Curve-Sensitive Cuttings*

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Abstract

We introduce $(1/r)$-cuttings for collections of surfaces in 3-space that are sensitive to an additional collection of curves. Specifically, let S be a set of n surfaces in $\mathbb{R}^3$ of constant description complexity and let C be a set of m curves in $\mathbb{R}^3$ of constant description complexity. Let $1 \leq r \leq \min\{m,n\}$ be a given parameter. We show the existence of a $(1/r)$-cutting Ξ of S of size $O(r^{3+\varepsilon})$, for any $\varepsilon > 0$, such that the number of crossings between the curves of C and the cells of Ξ is $O(m^{1+\varepsilon}r)$. The latter bound improves, by roughly a factor of r, the bound that can be obtained for cuttings based on vertical decompositions. We view curve-sensitive cuttings as a powerful tool for various scenarios that involve curves and surfaces in three dimensions. As a preliminary application, we use the construction to obtain a bound of $O(m^{1/2+\varepsilon}n^{2+\varepsilon})$, for any $\varepsilon > 0$, on the complexity of the multiple zone of m curves in the arrangement of n surfaces in 3-space. After the conference publication of this paper [17], curve-sensitive cuttings were applied to derive an algorithm for efficiently counting triple intersections among planar convex objects in three dimensions [13], and we expect additional applications to arise in the future.

1 Introduction

Motivation. $(1/r)$-cuttings (see below for definitions) have attracted considerable attention in the computational geometry community, as they turned out to be crucial to the solution of many central problems in the field [4, 5, 6, 8, 9, 10, 15, 18, 19]. For some applications, special properties possessed by the cutting can lead to improved results. For instance, the tree structure of *hierarchical cuttings* [5] is of great help in numerous settings [3, 19].

Many applications require constructing cuttings for collections of non-linear surfaces [7, 16, 20]. Furthermore, increasing attention is being directed to problems that involve non-linear curves and surfaces in 3-space [12]. Motivated by these considerations, we construct a

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$(1/r)$-cutting for a collection of surfaces in 3-space, such that the cutting is sensitive, in the sense defined below, to a collection of curves given as additional input to the construction.

We apply this cutting to obtain a bound of $O(m^{1/2+\varepsilon}n^{2+\varepsilon})$, for any $\varepsilon > 0$, on the complexity of the multiple zone of m curves in the arrangement of n surfaces in 3-space, all of constant description complexity. The multiple zone is defined as the collection of all cells of the arrangement of the given surfaces that are crossed by at least one of the curves. It is a generalization of both the concept of the zone of a curve in an arrangement [2, 14] and the widely studied notion of many faces/cells in arrangements [1].

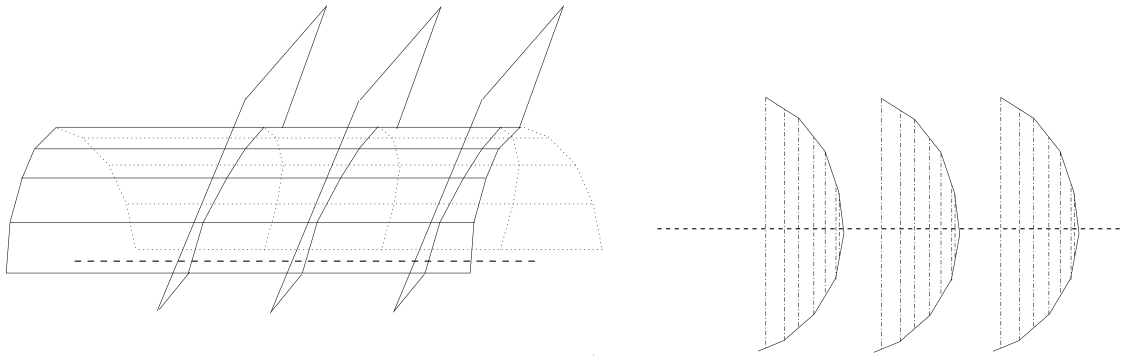
We expect curve-sensitive cuttings to find multiple additional uses in contexts that involve the interaction of curves and surfaces. It has already been applied, after the conference publication of this paper [17], to derive an algorithm for efficiently counting triple intersections among planar convex objects in three dimensions [13].

Overview. Let S be a set of n surfaces in $\mathbb{R}^3$ of constant description complexity, and let C be a set of m curves in $\mathbb{R}^3$ of constant description complexity; that is, each surface and curve is defined as a Boolean combination of a constant number of polynomial equalities and inequalities of constant maximum degree. Let $1 \le r \le \min\{m, n\}$ be a given parameter. A $(1/r)$-*cutting* of S is a subdivision of 3-space into connected cells, each of constant description complexity, so that each cell is crossed by at most n/r surfaces of S. We wish to construct a $(1/r)$-cutting Ξ of S of size near $O(r^3)$, so that the number of pairs (c, τ), where $c \in C$, τ a cell of Ξ, and $c \cap \tau \neq \emptyset$, is near $O(mr)$; that is, the average number of cells of Ξ crossed by a curve of C is near $O(r)$.

A standard method (in fact, the only general-purpose method known to date) for constructing a $(1/r)$-cutting for arrangements of non-linear surfaces is to take an appropriate random sample R of the surfaces of S, and to construct the *vertical decomposition* of the arrangement $\mathcal{A}(R)$ of R [20]. The construction of this decomposition proceeds in two stages. First, for every edge of $\mathcal{A}(R)$ and every vertical tangency curve (also known as the *silhouette*) on every surface of R, we erect a 2-dimensional vertical *visibility wall*, defined as the union of all z-vertical segments that have an endpoint on this edge (or curve) and are interior-disjoint from all surfaces of R. This first stage results in a decomposition of $\mathcal{A}(R)$ into vertical pseudo-prisms, such that the floor of each prism, if it exists, is contained in a single surface of R, and similarly for the ceiling of each prism.

