented by Koenderink and van Doorn [KvD02]. Recently, Pottmann and Liu [PL07] studied discrete surfaces of isotropic geometry with applications in architectural design.

1.2 Contributions and Overview

The contributions of the present paper are as follows:

1. A brief overview of Laguerre geometry, isotropic geometry, and their relation is provided in Section 2.
2. In Section 3, we prove our main result: L-minimal surfaces appear as graphs of biharmonic functions in the isotropic model of Laguerre geometry. We also discuss generations of L-minimal surfaces via Minkowski sums.
3. Section 3.3 describes the relation to linear elasticity: L-minimal surfaces are equivalent to Airy stress surfaces, and isotropic Möbius transformations map Airy surfaces to Airy surfaces. Moreover, we interpret Airy surfaces as isotropic Willmore surfaces.
4. The relation between L-minimal surfaces of the spherical type, isotropic minimal surfaces (graphs of harmonic functions), and Euclidean minimal surfaces is described and illustrated by examples in Section 3.4. We also elaborate on the connections to geometrical optics.
5. The computation of L-minimal surfaces via thin plate splines and numerical solutions of the biharmonic equation are addressed in Section 4.
6. Finally, Section 5 uses the metric duality in isotropic space to derive an isotropic counterpart to Laguerre minimal surfaces and certain Lie transforms of L-minimal surfaces in E^3. Such surfaces are shown to be Euclidean anticaustics of graph surfaces $z = f(x, y)$ of biharmonic functions f for z-parallel light rays.

2 Fundamentals

2.1 Isotropic Geometry

Motions and metric. Isotropic geometry is based on the following group G_6 of affine transformations $(x, y, z) \mapsto (x', y', z')$ in $\mathbb{R}^3$,

$$\begin{aligned} x' &= a + x\cos\phi - y\sin\phi, \\ y' &= b + x\sin\phi + y\cos\phi, \\ z' &= c + c_1 x + c_2 y + z, \end{aligned} \tag{2}$$

which are called *isotropic congruence transformations (i-motions)*. We see that motions in isotropic space I^3 appear as Euclidean motions in the projection onto the xy-plane; the result of this projection $\mathbf{p} = (x, y, z) \mapsto \mathbf{p}' = (x, y, 0)$ is henceforth called the *top view*. Many metric properties in isotropic 3-space I^3 (invariants under G_6) are thus Euclidean invariants in the top view. For example, the *i-distance* of two points $\mathbf{x}_j = (x_j, y_j, z_j),\ j = 1, 2$, is defined as the Euclidean distance of their top views $\mathbf{x}_j'$,

$$\|\mathbf{x}_1 - \mathbf{x}_2\|_i := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{3}$$

Two points (x, y, z_j) with the same top view are called *parallel points*; they have i-distance zero, but they need not agree. Since the i-metric (3) degenerates along z-parallel lines, these are called *isotropic lines*. *Isotropic angles* between straight lines are measured as Euclidean angles in the top view.

Planes, circles and spheres. There are two types of planes in I^3.

(i) *Non-isotropic planes* are planes non-parallel to the z-direction. In these planes we basically have an Euclidean metric: This is not the one we are used to, since we have to make the usual Euclidean measurements in the top view. An *i-circle (of elliptic type)* in a non-isotropic plane P is an ellipse, whose top view is an Euclidean circle. Such an i-circle with center $\mathbf{m} \in P$ and radius r is the set of all points $\mathbf{x} \in P$ with $\|\mathbf{x} - \mathbf{m}\|_i = r$ (see Figure 2).

(ii) *Isotropic planes* are planes parallel to the z-axis. There, I^3 induces an isotropic metric. An *isotropic circle (of parabolic type)* is a parabola with z-parallel axis and thus it lies in an isotropic plane (see Figure 2).

An i-circle of parabolic type is not the iso-distance set of a fixed point, but it may be seen as a curve with constant isotropic curvature: A curve c in an isotropic plane P (w.l.o.g. we set $P: \ y = 0$) which does not possess isotropic tangents can be written as graph $z = f(x)$. Then, the isotropic curvature of c at $x = x_0$ is given by the second derivative $\kappa_i(x_0) = f''(x_0)$. For an i-circle of parabolic type f is quadratic and thus κ_i is constant.

There are also two types of *isotropic spheres*. An *i-sphere of the cylindrical type* is the set of all points $\mathbf{x} \in I^3$ with $\|\mathbf{x} - \mathbf{m}\|_i = r$. Speaking in an Euclidean way, such a sphere is a right circular cylinder with z-parallel rulings; its top view is the Euclidean circle with center $\mathbf{m}'$ and radius r. The more interesting and important type of spheres are the *i-spheres of parabolic type*,

$$z = \frac{A}{2}(x^2 + y^2) + Bx + Cy + D, \ A \neq 0. \tag{4}$$

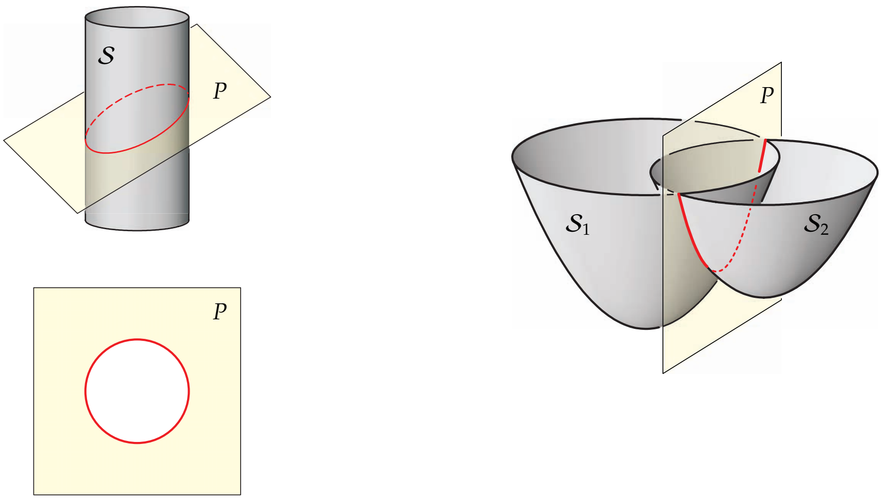

Fig. 2 (Left) An i-circle of elliptic type is the intersection curve of an i-sphere $\mathcal{S}$ of cylindrical type and a non-isotropic plane P. When viewed from the top, the i-circle is an Euclidean circle. (Right) An i-circle of parabolic type is a parabola with z-parallel axis. This curve appears as the intersection curve of two i-spheres, $\mathcal{S}_1$ and $\mathcal{S}_2$, of parabolic type. This is equivalent to the intersection curve of an i-sphere of parabolic type and an isotropic plane P. For more details, please refer to Section 2.

¿From an Euclidean perspective, they are paraboloids of revolution with z-parallel axis. The intersections of these i-spheres with planes P are i-circles. If P is non-isotropic, then the intersection is an i-circle of elliptic type. If P is isotropic, the intersection curve is an i-circle of parabolic type.

Curvature theory of surfaces. We confine our discussion to regular surfaces Φ without isotropic tangent planes. Thus, we may write Φ in explicit form,

$$\Phi: \; z = f(x, y). \tag{5}$$

Normal sections of Φ at a point $\mathbf{p}$ may be defined as intersections with isotropic planes through $\mathbf{p} = (x_0, y_0, f(x_0, y_0))$. Their isotropic curvatures at $\mathbf{p}$ are the second directional derivatives of f at (x_0, y_0). Hence, if $\mathbf{r} = (r_1, r_2)$ is a unit vector, the isotropic normal curvature in direction $\mathbf{r}$ is

$$\kappa_n(\mathbf{r}) = \mathbf{r}^T \cdot \nabla^2 f \cdot \mathbf{r}, \tag{6}$$

where $\nabla^2 f$ denotes the Hessian of f,

$$\nabla^2 f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

The extremal values κ_1, κ_2 of $\kappa_n(\mathbf{r})$ are given by the eigenvalues of $\nabla^2 f$ and called *i- principal curvatures*. The corresponding orthogonal directions $\mathbf{r}_1, \mathbf{r}_2$ are the eigenvectors of $\nabla^2 f$. They are the top views of conjugate surface tangents of Φ at $\mathbf{p}$, which are called *i-principal directions*. The integral curves of the field of principal directions are called *isotropic principal curvature lines*. They constitute exactly that conjugate curve network on S which appears as orthogonal network in the top view.

With the i-principal curvatures κ_1, κ_2 one defines *isotropic curvature* (or *relative curvature*)

$$K = \kappa_1 \kappa_2 = \det(\nabla^2 f) = f_{xx} f_{yy} - f_{xy}^2, \tag{7}$$

and *isotropic mean curvature H*,

$$2H = \kappa_1 + \kappa_2 = \operatorname{trace}(\nabla^2 f) = f_{xx} + f_{yy} = \Delta f. \tag{8}$$

Isotropic minimal surfaces are characterized by $H = 0$, and thus they are graphs of harmonic functions f ($\Delta f = 0$). They possess many properties which are analogous to their Euclidean counterparts [Str76, Str77, Sac90]. Later in Section 3.4, we will relate i-minimal surfaces to Euclidean minimal surfaces, and to special L-minimal surfaces.

