Ready, Set, Go!

The Voronoi Diagram of Moving Points that Start from a Line

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Abstract

It is an outstanding open problem of computational geometry to prove a near-quadratic upper bound on the number of combinatorial changes in the Voronoi diagram of points moving at a common constant speed along linear trajectories in the plane. In this note we observe that this quantity is $\Theta(n^2)$ if the points start their movement from a common line.

1 Introduction

Given a set P of n points in the plane, moving at a common constant speed along linear trajectories, we wish to estimate the number of changes that occur through time in the combinatorial structure of the Voronoi diagram $\mathrm{Vor}(P)$ of P. The best known lower bound of $\Omega(n^2)$ for this quantity is easily obtained by considering P that consists of two collections of $n/2$ points moving in opposite directions along two parallel lines. The best known upper bound is $O(n^3)$ [2].

This problem dates back to Atallah [4] and is one of the best known open problems in computational geometry [7, Problem 2]. Despite intensive work in the past two decades, the order or magnitude gap between the known lower and upper bounds remains. If the points have distinct speeds or move along non-linear trajectories the gap is even wider (although still near-linear) [1, 14].

The right upper bound is commonly conjectured to be quadratic or near-quadratic. Proving this would not only lead to breakthroughs in the study of kinetic data structures [5] but would also facilitate the application of Voronoi diagrams of moving points to many combinatorial optimization problems. Although numerous schemes have been developed for maintaining such diagrams through time [2, 8, 10], they all suffer from the lack of good bounds on the number of combinatorial changes.

The lack of progress on this front has led researchers to consider the special cases of the problem. If only $k < n$ points of P move while the other remain stationary, Roos [12] has shown the number of changes to be $O(k^3(n-k)^2)$, following an earlier result of Albers et al. [2]. When P consists of k rigidly moving groups of n/k points, Huttenlocher et al. [11] have shown this quantity to be $O(n^2 k\alpha(k))$, improving a weaker bound by Aonuma et al. [3].

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($\alpha(\cdot)$ denotes the inverse Ackermann function [1].) Both of these bounds become $O(n^2)$ when $k = O(1)$. Finally, Chew [6] has shown the number of changes to be $O(n^2\alpha(n))$ if the underlying metric is L_1 or L_∞ instead of the standard Euclidean.

In this note we prove the tight bound $\Theta(n^2)$ on the number of combinatorial changes of $\mathrm{Vor}(P)$ if all the points start their movement from a common line or pass through a common line at some instance of time. The proof uses the linearization technique as in [2, 9] and is rather simple, but our observation is to the best of our knowledge novel. It strongly supports the conjecture described above and is applicable in its own right.

2 The Analysis

Theorem 2.1 *Let $P = P(t)$ be a collection of points $p_i = p_i(t)$, $i = 1, \ldots, n$, moving with time $-\infty < t < \infty$ at a common constant speed along linear trajectories in the plane $\mathbb{R}^2$, such that when $t = 0$ all the points lie on a common line. The number of changes in the combinatorial structure of* $\mathrm{Vor}(P)$ *through time is $\Theta(n^2)$.*

Proof: Parameterize $p_i(t)$ as $(x_i(t), y_i(t))$. We can assume without loss of generality that all $p_i(0)$ lie on the x-axis. Hence $y_i(0) = 0$ and we can represent $p_i(t)$ as $(a_i t + b_i, c_i t)$ for appropriate constants a_i, b_i, c_i. Define the *distance function* $f_i(t, x, y)$ of $p_i(t)$ to be the Euclidean distance of a point (x, y) to $p_i(t)$:

$$f_i(t, x, y) = \sqrt{(x - x_i(t))^2 + (y - y_i(t))^2}.$$

For fixed $t = t_0$, the Voronoi diagram of $P(t_0)$ is the orthogonal projection of the bivariate lower envelope [1] (in terms of x and y) of the collection of all $f_i(t_0, x, y)$, $i = 1, \ldots, n$, onto $\mathbb{R}^2$. The evolution of this Voronoi diagram through time is encoded in the trivariate lower envelope (in terms of x, y and t) of the collection of all $f_i(t, x, y)$, $i = 1, \ldots, n$, and every combinatorial change in the Voronoi diagram uniquely corresponds to a vertex in the latter lower envelope. By algebraic manipulation we obtain

$$f_i^2(t, x, y) = \big(x - (a_i t + b_i)\big)^2 + \big(y - c_i t\big)^2 = \\ x^2 + y^2 + (a_i^2 + c_i^2)t^2 + 2a_i b_i t - 2a_i x t - 2c_i y t - 2b_i x + b_i^2.$$

Notice that $a_i^2 + c_i^2$ is the squared speed of p_i and is the same for all points. We can thus assume without loss of generality that $a_i^2 + c_i^2 = 1$. The lower envelope of the collection of $f_i(t, x, y)$ is combinatorially equivalent to the lower envelope of the collection of $g_i(t, x, y) = f_i^2(t, x, y) - x^2 - y^2 - t^2$. We have

$$g_i(t, x, y) = 2a_i b_i t - 2a_i x t - 2c_i y t - 2b_i x + b_i^2.$$

Consider the 4-dimensional space parameterized by t, xt, yt and x. $g_i(t, x, y)$ is a linear function defined over this space. The lower envelope of the collection of all $g_i(t, x, y)$ can thus be linearized to the lower envelope of a collection of n 4-dimensional hyperplanes in $\mathbb{R}^5$. The complexity of the latter lower envelope is $O(n^2)$ [13], which implies the upper bound stated in the theorem. The matching lower bound construction is an easy exercise. ■

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