A Quasi-Polynomial Time Approximation Scheme for Minimum Weight Triangulation (Extended Abstract)

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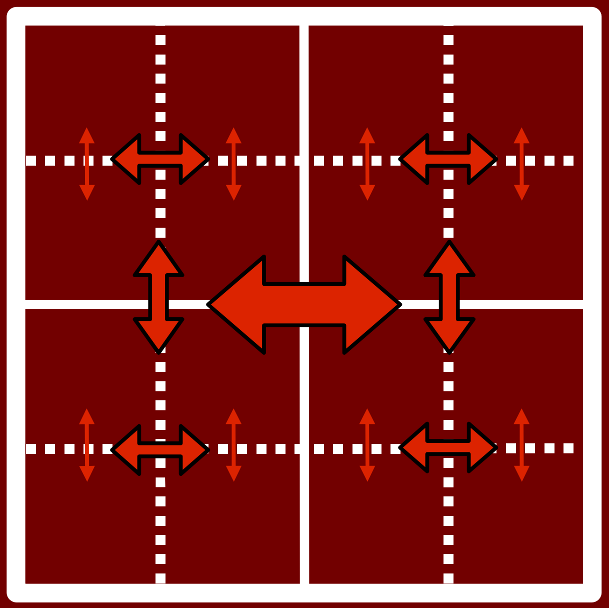
The Minimum Weight Triangulation Problem

- Given a point set P in R^2
- Compute the triangulation of the convex hull that minimizes the sum of edge lengths in the L_2-metric
- Recently proven to be NP-hard [Mulzer and Rote, 2006]

- This paper gives the first QPTAS for the problem (finds a $(1+\varepsilon)$-approximation in $n^{O((\log n)^8)}$ time)
- "Extended abstract" does not provide proofs of several important lemmas

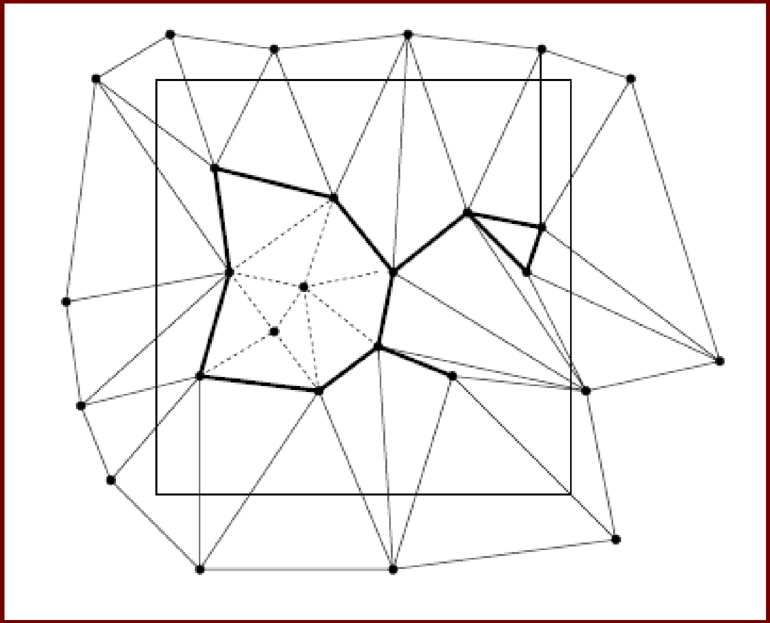
High-Level Algorithmic Strategy I

- Define a square Q_0 around P
- Recursively halve Q_0 to squares of length ≤ 1
- Find a small set of good triangulations for each square/rectangle created by the decomposition
- Optimally merge the smaller triangulations to find a small set of larger triangulations