Metric duality. Isotropic geometry enjoys a metric duality which may be realized by the polarity with respect to the isotropic unit sphere,

$$\Sigma : z = \frac{1}{2}(x^2 + y^2). \tag{9}$$

It maps a point $\mathbf{p} = (p_1, p_2, p_3)$ to the plane P with equation $z = p_1 x + p_2 y - p_3$. Points $\mathbf{p}, \mathbf{q}$ with i-distance d are mapped to planes P, Q with i-angle d and vice versa. Parallel points correspond in the duality to parallel planes. A surface $\Phi : z = f(x,y)$, seen as set of contact elements (points plus tangent planes) corresponds to a surface Φ^*, parameterized by

$$x^* = f_x(x,y),\ y^* = f_y(x,y),\ z^* = xf_x + yf_y - f. \tag{10}$$

Contact elements along i-principal curvature lines of Φ and Φ^* correspond in the duality. Note that Φ^* may have singularities which correspond to parabolic surface points of Φ ($K = 0$). This is reflected in the following relations between the isotropic curvature measures of dual surface pairs [Str78]:

$$H^* = H/K \qquad K^* = 1/K. \tag{11}$$

Thus, the dual surface to an i-minimal surface is also i-minimal. For further properties of the metric duality, see [Sac90].

Isotropic sphere transformations. In I^3, there exists a counterpart to Möbius geometry which is of great importance for the present paper. One puts *i-*spheres of parabolic type and non-isotropic planes into the same class $\mathcal{S}$ of *isotropic Möbius spheres*; they are given by (4), including $A = 0$. One adds a single ideal point ∞ to $\mathbb{R}^3$ to get the point set of Euclidean Möbius geometry. In *isotropic Möbius (i-M) geometry*, we add to $\mathbb{R}^3$ infinitely many ideal points, namely the set $\mathbb{R}$. Each isotropic sphere with equation (4) is extended by the "point" $A \in \mathbb{R}$ (which equals the constant i-mean curvature H of that sphere). We denote the extended point set $I^3 \cup \mathbb{R}$ by $\mathcal{P}$. An intersection curve of two i-M-spheres is called an *i-M-circle*; it may be an i-circle of elliptic or parabolic type, or a non-isotropic straight line. There exists a group M^i_{10} of *isotropic Möbius (i-M) transformations* which acts bijectively in $\mathcal{P}$, in the set $\mathcal{S}$ of i-M-spheres and in the set $\mathcal{C}$ of i-M-circles. The 10-dimensional group M^i_{10} is isomorphic to the group of Euclidean Laguerre transformations (see subsection 2.2). The top view of an i-M-transform is a planar Euclidean Möbius transformation. The basic i-M-transforms which generate the whole group are inversions with respect to i-spheres. The inversion (reflection) at an i-M-sphere $S : z = A(x^2 + y^2)/2 + \ldots =: s(x,y)$ is given by $(x,y,z) \mapsto (x,y,2s(x,y) - z)$. The top view remains unchanged, while in the z-direction we have a reflection at the corresponding point of S. An inversion with respect to an i-sphere S of cylindrical type, for brevity say $S :\ x^2 + y^2 = 1$, is defined as

$$(x,y,z) \mapsto (x,y,z)/(x^2 + y^2). \tag{12}$$

In the top view it appears as ordinary inversion with respect to a Euclidean circle, but also in I^3 corresponding points lie collinear with the inversion center (here $(0,0,0)$).

2.2 Laguerre Geometry

Laguerre geometry is the geometry of oriented planes and oriented spheres in Euclidean E^3 [Bla29, Cec92]. We may write an oriented plane P in Hesse normal form as $\mathbf{n}^T \cdot \mathbf{x} + h = 0$, where the unit normal vector $\mathbf{n}$ defines the orientation; $\mathbf{n}^T \cdot \mathbf{x} + h$ is the signed distance of the point $\mathbf{x}$ to P. An oriented sphere S, with center $\mathbf{m}$ and signed radius R, is tangent to an oriented plane P if the signed distance of $\mathbf{m}$ to P equals R, i.e. $\mathbf{n}^T \cdot \mathbf{m} + h = R$. Points are viewed as oriented spheres with radius zero. A *Laguerre transformation (L-transform)* is a mapping which is bijective on the sets of oriented planes and oriented spheres, respectively, and preserves tangency between plane and sphere.

L-transforms are more easily understood if we use the so-called *cyclographic model* of Laguerre geometry. There, an oriented sphere S is represented as point $\mathbf{S} := (\mathbf{m}, R) \in \mathbb{R}^4$. An oriented plane P in E^3 may be interpreted as the set of all oriented spheres which are tangent to P. Mapping P via this set of spheres into $\mathbb{R}^4$, one finds a hyperplane in $\mathbb{R}^4$ which is parallel to a tangent hyperplane of the cone $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$. In the cyclographic model, an L-transform is seen as a special affine map (Lorentz transformation),

$$\mathbf{S}' = \mathbf{a} + L \cdot \mathbf{S}, \tag{13}$$

where L denotes the matrix of a linear map $\mathbb{R}^4 \to \mathbb{R}^4$ which preserves the inner product,

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4. \tag{14}$$

With the diagonal matrix $D := diag(1, 1, 1, -1)$ we have $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \cdot D \cdot \mathbf{y}$, and the condition on L reads:

$$L^T \cdot D \cdot L = D. \tag{15}$$

Let us return to the standard model in E^3. A pencil of parallel oriented planes has the same normal vector $\mathbf{n}$ (image point on the "Gaussian" sphere S^2). An L-transform keeps the parallelity of oriented planes and induces a Möbius transformation of the Gaussian sphere S^2. Note that, in general, a L-transform does not preserve points, since those are seen as special spheres and may be mapped to other spheres. A simple example of an L-transform is the offset operation (given by equation (13), with L as identity matrix and $\mathbf{a} = (0, 0, 0, d)$), which adds a constant d to the radius of each sphere.

The isotropic model of Laguerre geometry. The following remarkable relation between Laguerre geometry and isotropic Möbius geometry is central for our study. We may use $(\mathbf{n}, h)$ as coordinates of an oriented plane P. However, these four coordinates are not independent due to $\|\mathbf{n}\| = 1$. Thus, one replaces $\mathbf{n} = (n_1, n_2, n_3)$ (point of S^2) by its image point $(n_1/(n_3 + 1), n_2/(n_3 + 1), 0)$ under the stereographic projection of S^2 from $(0, 0, -1)$ onto $z = 0$. It is then convenient (for a more geometric explanation please see [PP98b]) to view the point,

$$P^i := \frac{1}{n_3 + 1}(n_1, n_2, h) \tag{16}$$

as image of the oriented plane $P : n_1 x + n_2 y + n_3 z + h = 0$. For a plane P with normal vector $(0, 0, -1)$, P^i in (16) is not defined; in this case we define the real number $h \in \mathcal{P}$ (ideal point) as image. In this way, the set of oriented planes in E^3 is mapped to the point set $\mathcal{P}$ of isotropic Möbius geometry. One speaks of the *isotropic model of Laguerre geometry*.

Parallel oriented planes P and Q appear in the isotropic model as parallel points P^i and Q^i, respectively. The oriented tangent planes of an oriented sphere S with center $\mathbf{m} = (m_1, m_2, m_3)$ and radius R are seen in the isotropic model as points of an isotropic Möbius sphere S^i:

$$S^i : \; z = \frac{R + m_3}{2}(x^2 + y^2) - m_1 x - m_2 y + \frac{R - m_3}{2}. \tag{17}$$

The common tangent planes of two oriented spheres (tangent planes of an oriented cone of revolution or a limit case of it) correspond to the common points of two i-M-spheres (= i-M-circle). A non-developable surface Φ viewed as set of oriented tangent planes is mapped to a surface Φ^i in the isotropic model. Tangent planes along an Euclidean principal curvature line of Φ are mapped to points of an isotropic principal curvature line of Φ^i. Underlying all these facts is the central result that a L-transform corresponds to an i-M-transform in the isotropic model. Hence, the groups of L-transforms and i-M-transforms are isomorphic.

Bonnet coordinates. Blaschke [Bla24, Bla29] used coordinates of Bonnet in his study of Laguerre minimal surfaces. These complex coordinates (u, v, w) of an oriented plane P are closely related to the representation $P^i = (x, y, z)$ of P in the isotropic model, a fact which we did not find in the literature:

$$(x, y, z) = \frac{1}{2}(u + v, i(v - u), w) \quad \text{and} \quad (u, v, w) = (x + iy, x - iy, 2z). \tag{18}$$

Real planes correspond to conjugate complex (u, v). The first two Bonnet coordinates (u, v) define the unit normal vector $\mathbf{n}$ of the plane P via

$$(n_1, n_2, n_3) = \frac{1}{1 + uv}(u + v, i(v - u), 1 - uv). \tag{19}$$

Compared to Bonnet coordinates, the isotropic model has the advantage of being defined over $\mathbb{R}$, and still provides a very simple approach to Laguerre minimal surfaces; this will be seen in the next section.

3 L-minimal surfaces in the isotropic model of Laguerre geometry

3.1 L-minimal surfaces

Laguerre transformations map oriented planes to oriented planes, but they do not preserve points. Thus, in Laguerre geometry one views a surface Φ as set of its oriented tangent planes. We will use oriented *contact elements*, defined by the pairs $(\mathbf{p}, T)$ with $\mathbf{p} \in \Phi$ and T as oriented tangent plane of Φ at $\mathbf{p}$; but we view the surface element as the set of all oriented spheres which touch T at $\mathbf{p}$. Clearly these spheres are centered at the surface normal. An L-transform maps such a parabolic pencil of spheres again to a parabolic pencil of spheres. In this way L-transforms act on contact elements. A surface defined as set of contact elements is called a *Legendre surface*. In the following, surfaces in E^3 will be Legendre surfaces unless noted otherwise.