In the second stage of the construction we refine the decomposition as follows. For every prism as above, consider its projection onto the xy-plane. It is a 2-dimensional semi-algebraic set, which we decompose in the plane by erecting 0, 1, or 2 y-vertical (possibly infinite) visibility segments on each of its vertices and y-vertical tangency points on its edges, where a visibility segment is defined as a maximal y-vertical segment that has an end-point on this vertex (or tangency point) and is interior-disjoint from the boundary of the considered semi-algebraic set. We then erect z-vertical 2-dimensional walls inside the original prism, defined as its intersection with the z-vertical walls spanned by all the y-vertical segments erected by the planar decomposition. Repeating this process for each of the above prisms decomposes $\mathcal{A}(R)$ into cells of constant description complexity.

We can choose R as a single sample from S of size $ar \log r$, for an appropriate absolute constant a. It can be then argued that, with high probability, the resulting vertical decomposition of $\mathcal{A}(R)$ is indeed a $(1/r)$-cutting. This is a consequence of the probabilistic analyses of Haussler and Welzl [15] and of Clarkson [9]. Using a variant of the method of Chazelle and Friedman [6] or of Chazelle [5] slightly reduces the size of the resulting cutting.



(a) (b)

Figure 1: A curve (the x-axis, shown dashed) crossing a quadratic number of cells of the vertical decomposition. (a) A side view of the input set. (b) A view from above of the second-step subdivision of the cells mentioned in the text.

Unfortunately, vertical decompositions may fail to satisfy our requirement concerning the number of crossings between the curves of C and the cells of the cutting. In fact, a curve may cross nearly $\Omega(r^2)$ such cells. An example is shown in Figure 1, where R is a collection of r planes. Half of them are parallel to the x-axis and pass above it, all appearing on the lower envelope of this group, which looks like a tunnel in the x-direction with a convex roof that is symmetric about the xz-plane. The remaining $r/2$ planes are all parallel to the y-axis, and form a fixed angle, say 45°, with the xy-plane. These latter planes are sufficiently separated from each other, so that their portions that lie above the xy-plane and below the lower envelope of the first group have pairwise disjoint xy-projections. The x-axis crosses $\Theta(r^2)$ cells of the vertical decomposition of these planes: Indeed, the first decomposition step creates (among others) $r/2$ cells whose top facet is the portion of some slanted plane of the second group that lies below the lower envelope of the first group. The second decomposition step subdivides each of these cells into $\Theta(r)$ subcells, and the x-axis crosses them all.

In contrast, the *undecomposed* arrangement of $\Theta(r \log r)$ surfaces is sensitive to the curves of C, because each curve crosses each surface at $O(1)$ points, so it crosses $O(r \log r)$ cells of the arrangement. However, the undecomposed arrangement is generally not a $(1/r)$-cutting. On the other hand, the decomposed arrangement is (with high probability) a $(1/r)$-cutting, but, as we have just seen, it may fail to be sensitive to C.

In this paper we describe a technique that achieves the better of both worlds, and constructs cuttings that satisfy the desired properties. The construction proceeds by taking a sample R of the surfaces, as described above, and decomposing $\mathcal{A}(R)$ into vertical prisms using the *first stage* of the vertical decomposition construction. Inside each prism we construct a decomposition that takes into account the parts of the curves of C that lie inside the prism. Specifically, we construct a hierarchical sequence of cuttings, somewhat reminiscent of the construction in Chazelle [5], that reduces the number of crossings between the curves of C and the boundaries of the cells of the cuttings. We are able to guarantee that the curves of C are not cut more than $O(m^{1+\varepsilon} r)$ times, for any $\varepsilon > 0$, overall.

Before describing our results in detail, we remark that we can construct an alternative curve-sensitive decomposition scheme for the special case where the surfaces are planes and the curves are lines (as in the example of Figure 1), by using the Dobkin-Kirkpatrick hierarchical decomposition [11] in each cell of $\mathcal{A}(R)$. This approach, however, does not extend to general curves and surfaces. (An expanded discussion of this remark is given in the application paper [13].)

2 A Curve-Sensitive Decomposition

In this section we present a new decomposition scheme that is a $(1/r)$-cutting for S and satisfies the desired bounds on the number of cells and on the number of curve-cell crossings. For simplicity of exposition, we will base our analysis on a single random sample of surfaces from S (rather than the more elaborate repeated-sampling scheme of [6]). Moreover, we consider samples of size r (rather than $\Theta(r \log r)$). This simplifies the calculations, but will only produce a $O(\log r/r)$-cutting. We get the desired cuttings by simply replacing r at the end of the analysis by the above larger sample size.

2.1 First Stage of the Decomposition

We begin with taking a random sample R of r surfaces of S, and a random sample R' of r curves of C. We form the arrangement $\mathcal{A}(R)$ of R, and apply to it the first step of vertical decomposition. That is, we erect vertical walls up and down from each curve of intersection of pairs of surfaces in R, as well as from the silhouette of each surface in R; the walls are extended until they hit another surface of R, or, failing that, all the way to $\pm\infty$. In addition, we erect similar vertical walls from each curve $c \in R'$, which are also extended to the first surface above and below.