The parabolic pencil of spheres defined by a contact element $(\mathbf{p}, T)$ contains the two *principal spheres* Π_1 and Π_2, whose signed radii are the principal curvature radii $R_i = 1/\kappa_i$, and whose centers are the principal curvature centers $\mathbf{m}_i = \mathbf{p} + R_i\mathbf{n}$. These principal spheres are L-invariant, while their radii are not. The *middle sphere* of Blaschke [Bla29] is also an element of the pencil, and defined as the oriented sphere with center $\mathbf{m} = (\mathbf{m}_1 + \mathbf{m}_2)/2$ and radius $R = (R_1 + R_2)/2 = H/K$. The middle sphere is L-invariant. This is most easily seen in the cyclographic model, where it appears as midpoint $\mathbf{M}$ of the two points $\mathbf{M}_i = (\mathbf{m}_i, R_i) \in \mathbb{R}^4$ representing the principal spheres. An L-transform is a special affine map in $\mathbb{R}^4$, and thus it preserves midpoints.

A surface Φ can be mapped to the surface $\Phi_M \subset \mathbb{R}^4$ which is formed by all points $\mathbf{M}$ corresponding to the middle spheres of Φ. The *L-Gauss image* of Φ is Φ_M. L-differential geometry of Φ is the differential geometry of the L-Gauss image in 4-dimensional Minkowski space $\mathbb{R}^4_1$, which is based on the inner product (14).

It is natural to define an *L-minimal surface* as a surface Φ whose L-Gauss image Φ_M is a local minimizer of the area functional Ω in $\mathbb{R}^4_1$. According to Blaschke [Bla29], the area functional is given by (1) in terms of Euclidean invariants of Φ. Using the principal radii R_i and the surface element $dA_s = KdA$ of the Gaussian spherical image, it may also be written as:

$$\Omega = \frac{1}{4}\int (R_1 - R_2)^2 dA_s. \tag{20}$$

Thus, Ω measures the deviation from a sphere. If Φ's tangent planes are parameterized by their Bonnet coordinates (u, v, w) via a function $w(u, v)$, Ω takes the following simple form [Bla29]:

$$\Omega = \frac{i}{2}\int\int w_{uu}w_{vv}dudv. \tag{21}$$

Expressing vanishing first variation leads to the following Euler-Lagrange equation which characterizes L-minimal surfaces,

$$w_{uuvv} = \frac{\partial^4 w}{\partial^2 u \partial^2 v} = 0. \tag{22}$$

An Euclidean minimal surface Φ is also L-minimal, which is easily seen as follows: Because of $R_1 + R_2 = 0$, the middle spheres are the surface points $\mathbf{p}$ and thus Φ_M agrees with $\Phi \in E^3$. In E^3, $\mathbb{R}^4_1$ induces the canonical Euclidean metric. Hence, the surface area Ω is the Euclidean one, which proves the L-minimality of Φ. By the L-invariance, any L-transform and in particular any offset of an Euclidean minimal surface is also an L-minimal surface.

3.2 L-minimal surfaces in the isotropic model

After all these preparations it is very easy to derive those surfaces Φ^i which represent L-minimal surfaces Φ in the isotropic model of Laguerre geometry. We have to use relations (18) between Bonnet coordinates (u, v, w) of a plane and its image point $P^i = (x, y, z)$ in the isotropic model. We set

$$2z = w(u, v) = w(x + iy, x - iy) = 2f(x, y).$$

With $\partial x/\partial u = \partial x/\partial v = 1/2$ and $\partial y/\partial u = -\partial y/\partial v = -i/2$, we obtain $w_u = f_x - if_y$, $w_v = f_x + if_y$ and

$$w_{uu} = \frac{1}{2}(f_{xx} - f_{yy} - 2if_{xy}),\ w_{vv} = \frac{1}{2}(f_{xx} - f_{yy} + 2if_{xy}).$$

Hence the integrand in (21) equals

$$w_{uu}w_{vv} = \frac{1}{4}[(f_{xx} - f_{yy})^2 + 4f_{xy}^2] = H_i^2 - K_i. \tag{23}$$

Here $H_i = \Delta f/2$ and $K_i = f_{xx}f_{yy} - f_{xy}^2$ denote the isotropic curvatures of the surface $\Phi^i : z = f(x, y)$ which represents Φ in the isotropic model.

Noting $dudv = -2idxdy = -2idA_i$ with dA_i as isotropic area element, we finally obtain the functional Ω expressed in terms of the isotropic geometry of Φ^i,

$$\Omega = \int (H_i^2 - K_i)dA_i = \frac{1}{4}\int\int [(f_{xx} - f_{yy})^2 + 4f_{xy}^2]dxdy. \tag{24}$$

This is the isotropic counterpart of the Willmore energy $E = \int(H^2 - K)dA$ of (Euclidean) Möbius geometry. Just as E is invariant under Möbius transforms, Ω is invariant under isotropic Möbius transforms, since these represent L-transforms in the isotropic model. In the isotropic model, the Euler-Lagrange equation (22) of L-minimal surfaces is the biharmonic equation,

$$\Delta^2 f = \Delta(\Delta(f)) = 0. \tag{25}$$

Therefore, the surface $\Phi^i : z = f(x, y)$ is the graph of a biharmonic function f. Let us summarize our results:

Theorem 1 *A Laguerre minimal surface Φ, defined as minimizer of the energy (1), appears in the isotropic model of Laguerre geometry as minimizer Φ^i of the isotropic Willmore energy (24). The surface Φ^i is the graph $z = f(x, y)$ of a biharmonic function f ($\Delta^2 f = 0$). Isotropic Möbius transformations map graphs of biharmonic functions onto graphs of biharmonic functions.*

The latter invariance result about biharmonic functions can be proved directly, but the computations are quite involved. It is the isotropic counterpart to the Möbius invariance of Willmore surfaces.

An analytical expression for the transfer from the isotropic model Φ^i to the Laguerre model $\Phi \in E^3$ can be derived easily, and reads as follows:

Corollary 2 *Let Φ^i be the graph surface of the biharmonic function f. Then the corresponding Laguerre minimal surface Φ is given by*

$$\frac{1}{x^2+y^2+1}\begin{pmatrix}(x^2-y^2-1)f_x+2xyf_y-2xf\\(y^2-x^2-1)f_y+2xyf_x-2yf\\2xf_x+2yf_y-2f\end{pmatrix}. \tag{26}$$

Given two biharmonic functions f_1 and f_2, any linear combination $f = \lambda_1 f_1 + \lambda_2 f_2$ is also biharmonic. Three parallel points $P^i_j = (x, y, f_j(x, y))$, j=1,2, and $P^i = (x, y, f(x, y))$ on their graph surfaces correspond to three parallel oriented planes P_1, P_2, P in E^3, whose signed distances h_1, h_2, h to the origin (support functions) satisfy the same linear combination $h = \lambda_1 h_1 + \lambda_2 h_2$; this follows from the relation $z = h/(n_3 + 1)$ in (16). The planes P_1, P_2, P are tangent to three L-minimal surfaces Φ_1, Φ_2 and Φ, respectively. If we have such a relation between parallel oriented tangent planes of surfaces, we speak of a *Minkowski linear combination* $\Phi = \lambda_1\Phi_1 \oplus \lambda_2\Phi_2$. Since multiplication of h with a constant describes a uniform scaling, the essential operation here is the sum $h = h_1 + h_2$, which describes the *Minkowski sum* or *convolution surface* $\Phi = \Phi_1 \oplus \Phi_2$ of the surfaces with support functions h_1 and h_2. Convolution curves and surfaces have been studied recently from various perspectives [FH07, SPJ06, PS07]. With $h = (h_1 + h_2)/2$, we get a description of convolution which is independent of the choice of the origin: For any two parallel oriented tangent planes T_1 and T_2 of Φ_1 and Φ_2, respectively, take the middle plane T; then all such middle planes envelope the convolution surface Φ.

Corollary 3 *Convolution surfaces of L-minimal surfaces are L-minimal.*

This generalizes a theorem by Weierstrass which states that the convolution surface of two minimal surfaces is a minimal surface. A sphere is L-minimal and offsetting means convolution with a sphere. Hence Corollary 3 also contains the known result that the offsets of minimal surfaces are L-minimal (special case of the L-invariance of L-minimal surfaces).

A biharmonic function f can be represented by two harmonic functions f_1, f_2 or g_1, g_2 as:

$$f(x, y) = xf_1(x, y) + yf_2(x, y),$$

or

$$f(x, y) = (x^2 + y^2 - c)g_1(x, y) + g_2(x, y).$$

Just as harmonic functions f_i may have branching points, and thus $z = f_i(x, y)$ can have several sheets (Riemann surface), the same holds for biharmonic functions. We understand a "graph of a function" in this general sense. For the corresponding L-minimal surfaces in E^3 this says that their Gauss mapping needs not be injective.

3.3 Airy surfaces

The solution of linear elastostatic boundary value problems in the (x, y)-plane may be based on Airy stress functions $f(x, y)$; these functions are solutions of the biharmonic equation (25). Strubecker [Str62] has shown that their graph surfaces $\Phi^i : z = f(x, y)$, also referred to as *Airy surfaces*, are most naturally treated in I^3 based on the motion group (2). First of all, Airy stress functions are only determined up to a linear part, which by (2) corresponds to the application of an i-motion to the Airy surface Φ^i. Isotropic normal curvatures of Φ^i are identical with the normal stresses of the underlying elastostatic state. Hence, the isotropic principal curvature directions correspond to the orthogonal directions of principal stresses. The top views of isotropic principal curvature lines on Φ^i are therefore the principal stress lines of the planar elastostatic state. Further equivalences concern isotropic geodesic torsion and shear stresses.

Theorem 1 provides a new link between Laguerre minimal surfaces and Airy surfaces, which may be summarized as follows:

Corollary 4 *An Airy surface of a planar elastostatic state may be interpreted as isotropic Willmore surface and as representation of a Laguerre minimal surface in the isotropic model of Laguerre geometry. Therefore, Airy surfaces are invariant under isotropic Möbius transformations.*

The simplification of problems in linear elasticity by conformal mappings, in particular 2D Möbius transformations, is well known. The authors are not experts in elasticity, but they did not find the fact that Airy surfaces just change via an isotropic Möbius transformation if an Euclidean Möbius transformation is applied to a problem of planar linear elasticity.