Let $\mathcal{A}_1 = \mathcal{A}_1(R, R')$ denote the resulting decomposition. Note that each cell τ of $\mathcal{A}_1$ is a vertical prism-like cell: the intersection of each vertical line with τ is connected. However, the xy-projection τ^* of τ can have arbitrary shape and complexity.

For each cell τ of $\mathcal{A}_1$, let ξ_τ denote its combinatorial complexity (i.e., the number of vertices, edges and faces on its boundary), and let C_τ denote the set of all connected components of the nonempty intersections between τ and the curves of C. Let $\lambda_s(r)$ denote, as usual, the maximum length of a Davenport-Schinzel sequence of order s on r symbols [20], and put $\beta_s(r) = \lambda_s(r)/r$, which is thus an extremely slow growing function of r. We have

Lemma 2.1. *(a) The number of cells of $\mathcal{A}_1$ and their overall combinatorial complexity are both $O(r^3\beta_s(r))$, for an appropriate parameter s that depends on the complexity of the curves of C and the surfaces of S.*

(b) $\sum_{\tau\in\mathcal{A}_1} |C_\tau| = O(mr\beta_s(r))$.

(c) With high probability, for each cell $\tau \in \mathcal{A}_1$, we have $|C_\tau| = O\left(\frac{m\xi_\tau}{r}\log r\right)$.

Proof: Let γ be a fixed curve, which is either a curve in C, or an intersection curve of two surfaces in R, or the silhouette of a surface in R. Let V_γ denote the vertical 2-manifold (wall) spanned by γ. Let V_γ^+ (resp., V_γ^-) denote the portion of V_γ that lies above (resp., below) γ. Let $\mathcal{A}^+$ (resp., $\mathcal{A}^-$) denote the cross section of $\mathcal{A}(R)$ with V_γ^+ (resp., V_γ^-). By construction, any point at which γ crosses the boundary of some cell of $\mathcal{A}_1$ must either be a vertex of the lower envelope of $\mathcal{A}^+$, or a vertex of the upper envelope of $\mathcal{A}^-$ (or of both, if the vertex lies on γ itself), or a point that lies vertically above or below a point on another curve of R' (so that the two points are *vertically visible* in $\mathcal{A}(R)$). The complexity of each envelope is $O(\lambda_s(r)) = O(r\beta_s(r))$, for an appropriate constant s [20], and the number of times γ passes above or below a curve of R' is $O(r)$. This readily implies the first two parts of the lemma: Part (b) is an immediate consequence, while part (a) follows by applying this bound to each of the $O(r^2)$ intersection and $O(r)$ silhouette curves arising in the sample.

To prove part (c), let us construct (only for the sake of the proof) the second vertical decomposition step within τ. It partitions τ into $O(\xi_\tau)$ subcells of constant description complexity. By the ε-net theory of Haussler and Welzl [15], with high probability each of these cells is crossed by at most $O\left(\frac{m}{r}\log r\right)$ curves of C. This follows from the facts that none of these cells is crossed by any of the sample curves, and that the corresponding range space has finite VC-dimension, a property that is rather routine (albeit somewhat tedious) to establish (see, e.g., [20]). Multiplying the number $O(\xi_\tau)$ of subcells by $O\left(\frac{m}{r}\log r\right)$ yields part (c). □

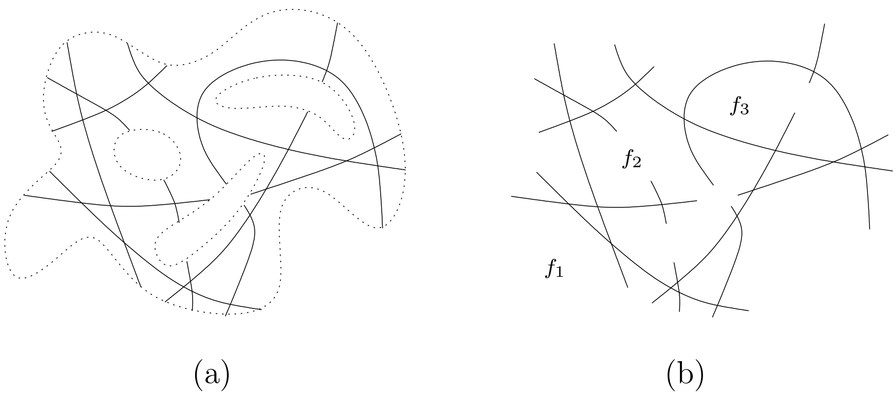


Figure 2: Step 2 of the decomposition. (a) The curves of $Q \subseteq C^*_{\tau_0}$ (solid) and $\partial\tau^*$ (dotted). (b) The external faces of $\mathcal{A}(Q)$; note that f_2 contains two components of $\partial\tau^*$.

2.2 Second Stage of the Decomposition

After constructing the decomposition $\mathcal{A}_1$, we perform a second decomposition step, which decomposes each cell τ of $\mathcal{A}_1$ as follows. Put $m_\tau = |C_\tau|$. Let C_τ^* denote the set of the xy-projections of the arcs of C_τ. Let X_τ denote the number of intersections between the curves of C_τ^*. This is also equal to the number of *vertical visibility segments* between pairs of curves of C_τ, where such a segment is parallel to the z-axis, and connects a point on one curve to a point on the other (and is thus fully contained in τ). We clearly have

$$\sum_{\tau \in \mathcal{A}_1} X_\tau = O(m^2). \tag{1}$$

Let $\partial\tau^*$ denote the boundary of τ^*. Since τ^* need not be simply connected, $\partial\tau^*$ may consist of more than one connected component. The potential existence of many components of $\partial\tau^*$ is the main source of technical difficulty of the analysis of our decomposition.

If $X_\tau \geq m_\tau$, we carry out a preliminary decomposition stage that partitions τ^* into subcells, so that, within each subcell τ_0, the number of intersections between the curves of C^*_τ that cross τ_0 is roughly the square of the number of such curves. We employ a standard approach that proceeds as follows (see, e.g., [4]). Put $s = \lceil m_\tau^2/X_\tau \rceil$, and sample each curve of C^*_τ with probability s/m_τ. This produces a random sample R'' of expected size s. The expected complexity of $\mathcal{A}(R'')$ is $O(s + (s/m_\tau)^2 X_\tau) = O(s)$, since each intersection counted in X_τ becomes a vertex of $\mathcal{A}(R'')$ with probability $(s/m_\tau)^2$. We construct the vertical decomposition of $\mathcal{A}(R'')$, and can argue that, with high probability, it consists of $O(s)$ trapezoids, each of which is crossed by at most $O((m_\tau/s)\log s)$ curves of C^*_τ. These trapezoids are the cells τ_0 of the preliminary decomposition of τ^*. Each cell contains on average $X_\tau/s = O(X_\tau^2/m_\tau^2)$ crossings between curves of C^*_τ, which is roughly the square of the number $O((m_\tau/s)\log s) = O((X_\tau/m_\tau)\log s)$ of these curves that cross it. It is important to note that this decomposition is defined only in terms of the curves in C^*_τ, and is thus not necessarily confined to within τ^*. This concludes the description of the preliminary decomposition of τ^* that is constructed only if $X_\tau \geq m_\tau$.

We now apply an additional decomposition step to each cell τ_0 of this preliminary cutting. (If $X_\tau < m_\tau$, $\tau_0 = \tau^*$.) This decomposition consists of a recursively constructed hierarchical sequence of cuttings of the subset $C^*_{\tau_0}$ of those curves of C^*_τ that cross τ_0, clipped to within τ_0. This decomposition is somewhat reminiscent of the hierarchical cutting construction of Chazelle [5]. We begin by choosing a sufficiently large *constant* r', to be used throughout the construction. Put $m_{\tau_0} = |C^*_{\tau_0}|$.

First level in the hierarchy. We draw a random sample Q of r' arcs of $C^*_{\tau_0}$, and compute all the faces of the planar arrangement $\mathcal{A}(Q)$ that contain components of $\partial\tau^*$. Clearly, each component of $\partial\tau^*$ lives in a single (not necessarily distinct) face of $\mathcal{A}(Q)$. We refer to such faces as the *external* faces of $\mathcal{A}(Q)$. Note also that, as defined, those faces are not confined to within τ_0 nor within τ^*. That is, $\partial\tau^*$ is not part of $\mathcal{A}(Q)$ and does not delimit any face of it. However, each component γ of $\partial\tau^*$ bounds a connected component of the complement of τ^* which is fully disjoint from all the arcs of Q (or of $C^*_{\tau_0}$ for that matter). See Figure 2.

For each external face f of $\mathcal{A}(Q)$, we compute the 2-dimensional vertical decomposition of f into vertical pseudo-trapezoids (see, e.g., [20]), which we refer to shortly as *trapezoids* or *subcells*. With high probability, each resulting subcell σ is crossed by at most $\frac{am_{\tau_0}}{r'}\log r'$ curves of $C^*_{\tau_0}$, for an appropriate absolute constant a [9, 15]. For each connected component γ of $\partial\tau^*$, the face f_γ of $\mathcal{A}(Q)$ that contains γ consists of $O(r'\beta_s(r'))$ subcells [20], so the total number of crossings between the arcs of $C^*_{\tau_0}$ and these subcells is $O(m_{\tau_0}\beta_s(r')\log r')$. Let κ_{τ_0} denote the number of distinct external faces of $\mathcal{A}(Q)$. Then we get a total of $O(\kappa_{\tau_0} r'\beta_s(r'))$ external trapezoids,[1] and the total number of crossings between the arcs of $C^*_{\tau_0}$ and these subcells is $O(\kappa_{\tau_0} m_{\tau_0}\beta_s(r')\log r')$.

An obvious upper bound on κ_{τ_0} is $1 + h_{\tau_0}$, where h_{τ_0} denotes the number of internal connected components of $\partial\tau^*$ that are fully contained in τ_0 (boundary components that cross $\partial\tau_0$ all lie in the single unbounded face of $\mathcal{A}(Q)$), but we will use in the following

[1]In general, better bounds are known for the complexity of κ_{τ_0} faces in an arrangement of r' curves (see, e.g., [8]), but the cruder bound that we use suffices for our purposes.