In the following, we will use geometric interpretations only, but one should keep in mind that any computational or geometric problem for L-minimal surfaces has a meaning in terms of linear elasticity.

3.4 L-minimal surfaces of the spherical type and isotropic minimal surfaces

We will now characterize those L-minimal surfaces $\Phi \subset E^3$ whose corresponding surfaces $\Phi^i \in I^3$ are graphs of harmonic functions, i.e., isotropic minimal surfaces.

As a preparation, we present some formulae for the middle spheres of a surface Φ, which can be derived with (18) from results by Blaschke [Bla29]. If we know the image $\Phi^i : z = f(x, y)$ of Φ, then the set of middle spheres $M = (m_1, m_2, m_3, R)$ of Φ can be parameterized as:

$$
\begin{aligned}
m_1 &= \frac{x}{2}\Delta f - f_x, \quad m_2 = \frac{y}{2}\Delta f - f_y, \\
m_3 &= \frac{1 - x^2 - y^2}{4}\Delta f + x f_x + y f_y - f, \\
R &= \frac{1 + x^2 + y^2}{4}\Delta f - x f_x - y f_y + f. \qquad (27)
\end{aligned}
$$

In the isotropic model I^3, the middle sphere M to parameters (x_0, y_0) is the i-M-sphere M^i,

$$z = f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) + \frac{1}{4}[(x - x_0)^2 + (y - y_0)^2] \Delta f(x_0, y_0). \tag{28}$$

A surface $\Phi \subset E^3$ all whose middle spheres are tangent to a fixed oriented plane, is called a *Laguerre minimal surface of the spherical type*. We can now prove a new characterization of these surfaces with help of the isotropic model:

Theorem 5 *A surface $\Phi \subset E^3$ whose middle spheres are tangent to an oriented plane Π (L-minimal surface of the spherical type) appears in the isotropic model as isotropic minimal surface Φ^i (graph of a harmonic function), if one sets up the transfer to I^3 such that Π gets mapped to the ideal point 0. Conversely, any isotropic minimal surface $\Phi^i \subset I^3$ determines an L-minimal surface $\Phi \subset E^3$.*

Proof We introduce a coordinate system, such that the oriented plane Π is $(n_1, n_2, n_3, h) = (0, 0, -1, 0)$ (xy-plane with the orientation $(0, 0, -1)$). It gets mapped into the ideal point 0 of the isotropic model. Spheres M which touches Π satisfy $m_3 + R = 0$. By equation (17), the corresponding i-M-spheres $M^i \subset I^3$ are planes. According to equation (28), this happens when $\Delta f = 0$, and thus Φ^i is an i-minimal surface. Conversely, an i-minimal surface Φ^i ($\Delta f = 0$) determines planes M^i (tangent planes of Φ^i) as the images of middle spheres M. Hence, the spheres M are tangent to Π and Φ is L-minimal of the spherical type. □

A more geometric argument uses the fact that the principal spheres of $\Phi \subset E^3$ correspond to the isotropic principal spheres of $\Phi^i \subset I^3$. If the latter are $z = f_1(x, y)$ and $z = f_2(x, y)$, the i-image of the middle sphere M must be $z = (f_1 + f_2)/2$. The two isotropic principal spheres of an i-minimal surface $\Phi^i : z = f(x, y)$ at a point $\mathbf{p}_0 = (x_0, y_0, f(x_0, y_0))$ are symmetric with respect to the tangent plane at $\mathbf{p}_0$ (this is an isotropic counterpart of an equivalent property of the principal spheres of an Euclidean minimal surface). Therefore, the tangent plane of Φ^i corresponds to the middle sphere M of Φ. This shows that all middle spheres M are tangent to Π. Clearly, any image Φ^i of an L-minimal surface under an i-M-transform also determines an L-minimal surface Φ of the spherical type.

Further there is a closely related connection between L-minimal surfaces of the spherical type and i-minimal surfaces:

Theorem 6 *The centers of the middle spheres M of an L-minimal surface Φ of the spherical type form an isotropic minimal surface Φ^*; the isotropic lines of the isotropic space are orthogonal to the plane Π which is tangent to all middle spheres M. Therefore, Φ is an anticaustic of Φ^* for light rays orthogonal to Π. Any Euclidean anticaustic of an isotropic minimal surface for light rays in isotropic direction is a L-minimal surface of the spherical type.*

Proof We use the same coordinate system as in the proof of Theorem 5 and therefore, the surface Φ^i is the graph of a harmonic function f. Using $\Delta f = 0$, equation (27) yields the following parametrization of the midpoint set $\mathbf{m}(x, y)$ of Φ's middle spheres,

$$\mathbf{m}(x, y) = (-f_x, -f_y, xf_x + yf_y - f). \tag{29}$$

Equation (10) shows that this surface Φ^* results from the dual surface of the isotropic minimal surface $\Phi^i : z = f(x, y)$ by a reflection at the z-axis. Hence, Φ^* is also i-minimal (cf. equation (11)). Since Φ is the second envelope of all spheres M which are centered at Φ^* and tangent to Π, its normals are obtained by reflecting the lines orthogonal to Π (in our coordinate system, z-parallel) at the surface Φ^* and therefore Φ has to be seen as an anticaustic of Φ^*. The other anticaustics are the offsets of Φ and therefore also L-minimal surfaces of the spherical type. □

We will say more about geometrical optics and its relation to metric duality in I^3 in Section 5, where we will present a generalization of the present result (Theorem 11).

The envelope Φ_E of bisecting planes of the pairs of principal curvature centers of a surface Φ is called the *middle evolute*. A theorem by Kommerell [Kom11] states that *Φ is L-minimal if and only if the middle evolute Φ_E is an Euclidean minimal surface*. The bisecting plane of the two principal curvature centers $\mathbf{m}_1, \mathbf{m}_2$ of $\mathbf{p} \in \Phi$ passes through the center $\mathbf{m}$ of the middle sphere and is orthogonal to $\mathbf{p} - \mathbf{m}$. For a surface Φ of Theorem 6, $\mathbf{m}$ lies on an isotropic minimal surface and $\mathbf{p} - \mathbf{m}$ lies on the ray obtain by reflecting the isotropic line at Φ^*. This yields the following construction of Euclidean minimal surfaces from isotropic ones, which we did not find in the literature:

Corollary 7 *Let Φ^* be an isotropic minimal surface, i.e., the graph $z = f^*(x, y)$ of a harmonic function f^*. For any point $\mathbf{m} \in \Phi^*$, reflect the xy-parallel plane through $\mathbf{m}$ at the tangent plane of Φ^* at $\mathbf{m}$. Then, the reflected planes envelope an Euclidean minimal surface Φ_E.*

An explicit analytic expression for this construction is complicated, and not very insightful. However, if the dual surface is parameterized as a graph of a harmonic function f, the Euclidean minimal surface Φ_E can be computed as follows: For a given Laguerre minimal surface $\Phi \subset \mathbb{R}^3$ the middle evolute can be computed from its isotropic model $\Phi^i \subset I^3$ using (27), and the fact that Φ^i is the graph of a harmonic function f. We obtain the following expression for the middle evolute of Φ:

$$\begin{pmatrix} -f_x - \frac{1}{2}(y - 3x^2y + y^3)f_{xy} + \frac{1}{2}(-x + x^3 - 3xy^2)f_{xx} \\ -f_y - \frac{1}{2}(x - 3xy^2 + x^3)f_{xy} - \frac{1}{2}(-y + y^3 - 3x^2y)f_{xx} \\ 2xyf_{xy} + xf_x + yf_y + (x^2 - y^2)f_{xx} - f \end{pmatrix}. \tag{30}$$

To get an analytical expression for the construction in Corollary 7, we have to find the isotropic representation of the Laguerre minimal surface whose midpoint set is Φ^*. From the proof of Theorem 7, it follows that this surface corresponds to the dual surface Φ of Φ^*, reflected at the z-axis.

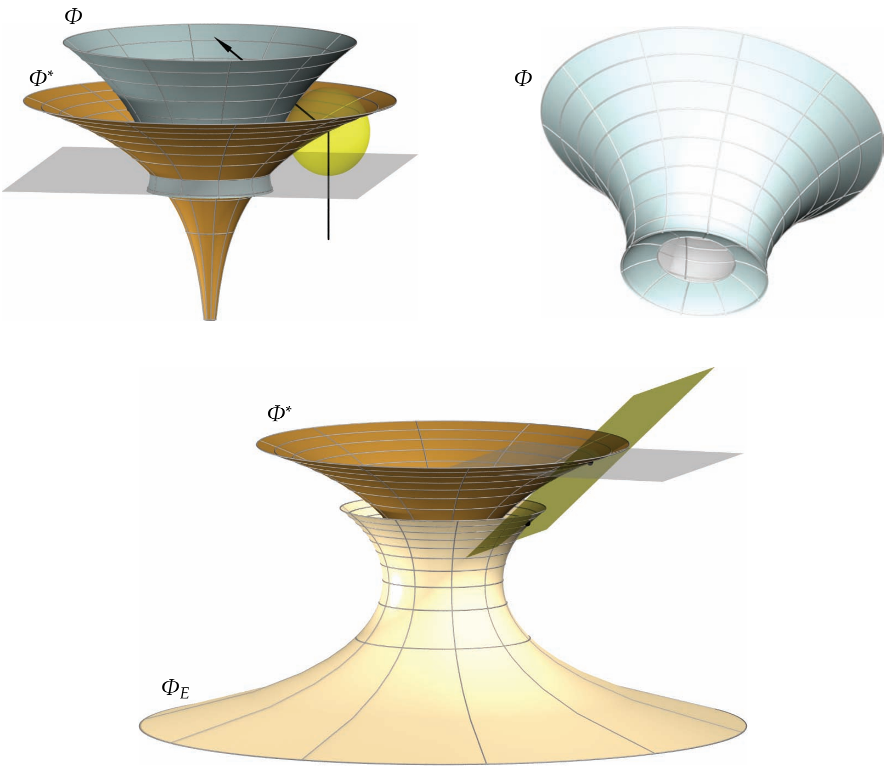

Fig. 3 (Top) Rotational L-minimal surface Φ of the spherical type as anticaustic of the logarithmoid (i-minimal surface) Φ^*. (Bottom) The middle evolute Φ_E of Φ is a catenoid (rotational Euclidean minimal surface) and can be directly derived from Φ^* by the reflection procedure described in Corollary 7.