analysis a more refined bound. The need for a refined analysis comes from the observation that, at this initial stage of the hierarchy, the total number of faces of $\mathcal{A}(Q)$ is only a constant (at most $O((r')^2)$), whereas h_{τ_0} can be much larger. Note that, trivially,

$$h_\tau \equiv \sum_{\tau_0} h_{\tau_0} \leq \xi_\tau. \tag{2}$$

We also have $h_\tau = O(r)$, because we can charge each internal component of $\partial\tau^*$ either to a complete connected component of an intersection curve between the surface of R forming the floor of τ with another surface in R, or to a similar intersection component involving the surface forming the ceiling of τ, or to a complete connected component of the silhouette of some surface of R (which 'floats in the middle' of τ), and the overall number of such components is clearly $O(r)$. In fact, applying this analysis for all the cells τ of $\mathcal{A}_1$, we obtain

$$\sum_{\tau\in\mathcal{A}_1} h_\tau = O(r^2). \tag{3}$$

In addition to decomposing the external faces as described above, we also partition the remainder portion of $\mathcal{A}(Q)$ (its 'internal' portion) into vertical trapezoids. In doing so, we erase all the edges of $\mathcal{A}(Q)$ that are contained in the interior of the internal portion, and retain only the edges that also bound the external faces. Thus the number of trapezoids into which the internal portion is partitioned is also $O(\kappa_{\tau_0} r'\beta_s(r'))$. The total number of crossings between the arcs of $C^*_{\tau_0}$ and these internal subcells is $O(\kappa_{\tau_0} m_{\tau_0} r'\beta_s(r'))$. (Here we can no longer claim that each internal trapezoid is crossed by only a small number of curves, because it is not necessarily disjoint from the sampled arcs, so this bound is larger than the bound claimed for external trapezoids, by nearly a factor of r'.)

Second level in the hierarchy. We now apply a second partitioning step within each external trapezoid σ that has a nonempty intersection with $\partial\tau^*$. (All other external and internal trapezoids are not decomposed any further.) Let C^*_σ denote the set of connected components of the intersections of the curves in $C^*_{\tau_0}$ with σ. As in the preceding step, σ is not necessarily contained in τ^*; however, each arc in C^*_σ lies fully in $\sigma \cap \tau^*$. See Figure 3.

We draw a random sample Q_σ of r' curves of C^*_σ, and compute all the faces of the planar arrangement $\mathcal{A}(Q_\sigma)$ that contain components of $\partial\tau^*$. As above, each component of $\partial\tau^*$ lives in a single ('external') face of $\mathcal{A}(Q_\sigma)$. Again, those faces are not necessarily distinct. This time, however, all external faces, with the exception of the unbounded one, are confined within σ. Boundary components γ of $\partial\tau^*$ that intersect σ are of two types: those that are fully contained in the interior of σ, and those that cross $\partial\sigma$. All components γ of the second type lie in the same (unbounded) face of $\mathcal{A}(Q_\sigma)$. Let h_σ, κ_σ denote, respectively, the number of components γ of the first type, and the number of distinct external faces of $\mathcal{A}(Q_\sigma)$. Clearly, $\kappa_\sigma \leq 1 + h_\sigma$, and $\sum_\sigma h_\sigma \leq h_\tau$ (where the sum extends over all σ and all τ_0). Again, however, we will have to use a more refined bound for k_σ in what follows.

For each external face f of $\mathcal{A}(Q_\sigma)$, we compute the 2-dimensional vertical decomposition of f. With high probability, each resulting subcell σ' is crossed by at most

$$\left(\frac{a\log r'}{r'}\right)^2 m_{\tau_0}$$

curves of C^*_σ. For each connected component γ of $\partial\tau^*$, the face f_γ of $\mathcal{A}(Q_\sigma)$ that contains γ consists of $O(r'\beta_s(r'))$ subcells, so the total number of crossings between the arcs of C^*_σ

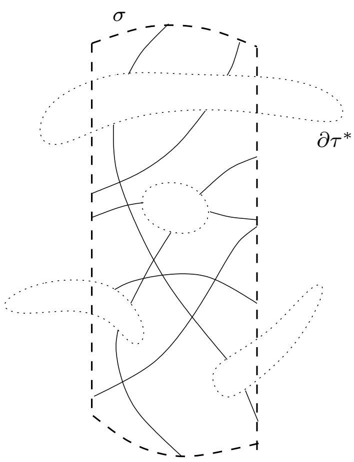


Figure 3: An external trapezoid σ (dashed), the portions of $\partial\tau^*$ that meet σ (dotted), and the arcs in C^*_σ (solid).

and these subcells is

$$O(m_{\tau_0}\beta_s(r')\log^2 r'/r').$$

Summing over all boundary components of $\partial\tau^*$ that meet σ, we get a total of $O(\kappa_\sigma r'\beta_s(r'))$ external trapezoids, and the total number of crossings between the arcs of C^*_σ and these subcells is $O(\kappa_\sigma m_{\tau_0}\beta_s(r')\log^2 r'/r')$. Summing these bounds over all external trapezoids σ, we obtain bounds for the overall number of external trapezoids in the second hierarchical partitioning step, and the total number of crossings between arcs in $C^*_{\tau_0}$ and these trapezoids. These bounds are, respectively,

$$\sum O(\kappa_\sigma r'\beta_s(r')) \tag{4}$$

and

$$\sum O(\kappa_\sigma m_{\tau_0}\beta_s(r')\log^2 r'/r'),$$

where these sums are over every σ that is an external trapezoid in $\mathcal{A}(Q)$.