Corollary 8 *Let Φ^* be an isotropic minimal surface (graph of the harmonic function $f^*(x, y)$) and Φ^i be the reflection at the z-axis of its dual surface (graph of the harmonic function $f(x, y)$). Then Φ^i is the isotropic representation of the anticaustic Φ of Φ^* for light rays in isotropic direction. The middle evolute Φ_E of Φ is given by (30) and is the Euclidean minimal surface derived from Φ^* with the reflection procedure described in Corollary 7.*

Example 1 Let us discuss *rotational L-minimal surfaces*. To obtain those of the spherical type, we note that the plane Π must be orthogonal to the rotational axis A, and hence we use the coordinate system from above with A as the z-axis. The centers of middle spheres form a rotational i-minimal surface Φ^*. The latter surface is well-known and easily found via polar coordinates (r, ϕ): We set $\Phi^* : \ z = g(r)$, use the polar representation of the Laplace operator,

$$\Delta = \frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + r^{-2}\frac{\partial^2}{\partial \phi^2}, \tag{31}$$

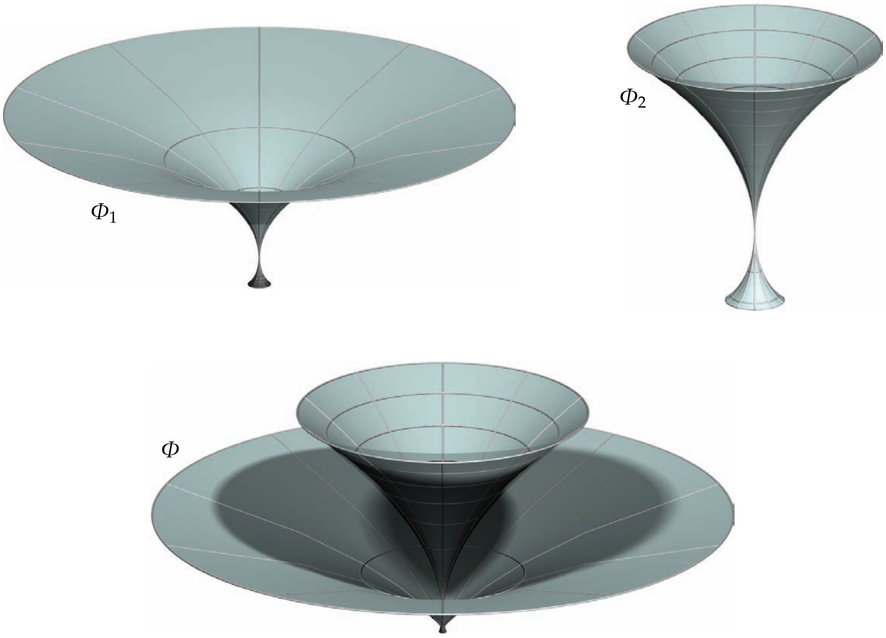

Fig. 4 A general rotational L-minimal surface Φ is the convolution surface of two *L*-minimal surfaces Φ_1, Φ_2 of the spherical type.

and solve the resulting equation $\Delta g = g_{rr} + r^{-1} g_r = 0$,

$$g(r) = c_1 \ln r + c_2. \tag{32}$$

Let us set $c_1 = 1, c_2 = 0$ w.o.l.g. Hence, the i-minimal surface of revolution Φ^* is obtained by rotating a logarithmic curve $z = \ln r$ about the z-axis and therefore called a *logarithmoid* (see Fig. 3).

To obtain Φ from Φ^*, we compute the envelope of the middle spheres, which are parameterized as $(\mathbf{m}, R) = (r\cos\phi, r\sin\phi, \ln r, -\ln r)$. This can again be done by restricting to a plane through the z-axis. The envelope consists of $\Pi : \ z = 0$ and the desired surface Φ, whose profile in the (x, z)-plane is parameterized via the polar radius r of Φ^* as

$$(x, z) = \left(r - \frac{2r\ln r}{r^2 + 1}, \frac{2r^2 \ln r}{r^2 + 1} \right).$$

The parameter change $t = \ln r$ yields the representation [MN95],

$$(x, z) = \frac{1}{e^t + e^{-t}} (1 - 2t + e^{2t}, 2te^t). \tag{33}$$

Note that this curve is an anticaustic of the logarithmic curve for light rays parallel to the z-axis (Fig. 3).

The reflection procedure from Corollary 7 transforms Φ^* into a rotational Euclidean minimal surface Φ_E, the well-known catenoid. The analytic verification is simple and yields the catenary $x = \cosh(z+1)$ as profile curve.

To determine *general rotational L-minimal surfaces*, we use the isotropic model. One can keep rotational symmetry, and thus we have to find rotational surfaces $\Phi^i: z = G(r)$ in I^3 which are graphs of biharmonic functions. The solution of $\Delta^2 G = 0$ is

$$G(r) = (c_1 + c_2 r^2)\ln r + c_3 r^2 + c_4. \tag{34}$$

The latter part $z = c_3 r^2 + c_4$ describes an i-sphere $S^i \subset I^3$, and a Euclidean sphere $S \subset E^3$. Therefore, the surface $\Phi \subset E^3$ defined by (34) is obtained by convolution of S with the surface Φ', which belongs to

$$G(r) = (c_1 + c_2 r^2)\ln r. \tag{35}$$

Hence, Φ is an offset of Φ', and thus L-equivalent to Φ'. Since we are interested in L-geometric properties, we restrict our considerations to the case (35) and denote the corresponding surfaces in E^3 again Φ. The transformation to E^3 may be performed via the middle spheres. We base their parametrization (27) on a polar coordinate representation $z = G(r,\phi)$ of Φ^i,

$$\begin{aligned} m_1 &= B\cos\phi + r^{-1}G_\phi \sin\phi,\ m_2 = B\sin\phi - r^{-1}G_\phi\cos\phi, \\ m_3 &= \frac{1-r^2}{4}\Delta G + rG_r - G,\ z = \frac{1+r^2}{4}\Delta G - rG_r + G, \end{aligned} \tag{36}$$

with

$$B = \frac{1}{2}(rG_{rr} + r^{-1}G_{\phi\phi} - G_r),\ \Delta G = G_{rr} + r^{-1}G_r + r^{-2}G_{\phi\phi}. \tag{37}$$

We insert G from (35) and due to rotational symmetry we compute only the middle spheres $(m_1, 0, m_3, R)$ centered in the (x,z)-plane,

$$m_1 = c_2 r - \frac{c_3}{r},\ m_3 = c_1(1-\ln r) + c_2(1+\ln r),\ R = -c_1(1-\ln r) + c_2(1+\ln r). \tag{38}$$

We see that the corresponding L-minimal surface Φ_1 with $c_2 = 0$ is of the spherical type: Because of $m_3 + R = 0$, all middle spheres are tangent to $\Pi: (n_1, n_2, n_3, h) = (0,0,-1,0)$. By Theorem 5 this is expected, since $c_2 = 0$ corresponds to an i-minimal surface Φ_1^i, namely the logarithmoid $z = c_1 \ln r$. However, also the L-minimal surface Φ_2 to $c_1 = 0$, i.e., $\Phi_2^i:$ $z = c_2 r^2 \ln r$ is of the spherical type. Due to $m_3 - R = 0$, all middle spheres are tangent to Π' with $(n_1, n_2, n_3, h) = (0,0,1,0)$. This is not a contradiction to Theorem 5. Performing in E^3 a reflection at $z = 0$, Π' gets mapped to Π and now the theorem can be applied to the reflected copy Φ_2' of Φ_2. This reflection in E^3 corresponds to the isotropic inversion (12) in I^3, which in polar coordinates reads $z' = z/r^2,\ r' = 1/r$. Applying this to $z = c_2 r^2 \ln r$, we obtain $z' = -c_2 \ln r'$, which is an i-minimal surface, namely the representation of Φ_2' in the isotropic model. Hence, the fundamental

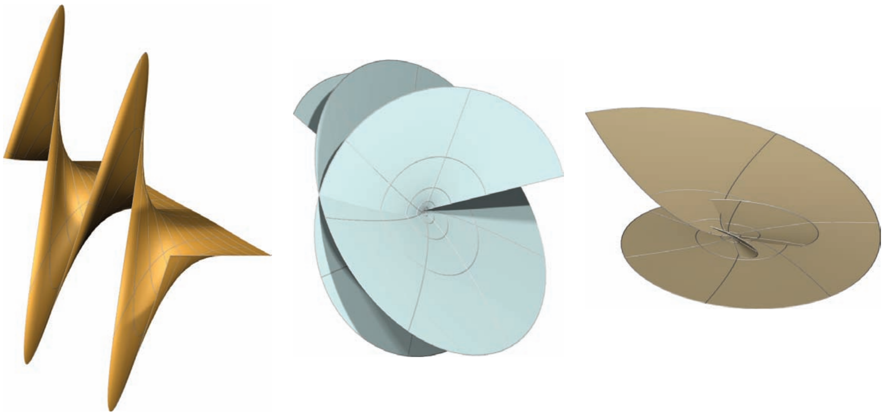

Fig. 5 Isotropic Scherk minimal surface (left), the corresponding Laguerre minimal surface of the spherical type Φ (middle), and the middle evolute Φ_E (right).

solution $z = r^2 \ln r$ of the biharmonic equation, which forms the basis of thin plate splines (see [Duc77] and Section 4), just yields an L-minimal surface of the spherical type.

General rotational L-minimal surfaces Φ, which belong to $\Phi^i : z = (c_1 + c_2 r^2) \ln r$ are not of the spherical type. However, we have shown that they are *convolution surfaces of two L-minimal surfaces* Φ_1, Φ_2 *of the spherical type* (Fig. 4). According to Kommerell's theorem [Kom11], all these surfaces Φ must have a *catenoid* as middle evolute Φ_E. Since Euclidean minimal surfaces are L-minimal, the catenoid also appears among the surfaces Φ; it is identical with its middle evolute.

Example 2 We study the Laguerre minimal surfaces corresponding to two well-known isotropic minimal surfaces: the isotropic Scherk minimal surface [Str77], and the isotropic Enneper surface [Str76].

The isotropic Scherk minimal surface Φ^i is given as graph of the function:

$$f(x, y) = e^{2x} sin(2y). \tag{39}$$

Using (26) we obtain Φ:

$$\Phi : \frac{2e^{2x}}{x^2 + y^2 + 1} \begin{pmatrix} (x^2 - y^2 - 1)\sin(2y) + 2xy\cos(2y) - x\sin(2y) \\ (y^2 - x^2 - 1)\cos(2y) + 2xy\sin(2y) - y\sin(2y) \\ 2x\sin(2y) + 2y\cos(2y) - \sin(2y) \end{pmatrix}. \tag{40}$$

The middle evolute Φ_E obtained using (30) is given by:

$$\Phi_E : 2e^{2x} \begin{pmatrix} -(y - 3x^2y + y^3)\cos(2y) + (-x + x^3 - 3xy^2 - 1)\sin(2y) \\ -(x - 3xy^2 + x^3)\cos(2y) + (y - y^3 + 3x^2y - 1)\sin(2y) \\ (4xy + y)\cos(2y) + (2x^2 - 2y^2 + x - \frac{1}{2})\sin(2y) \end{pmatrix}. \tag{41}$$

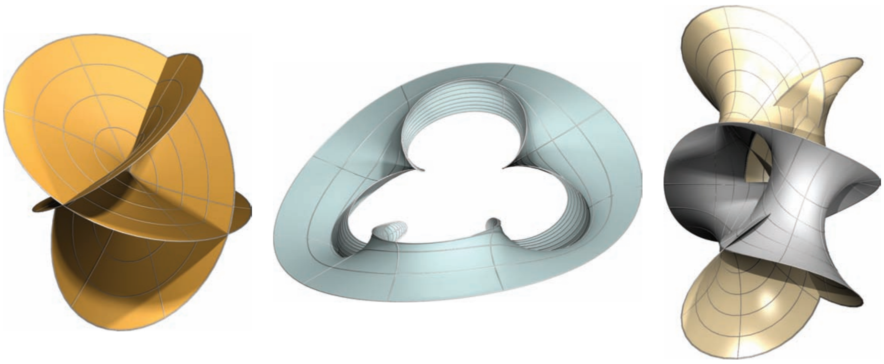

Fig. 6 Isotropic Enneper minimal surface (left), the corresponding Laguerre minimal surface of the spherical type Φ (middle), and the middle evolute Φ_E (right).

The second example is the isotropic Enneper surface given in polar coordinates as graph of the function:

$$\Phi^i : \ f(r,\varphi) = \frac{3}{2} r^{\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right). \tag{42}$$

The corresponding Laguerre minimal surface of the spherical type, Φ, is given by:

$$\Phi : \ \frac{1}{r^{1/3}(r^2+1)} \begin{pmatrix} \cos(\frac{\varphi}{3})(8r^2\cos(\frac{\varphi}{3})^4 - 10r^2\cos(\frac{\varphi}{3})^2 + r^2 - 1) \\ -\sin(\frac{\varphi}{3})(8r^2\cos(\frac{\varphi}{3})^4 - 6r^2\cos(\frac{\varphi}{3})^2 - r^2 - 1) \\ -r\sin(\frac{2\varphi}{3}) \end{pmatrix}. \tag{43}$$

The middle evolute is given by:

$$\Phi_E : \ \begin{pmatrix} -\frac{1}{6}r^{-1/3}(16\cos(\frac{\varphi}{3})^4 r^2 - 12\cos(\frac{\varphi}{3})^2 r^2 + r^2 - 5)\sin(\frac{\varphi}{3}) \\ \frac{1}{6}r^{-1/3}(16\cos(\frac{\varphi}{3})^4 r^2 - 20\cos(\frac{\varphi}{3})^2 r^2 + 5r^2 - 5)\cos(\frac{\varphi}{3}) \\ -\frac{5}{3}\sin(\frac{\varphi}{3})\cos(\frac{\varphi}{3}) r^{2/3} \end{pmatrix}. \tag{44}$$

4 Computational Issues

The present section concerns the computational design of Laguerre minimal surfaces. From our previous results it follows, that the design of a Laguerre minimal surface can be realized by solving the biharmonic equation in the isotropic model. To this end we describe two approaches.

4.1 Thin plate splines

We address the following problem: Given a set of planes in E^3 denoted by $\mathcal{P} = \{P_i : i = 1, \ldots, N\}$, we want to find a Laguerre minimal surface Φ that is tangent to all planes $P_i \in \mathcal{P}$.

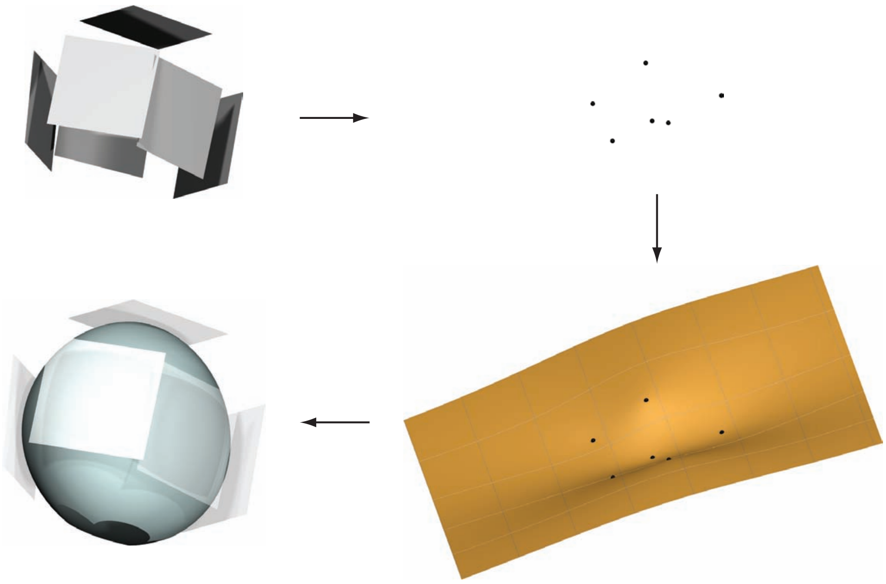

Fig. 7 The general construction scheme for a Laguerre minimal surface using thin plate spline interpolation. A set of prescribed tangent planes (top left) is mapped to a set of points (top right) in the isotropic model. A thin plate spline surface (bottom right) interpolating these points is constructed, and finally mapped back to extract the Laguerre minimal surface (bottom left). The prescribed tangent planes are indeed tangent to the generated Laguerre minimal surface.

Let us transform this problem into the isotropic model. The planes $P_i \in \mathcal{P}$ get mapped to a set of points Q, and the condition that Φ touches all planes in $\mathcal{P}$ translates to the condition that Φ^i interpolates the point set Q. Thus, in the isotropic model our problem reduces to the following: Given a set of data points find a biharmonic function that interpolating these points. We can get such a function using thin plate spline interpolation. It works as follows:

Given data points $\{(x_i, y_i, z_i) : i = 1, \ldots, N\}$, find a function of the form:

$$f(x, y) = a_0 + a_x x + a_y y + \sum_{i=1}^{N} w_i U(\|(x, y)^T - (x_i, y_i)^T\|), \quad U(r) := r^2 \ln(r),$$

such that $f(x_i, y_i) = z_i$ for all $i = 1, \ldots, N$. Clearly, this problem amounts to solving a linear system of equations. It is well-known that this system is regular if $(x_i, y_i) \neq (x_j, y_j)$ for all $i \neq j$ [Duc77]. This implies that as long as there are no parallel planes in our set $\mathcal{P}$, the interpolation problem can be solved uniquely.

Our first method for designing Laguerre minimal surfaces (Figure 7) can be summarized as follows:

1. Prescribe a set $\mathcal{P} = \{P_i : i = 1, \ldots, N\}$ as tangent planes in E^3.
2. Transform this set to a point set $(x_i, y_i, z_i)\; i = 1, \ldots, N$, in I^3.

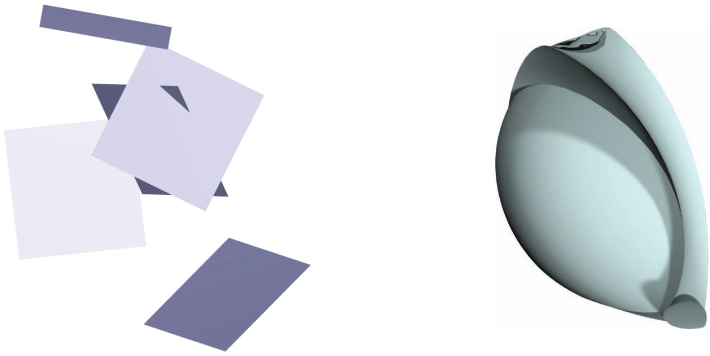

Fig. 8 Given a set of prescribed tangent planes (left), we obtain a Laguerre minimal surface (right) using the thin plate spline based approach. Notice the cusps in the constructed Laguerre minimal surface.