As above, we also partition the remainder internal portions of the arrangements $\mathcal{A}(Q_\sigma)$, over all trapezoids σ, into vertical trapezoids, using, as above, only the edges and vertices of these internal portions that bound also the external portions. Thus the overall number of these internal trapezoids is also bounded by (4), and the total number of crossings between arcs in $C^*_{\tau_0}$ and these internal trapezoids is at most

$$\sum_{\sigma \text{ an external trapezoid in } \mathcal{A}(Q)} O(\kappa_\sigma m_{\tau_0}\beta_s(r')\log r').$$

Recursive construction of the hierarchy. The above process is repeated recursively, each recursion stage refining the decomposition inside those 'external' trapezoids constructed in the previous stage that are still crossed by (or contain) boundary components of $\partial\tau^*$. Let $j = j_{\tau_0}$ be the smallest integer such that $(r')^j \ge \xi_\tau/s$. Note that this implies that $(r')^j = O(\xi_\tau/s)$. We stop the recursive decomposition process after j steps. By an appropriate extension of the preceding arguments, the overall number of external and internal trapezoids produced in the i-th step, for any $i = 1, \ldots, j$, is at most

$$\sum_{\sigma \text{ an external trapezoid in some } \mathcal{A}(Q_{\sigma'})} O(\kappa_\sigma r' \beta_s(r')), \tag{5}$$

where σ' is an external trapezoid constructed in the preceding $(i-1)$-st step which intersects $\partial\tau^*$. With high probability, each external trapezoid constructed at the i-th step is crossed by at most

$$O\left(\left(\frac{\log r'}{r'}\right)^i m_{\tau_0}\right)$$

curves of C^*_τ, and each such internal trapezoid is crossed by at most

$$O\left(\left(\frac{\log r'}{r'}\right)^{i-1} m_{\tau_0}\right)$$

curves. Hence, the number of crossings between the arcs of $C^*_{\tau_0}$ and the external trapezoids is at most

$$\sum_\sigma O\left(\kappa_\sigma m_{\tau_0} \beta_s(r') \frac{\log^i r'}{(r')^{i-1}}\right),$$

and the number of crossings between the arcs of $C^*_{\tau_0}$ and the internal trapezoids is at most

$$\sum_\sigma O\left(\kappa_\sigma m_{\tau_0} \beta_s(r') \frac{\log^{i-1} r'}{(r')^{i-2}}\right), \tag{6}$$

where these sums are over every σ that is an external trapezoid in some $\mathcal{A}(Q_{\sigma'})$. It clearly suffices to bound the number of crossings of the latter kind.

Let us first analyze the number of trapezoids in more detail. Let γ be a boundary component of $\partial\tau^*$. If at some step i, γ crosses the boundary of some external trapezoid(s), it has no effect on the quantities κ_σ from this step on (inclusive). If on the other hand γ remains confined to the interior of a single external trapezoid σ, then it may add 1 to κ_σ, but it will not affect κ_{σ_1}, for any other external trapezoid σ_1 produced at this step.

In the absence of any internal boundary components of $\partial\tau^*$, the number of trapezoids increases in each step by a factor of at most $O(r'\beta_s(r'))$, for a total of $O((r')^i\beta_s^i(r'))$ trapezoids produced in the i-th step. If an internal boundary component γ survives (in the above sense) up to step i, it generates an additional number of at most $O(ir'\beta_s(r'))$ trapezoids. (At each step, γ may cause $O(r'\beta_s(r'))$ new trapezoids to be constructed, but only one of them, namely, the one fully containing γ, if any, continues to interact with γ in the subsequent steps.) Hence, the overall number of trapezoids (external and internal) produced by the i-th step of the process is at most

$$O((r')^i\beta_s^i(r') + ih_{\tau_0} r'\beta_s(r')).$$

Summing this bound over $i = 1, \ldots, j = \lceil \log_{r'}(\xi_\tau/s) \rceil$, we obtain that the total number of trapezoids is

$$O((r')^j\beta_s^j(r') + j^2 h_{\tau_0} r'\beta_s(r')).$$

Since $(r')^j = O(\xi_\tau/s)$, we may write $(r')^j\beta_s^j(r') = O(\xi_\tau^{1+\varepsilon}/s)$, for any $\varepsilon > 0$, and write similarly $j^2 = O(\xi_\tau^\varepsilon)$. Therefore, remembering that $r' = O(1)$, the total number of trapezoids is

$$O\left(\left(\frac{\xi_\tau}{s} + h_{\tau_0}\right)\xi_\tau^\varepsilon\right).$$

Summing this bound over all $O(s)$ subcells τ_0 of the fixed cell τ, and using (2), we obtain the following bound on the overall number of trapezoids constructed during the recursion:

$$O(\xi_\tau^\varepsilon)\left(\xi_\tau + \sum_{\tau_0} h_{\tau_0}\right) = O(\xi_\tau^{1+\varepsilon}).$$

Next consider the bounds (6) on the number of curve-cell crossings. Assume first that $X_\tau > m_\tau$. We then bound the sum (6) in two different ways. First, assume that $(r')^i \le h_{\tau_0}^{1/2}$. Then we simply use the crude bound of $\kappa_\sigma = O((r')^2)$, which is a general bound on the number of faces in an arrangement of r' curves. In a similar manner, we trivially bound the overall number of trapezoids σ that can arise at all recursive steps up to level i by $O((r')^{2i-2})$, since the number of trapezoids in the vertical decomposition of the arrangement of the r' curves sampled at each step is $O((r')^2)$. Then (6) is at most

$$m_{\tau_0} \cdot O\left((r')^{2i}\beta_s(r')\frac{\log^{i-1} r'}{(r')^{i-2}}\right) = O\left(m_{\tau_0}(r')^{i(1+\varepsilon)}\right),$$

for any $\varepsilon > 0$, using the fact that r' is a constant. The sum of all these bounds, over all levels i satisfying $(r')^i \le h_{\tau_0}^{1/2}$, can be upper bounded by $O(m_{\tau_0}h_{\tau_0}^{(1+\varepsilon)/2})$, for any $\varepsilon > 0$.