3. Solve the thin plate spline (TPS) interpolation problem for the data (x_i, y_i, z_i).
4. Transform the TPS solution back to E^3.

This construction does not always lead to pleasing results, as the resulting Laguerre minimal surface may have cusps. Therefore the tangent planes have to be prescribed in a meaningful way in order to get nice surfaces (Figure 8).

4.2 The "Björling-Problem" for Laguerre minimal surfaces

A simply connected Euclidean minimal surface (i.e. a surface with mean curvature zero) can be characterized by its boundary curve together with tangent planes along this curve. It is natural to explore the following question: Which boundary conditions uniquely determine a Laguerre minimal surface? To answer this question, we reformulate the problem in terms of the isotropic model. Since Laguerre minimal surfaces are transformed into graphs of biharmonic functions, we need to find boundary conditions that characterize biharmonic functions. This problem is well-studied and the solution is as follows [Cia78]:

Theorem 9 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with Lipschitz-continuous boundary Γ. Then a biharmonic function on Ω is uniquely defined by its values on Γ and its derivative into the direction of the inner normal of Γ on Γ.*

In the Laguerre model this corresponds to the following statement:

Theorem 10 *A Laguerre minimal surface is uniquely determined by its boundary curve Γ_L and the tangent planes along Γ_L.*

Proof We transform this situation into the isotropic model. By (16) the family of tangent planes along Γ_L gets transformed to a curve Γ in isotropic space. In order to transform the boundary value problem in the Laguerre

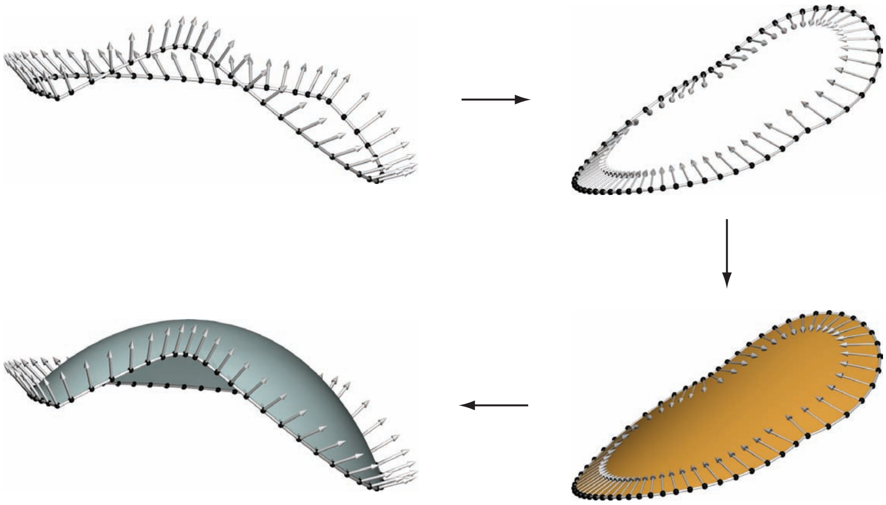

Fig. 9 Construction of a Laguerre minimal surface from a boundary strip. The figure shows the input condition (top left), the corresponding boundary strip in the isotropic model (top right), the solution Φ^i of the biharmonic boundary value problem (bottom right) and the corresponding Laguerre minimal surface Φ.

model into the isotropic boundary value problem of Theorem 9, we have to assign normal derivatives to the curve Γ from our initial data in the Laguerre model. This is done as follows:

By (17), every point P (=sphere of radius 0) of Γ_L corresponds to a paraboloid of revolution which clearly touches Γ at the point corresponding to the prescribed tangent plane in P. Now we assign normal derivatives to the curve Γ by taking the normal derivatives of the paraboloids at the corresponding points (Figure 10).

Solving the biharmonic equation with these boundary conditions yields the isotropic representation of the Laguerre minimal surface with the prescribed properties.□

The proof of Theorem 10 gives us a way to design a Laguerre minimal surface from a boundary strip (i.e. points with tangent planes, see Figure 9):

1. Transform the boundary conditions to the isotropic model as in the proof of Theorem 10.
2. Solve the biharmonic boundary value problem.
3. Transform the resulting surface back into the Laguerre model.

For the numerical solution of Step 2, we use a 13-point mask Δ_h^2 over a square grid given by:

$$\Delta_h^2 = \frac{1}{h^4}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -8 & 2 & 0 \\ 1 & -8 & 20 & -8 & 1 \\ 0 & 2 & -8 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

as a discretization of the Δ^2 operator, where h denotes the resolution of the underlying grid. The solution of the boundary value problem then amounts to solving a linear system. This naive discretization method has the advantage that the transfer from the isotropic model to the Laguerre model is particularly easy using (2), because over a square grid the x- and y-derivatives can be discretized naturally.

5 Metric duality, geometrical optics and Lie transforms of L-minimal surfaces

5.1 Isotropic L-minimal surfaces

Whereas Laguerre and Möbius geometry in E^3 are not equivalent, their isotropic counterparts are isomorphic. Isotropic Möbius geometry corresponds to isotropic Laguerre geometry via the metric duality in I^3. We briefly discuss the application of duality to isotropic Willmore surfaces Φ^i and then view this geometric transformation from E^3.

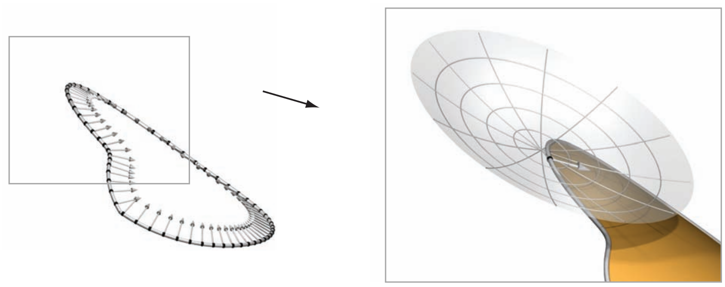

Fig. 10 Construction of normal derivatives in the isotropic model: A point in the Laguerre model is mapped to a paraboloid of revolution with z-parallel axis. The normal derivatives of the image paraboloids of the input curve Γ serve as input data for the boundary value problem.

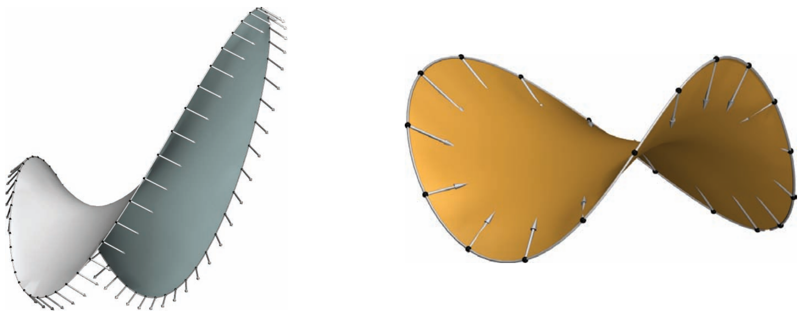

Fig. 11 Example of Laguerre minimal surface Φ with negative curvature (left) and the corresponding isotropic surface Φ^i (right).

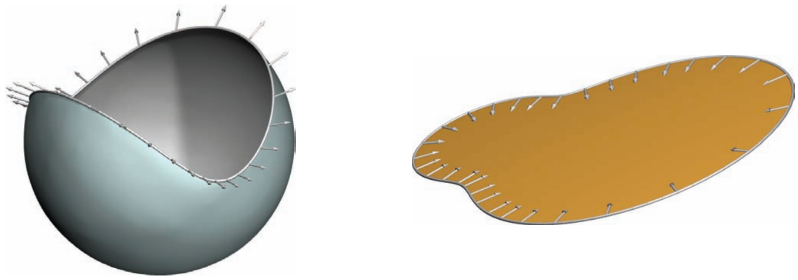

Fig. 12 Another example of a Laguerre minimal surface Φ (left) and the corresponding isotropic surface Φ^i (right).

Let Φ^i be an i-Willmore surface, i.e., the graph $z = f(x, y)$ of a biharmonic function f. Φ^i is a minimizer of the isotropic Willmore energy Ω from equation (24). Metric duality, realized via the polarity with respect to the i-sphere (9), maps Φ^i to a surface Φ^* with parametrization (10). The isotropic surface area elements dA_i, dA_i^* of Φ^i and Φ^*, respectively, satisfy the relation $dA_i = K_i^* dA_i^*$ where $K_i^* = 1/K_i$ is the isotropic curvature of Φ^*. Further using the relation $H_i = H_i^*/K_i^*$ between i-mean curvatures, $\Omega = \int (H_i^2 - K_i) dA_i$ can be expressed as

$$\Omega = \int \frac{(H_i^*)^2 - K_i^*}{K_i^*} dA_i^*. \tag{45}$$

This is exactly the isotropic counterpart of the functional Ω in (1) which is minimized by L-minimal surfaces of Euclidean geometry. Therefore, we have the expected result that the surfaces Φ^* are *isotropic L-minimal surfaces*.