Suppose next that $(r')^i > h_{\tau_0}^{1/2}$. Then we can upper bound $\sum_\sigma k_\sigma$ by $O(h_{\tau_0})$, so the overall bound for such an i is at most

$$O\left(m_{\tau_0}h_{\tau_0}\beta_s(r')\frac{\log^{i-1} r'}{(r')^{i-2}}\right) = O\left(\frac{m_{\tau_0}h_{\tau_0}}{(r')^{i(1-\varepsilon)}}\right),$$

for any $\varepsilon > 0$, and the sum of these bounds, over all levels i satisfying $(r')^i > h_{\tau_0}^{1/2}$, can also be upper bounded by $O(m_{\tau_0}h_{\tau_0}^{(1+\varepsilon)/2})$, for any $\varepsilon > 0$. (Here the actual terminal value j is immaterial.)

Summing the bound just derived over all cells τ_0 of $\mathcal{A}(R'')$, and replacing $\varepsilon/2$ by ε, we obtain an overall bound of

$$\sum_{\tau_0} O(m_{\tau_0}h_{\tau_0}^{1/2+\varepsilon}) = O\left(\frac{m_\tau \log s}{s} \cdot \left(\sum_{\tau_0} h_{\tau_0}\right)^{1/2+\varepsilon} \cdot s^{1/2-\varepsilon}\right),$$

where we remind the reader that the number of cells τ_0 is $O(s)$, and $m_{\tau_0} = O((m_\tau/s)\log s)$. Recalling that $s = \lceil m_\tau^2/X_\tau \rceil$, and using (2), we can write this as

$$O\left(X_\tau^{1/2+\varepsilon}h_\tau^{1/2+\varepsilon}\right),$$

for any $\varepsilon > 0$.

Suppose next that $X_\tau < m_\tau$. Then there is only a single cell $\tau_0 = \tau^*$. By cutting the curves at their intersection points, we may assume that the curves in C_τ^* are pairwise openly disjoint (the number of these pieces remains $O(m_\tau)$). In this case, the total size of each of the cuttings, throughout the recursive hierarchy, is only $O(r')$, so the overall number of trapezoids that are generated at level i of the hierarchy is only $O((r')^i)$. Bounding $\sum_\sigma k_\sigma$ by this quantity, (6) becomes $O(m_\tau \xi_\tau^\varepsilon)$, for any $\varepsilon > 0$.

In conclusion, the total number of curve-cell crossings within a cell τ of $\mathcal{A}_1$ is at most

$$O\left(X_\tau^{1/2+\varepsilon} h_\tau^{1/2+\varepsilon} + m_\tau \xi_\tau^\varepsilon\right), \tag{7}$$

for any $\varepsilon > 0$.

Completion. We now form the final 2-dimensional decomposition, by taking $\partial\tau^*$ into account. This has to be done wth some care, as follows. The hierarchy of trapezoids constructed so far is induced by various samples of (pieces of) curves from C_τ^*. Let Γ_τ denote the collection of all curve portions that constitute the floors and ceilings of all these trapezoids. By construction, no two curve portions in Γ_τ intersect transversally. (Some pairs, constituting, e.g., floors of trapezoids that are nested in the hierarchy, may partially overlap; this has no effect on the analysis about to be presented.) Clearly, the number of trapezoids is $\Theta(|\Gamma_\tau|)$.

Consider now the union Γ'_τ of Γ_τ with the set of arcs forming $\partial\tau^*$. The arcs of Γ'_τ are also pairwise openly disjoint. Form the vertical trapezoidal decomposition of Γ'_τ. The number of trapezoids in this decomposition is

$$O(|\Gamma'_\tau|) = O(\xi_\tau^{1+\varepsilon} + \xi_\tau) = O(\xi_\tau^{1+\varepsilon}).$$

We retain only those trapezoids that are fully contained in τ^* (the others are disjoint from τ^*).

We next consider the number of crossings between the curves of C_τ^* and the new trapezoids. Each such crossing can be charged to a crossing of a curve $\gamma \in C_\tau^*$ with the boundary of a new trapezoid σ (unless γ is fully contained in σ; the number of such latter pairs is clearly at most m_τ). If such a crossing occurs on the floor or ceiling of σ, then it is also a crossing with the boundary of an old trapezoid, and is thus counted in (7). If it occurs at a vertical wall erected from an endpoint q of some arc in Γ_τ, then the new wall is equal to or is shorter than the old wall erected from q. Hence the number of such crossings is also upper bounded by (7). The only remaining case is a vertical wall erected from some vertex of $\partial\tau^*$ or from an x-extreme point on some arc of $\partial\tau^*$. Such a wall is fully contained in an old external trapezoid, and is thus crossed by at most

$$O((\log r'/r')^j m_{\tau_0}) = O((\log r'/r')^j (m_\tau/s)\log s)$$

curves of C_τ^*. Hence the total number of crossings of this kind is (recall that $(r')^j = \Theta(\xi_\tau/s)$)

$$O(\xi_\tau^{1+\varepsilon}(\log r'/r')^j (m_\tau/s)\log s) = O(m_\tau \xi_\tau^\varepsilon),$$

for any $\varepsilon > 0$.

The new decomposition is clearly a partition of τ^* into subcells (trapezoids) of constant description complexity. Each of these subcells is lifted vertically in the z-direction to within τ, thereby obtaining a partition of τ itself. The collection of all these partitionings, over all cells τ of $\mathcal{A}_1$, constitutes our final decomposition.

Since each resulting (3-dimensional) cell has constant description complexity, it follows by the ε-net theory of Haussler and Welzl [15] that, with high probability, each of them is crossed by at most $\frac{a'n}{r}\log r$ surfaces of S, for an appropriate absolute constant $a' > 0$, so it is an $O((\log r)/r)$-cutting of S.