5.2 Metric duality in I^3 as a Lie transformation in E^3

We are interested in those surfaces of E^3 which correspond to the surfaces $\Phi^* \subset I^3$. As a preparation it is necessary to interpret duality in I^3 from E^3. This leads us into *Lie sphere geometry* [Bla29, Cec92], which subsumes both Laguerre geometry and Möbius geometry. In Lie geometry, we have the class of *Lie spheres*, which is the union of the sets of oriented planes, oriented spheres including points, and the ideal point ∞ of Möbius geometry. The fundamental relation between Lie spheres is *contact*, which is taken from Laguerre geometry and enriched by the following two cases: ∞ is in contact with all oriented planes and no oriented sphere; parallel oriented planes are in contact.

An analytical treatment is based on hexaspherical coordinates, which are six homogeneous coordinates $(x_0, \dots, x_5)$ assigned to Lie spheres. The hexaspherical coordinates of an oriented plane (n_1, n_2, n_3, h) are:

$$(x_0, \dots, x_5) = (-h, h, n_1, n_2, n_3, 1), \tag{46}$$

those of an oriented sphere (m_1, m_2, m_3, R) are defined as

$$(x_0, \ldots, x_5) = (1 + m_1^2 + m_2^2 + m_3^2 - R^2, 1 - (m_1^2 + m_2^2 + m_3^2 - R^2), 2m_1, 2m_2, 2m_3, 2R). \tag{47}$$

The ideal point ∞ corresponds to $(x_0, \ldots, x_5) = (1, -1, 0, 0, 0, 0)$. Hexaspherical coordinates of a Lie sphere are homogeneous, and thus define a point in 5-dimensional projective space P^5. Since hexaspherical coordinates satisfy the relation:

$$\langle X, X\rangle := -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = 0, \tag{48}$$

the image points of Lie spheres form a quadric (Lie quadric Λ) in P^5. Vanishing of the bilinear form to the quadratic form (48), $\langle X, Y\rangle = 0$ (conjugacy with respect to Λ), characterizes contact of the corresponding Lie spheres in E^3. *Lie transformations* are defined as contact preserving bijections on the set of Lie spheres. They correspond to automorphic projective mappings of Λ in P^5.

Duality δ in I^3 maps point (p_1, p_2, p_3) to plane $z = p_1x + p_2y - p_3$. Let us now interpret a point P^i in I^3 as the image of an oriented plane $P:\ (n_1, n_2, n_3, h)$ of E^3. By equation (16), the duality in I^3 reads:

$$P^i = \frac{1}{n_3+1}(n_1, n_2, h) \mapsto P^*:\ z = \frac{1}{n_3+1}(n_1x + n_2y - h).$$

The plane $P^* \subset I^3$ is a special i-M-sphere, and corresponds to an oriented sphere in E^3. Equation (17) tells us that this oriented sphere, which we denote by P^δ, has the following center and radius:

$$P^\delta:\ (m_1, m_2, m_3, R) = \frac{1}{n_3+1}(-n_1, -n_2, h, -h).$$

The center of P^δ is almost P^i; it is obtained by reflecting P^i at the z-axis. This is just an Euclidean congruence transformation and therefore we consider the sphere:

$$P':\ (m_1, m_2, m_3, R) = \frac{1}{n_3+1}(n_1, n_2, h, -h), \tag{49}$$

as image of the oriented plane P. Up to the reflection, the mapping $\alpha:\ P \mapsto P'$ is the transformation which is induced in E^3 by the metric duality of I^3. In fact, α is not just defined on oriented planes, but also on oriented spheres, since δ maps i-M-spheres to i-M-spheres. This suggests to treat α within Lie sphere geometry. We will now show that α is a Lie transformation. To do so, we have to convert to hexaspherical coordinates. Using homogeneity, we find for the oriented sphere P' the hexaspherical representation:

$$P':\ (x'_0, \ldots, x'_5) = (1, n_3, n_1, n_2, h, -h),$$

which fits into the involuntary projective mapping,

$$(x'_0, x'_1, x'_2, x'_3, x'_4, x'_5) = \lambda \cdot (x_5, x_4, x_2, x_3, x_1, x_0). \tag{50}$$

It is easily seen that it maps Λ onto itself, and therefore describes a Lie transformation.

Let us add a few elementary properties of Λ. All oriented planes P are mapped to spheres P' which satisfy $m_3 + R = 0$ and therefore are tangent to $\Pi:\ (n_1, n_2, n_3, h) = (0, 0, -1, 0)$. Hence, P' is the sphere which has center P^i and is tangent to Π. An elementary expression for the transformation α acting on oriented spheres follows from equations (47) and (50),

$$\begin{aligned} (m_1', m_2', m_3', R') &= \frac{1}{2(R+m_3)}(2m_1, 2m_2, 1-b, 1+b) \\ b &= m_1^2 + m_2^2 + m_3^2 - R^2. \end{aligned} \tag{51}$$

Fixed oriented spheres, $S = \alpha(S)$, satisfy the relations:

$$R + m_3 = 1,\ m_1^2 + m_2^2 = 2 - 4m_3,$$

and thus they are tangent to the oriented plane $(n_1, n_2, n_3, h) = (0, 0, -1, 1)$ and centered on the rotational paraboloid $x^2 + y^2 = 2 - 4z$. These fixed spheres are also tangent to the oriented sphere $(m_1, m_2, m_3, R) = (0, 0, -1/2, -1/2)$.

5.3 Lie transformed L-minimal surfaces and geometrical optics

Finally, we are able to interpret an isotropic L-minimal surface Φ^* in E^3, i.e., perform the transfer from the isotropic model to the standard model of Laguerre geometry. The corresponding surface in E^3 is called $\alpha(\Phi)$ and by the discussions of the previous subsection, it arises from the L-minimal surface $\Phi \subset E^3$ by application of the Lie transformation α.

We have also seen that the transfer from Φ^i to $\alpha(\Phi)$ is very simple: $\alpha(\Phi)$ is the envelope of all spheres which are centered at Φ^i and tangent to Π (xy-plane with orientation $(0, 0, -1)$). This has a simple interpretation within geometrical optics (Figure 13): Reflecting z-parallel lines at Φ^i we obtain lines orthogonal to $\alpha(\Phi)$ and thus $\alpha(\Phi)$ is an anticaustic of Φ^i for illumination with z-parallel lines.

We can summarize our results as follows:

Theorem 11 *An isotropic Laguerre minimal surface Φ^* (i-curvature K_i^* and i-mean curvature H_i^*) is a minimizer of the functional (45) and arises from an isotropic Willmore surface Φ^i (graph $z = f(x, y)$ of a biharmonic function f) by the metric duality in I^3. Φ^i corresponds to an L-minimal surface Φ in the standard model of Laguerre geometry and Φ^* corresponds to a Lie transformed version $\alpha(\Phi)$ of Φ. The surface $\alpha(\Phi)$ is an anticaustic of Φ^i for light rays parallel to the isotropic direction (z-axis).*

If Φ^i is i-minimal, Φ^* is also i-minimal and thus Theorem 6 is a special case of Theorem 11. Moreover, we see that the duality between i-minimal surfaces in I^3 leads to a *duality between L-minimal surfaces of the spherical type. The duality transformation is provided by the Lie transformation α.*

Fig. 13 The anticaustic map provides the transfer from an isotropic Willmore surface Φ^i (graph $z = f(x, y)$ of a biharmonic function f) to a Lie-transformed Laguerre minimal surface $\alpha(\Phi)$ (shown in purple). The latter surface is orthogonal to rays obtained by reflecting the z-parallel rays at Φ^i. The surface $\alpha(\Phi)$ is constructed as the envelope of spheres centered on Φ^i and touching the xy plane. One such touching sphere is shown on the left.

Lie transformations map principal curvature lines (as sets of contact elements) onto principal curvature lines. Since i-principal curvature lines of Φ^i correspond to principal curvature lines of Φ, we see that the anticaustic map transfers the i-principal curvature lines of Φ^i to the principal curvature lines of $\alpha(\Phi)$.

For applications, the class of surfaces $\alpha(\Phi)$ may have the following advantage over L-minimal surfaces Φ: If Φ^i has a sign change in Gaussian curvature along a curve (parabolic line), the dual surface Φ^* has a singularity there and the transfer back to E^3 (which is again a point-plane transformation) will result in a parabolic curve of $\alpha(\Phi)$. Since $\alpha(\Phi)$ is related to Φ^i by two point-plane transformations, modeling of parabolic lines is not difficult. On the other hand, there is only one point-plane transformation between Φ^i and Φ; modeling parabolic lines on Φ (singularities of Φ^i) is not just difficult, it is impossible. It follows already from the functional (1) that L-minimal surfaces Φ cannot have generic parabolic lines, $K = 0, H \neq 0$, which is a limitation for design. It may not be too limiting for architecture, but still we expect that the surfaces $\alpha(\Phi)$ should be even more useful.

6 Conclusion and Future Work

We have shown that L-minimal surfaces are easily accessible through their close relation to biharmonic functions. This provides a new approach to the computation of L-minimal surfaces and also gives us a new tool for their study. Various relations to geometrical optics and new constructions of L-minimal surfaces as well as Euclidean minimal surfaces are a result of our investigations.

Having this new approach to L-minimal surfaces at our disposal, we will present more results in forthcoming publications. Generalizing results by König [Kön26, Kön28] and Musso and Nicolodi [MN95], we will present a study of all 'ruled' L-minimal surfaces. These are those L-minimal surfaces which may be defined as envelope of a one-parameter family of oriented right circular cones. Our main goal for future research is a construction of discrete L-minimal surfaces. As a first contribution in that direction, we have found a construction of quadrilateral edge offset meshes (as defined in [PLW*07]) which discretize L-minimal surfaces. Quad meshes with the edge offset property are discrete versions of L-isothermic surfaces. Hence, the discretized surfaces are exactly those L-minimal surfaces which are also L-isothermic. These are L-transformed Euclidean minimal surfaces, L-minimal surfaces of the spherical type and L-transformed surfaces of Bonnet [Bla29].

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