Lemma 2.2. *(a) The total number of cells of the above decomposition is $O(r^{3+\varepsilon})$, for any $\varepsilon > 0$.*

(b) The total number of crossings between the curves of C and these cells is $O(mr^{1+\varepsilon})$, for any $\varepsilon > 0$.

Proof: (a) As shown above, the number of cells is $O(\sum_{\tau\in\mathcal{A}_1} \xi_\tau^{1+\varepsilon})$, which, by Lemma 2.1(a), is $O(r^{3+\varepsilon})$, for any $\varepsilon > 0$.

(b) By (7) and the preceding discussion, the number of crossings is

$$\sum_\tau O\left(X_\tau^{1/2+\varepsilon} h_\tau^{1/2+\varepsilon} + m_\tau \xi_\tau^\varepsilon\right),$$

for any $\varepsilon > 0$. Using (1) and (3), the Cauchy-Schwarz inequality, and Lemma 2.1(b), this can be upper bounded by

$$\sum_\tau O(X_\tau^{1/2+\varepsilon} h_\tau^{1/2+\varepsilon}) + \sum_\tau O(m_\tau r^\varepsilon) =$$

$$O(1)\cdot\left(\sum_\tau X_\tau\right)^{1/2+\varepsilon}\cdot\left(\sum_\tau h_\tau\right)^{1/2+\varepsilon} + O(mr^{1+\varepsilon}) =$$

$$= O(m^{1+2\varepsilon} r^{1+2\varepsilon}),$$

for any $\varepsilon > 0$, which, with an appropriate scaling of ε, can also be written as $O(m^{1+\varepsilon} r)$, for any $\varepsilon > 0$. □

By replacing r by $ar\log r$, for an appropriate absolute constant a, as discussed above, we obtain the following main result:

Theorem 2.3. *Let S be a set of n surfaces in $\mathbb{R}^3$ of constant description complexity, and let C be a set of m curves in $\mathbb{R}^3$ of constant description complexity. Let $1 \le r \le \min\{m, n\}$ be a given parameter. Then there exists a $(1/r)$-cutting Ξ of S of size $O(r^{3+\varepsilon})$, for any $\varepsilon > 0$, so that the number of crossings between the curves of C and the cells of Ξ is $O(m^{1+\varepsilon} r)$.*

Remark. With some additional work, we can also show that each cell of the cutting is crossed by $O(m/r)$ curves of C.

3 The Complexity of a Multiple Zone

Let S and C be as above. Define the zone $Z(C)$ of C in $\mathcal{A}(S)$ to be the collection of all cells of $\mathcal{A}(S)$ that are crossed by at least one curve of C.

Theorem 3.1. *The complexity of $Z(C)$ is $O(m^{1/2+\varepsilon} n^{2+\varepsilon})$, for any $\varepsilon > 0$.*

Proof: Fix a parameter r, and construct a C-sensitive $(1/r)$-cutting of $\mathcal{A}(S)$, consisting of $O(r^{3+\varepsilon})$ cells, each crossed by at most n/r surfaces of S, and the total number of crossings between these cells and the curves of C is at most $O(m^{1+\varepsilon}r)$.

Fix a cell τ of the cutting. Let S_τ (resp., C_τ) denote the set of surfaces of S (resp., curves of C) that cross τ, clipped to within τ. The complexity of $Z(C) \cap \tau$ can be upper bounded as follows: First, the zone of a single curve in an arrangement of N surfaces of constant description complexity is $O(N^{2+\varepsilon})$, for any $\varepsilon > 0$ [14]. Hence, the overall complexity of the $|C_\tau|$ separate zones of each of the curves in C_τ in $\mathcal{A}(S_\tau)$ is at most $O(|C_\tau||S_\tau|^{2+\varepsilon})$. In addition, portions of the boundary of the external cell of $\mathcal{A}(S_\tau)$ may also belong to $Z(C)$, because they may bound cells of $\mathcal{A}(S)$ that are crossed by curves of C that do not cross τ. The complexity of this external cell is $O(|S_\tau|^{2+\varepsilon})$. Hence, putting $m_\tau = |C_\tau|$, the overall complexity of $Z(C)$ is (we use the same ε both in the bounds in Theorem 2.3 and for the bound on the complexity of the zone of a curve)

$$O\left(\sum_\tau (m_\tau + 1)\left(\frac{n}{r}\right)^{2+\varepsilon}\right) = O\left(\frac{m^{1+\varepsilon}n^{2+\varepsilon}}{r^{1+\varepsilon}} + n^{2+\varepsilon}r\right),$$

where we use Theorem 2.3 to infer that $\sum_\tau m_\tau = O(mr^{1+\varepsilon})$. Choosing $r = m^{1/2}$ completes the proof of the theorem. $\Box$

Remark: A lower bound for $Z(C)$ is $\Omega(m^{2/3}n^{5/3})$. To establish it, take a planar arrangement of $n/2$ lines that has m distinct faces of overall complexity $\Theta(m^{2/3}n^{2/3})$. Lift each of these lines to a vertical plane in three dimensions, and add to the resulting arrangement $n/2$ additional horizontal planes. The resulting collection of n planes is our set S. For the set C of curves, take m vertical lines, each intersecting the xy-plane at a point inside one of the m marked faces. The complexity of the multiple zone $Z(C)$ is easily seen to be $\Theta(m^{2/3}n^{5/3})$.

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