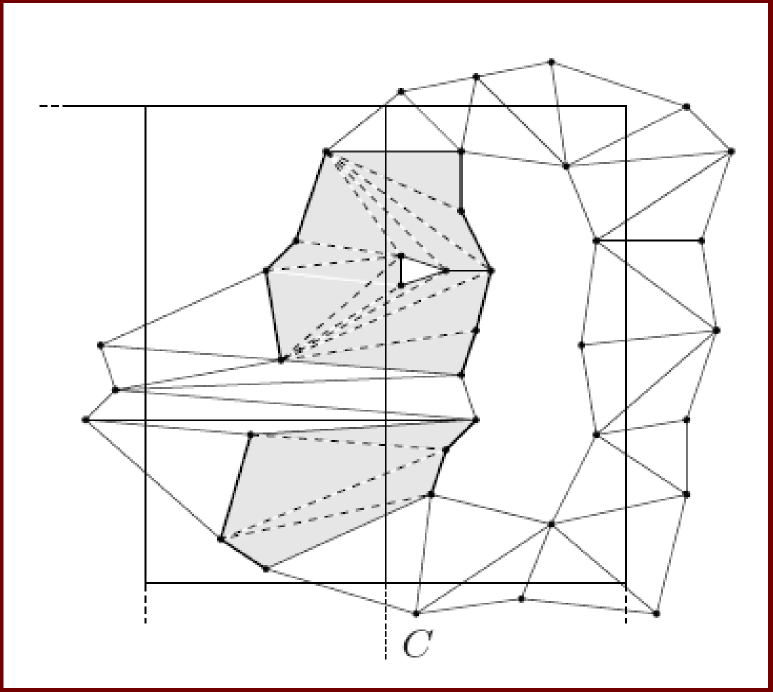
Local Hull

- Given:
  - Triangulation, T, of P
  - Rectangle, R
- Local hull H(R,T) is the set of edges in T such that:
  - e ∈ conv(P), or
  - e ∈ R and is an edge of a triangle with its third vertex outside R



Border Triangulations I

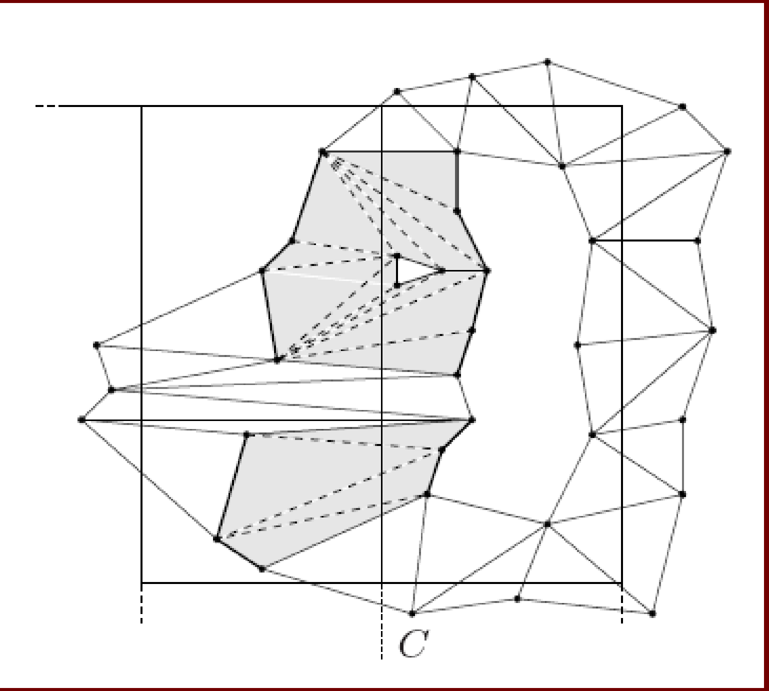
- Children of R (R′ & R″) have their own local hulls (H′ & H″)
- H′ and H″ are joined by border triangles, $\Delta_B(R,T)$, which fall inside R and cut the separator, C



- Edges of $\Delta_B(R,T)$ which cross C are "cut edges" (dotted)
- Other edges of $\Delta_B(R,T)$ are "border edges" (bold)

Border Triangulations II

- Contiguous border triangles form simple polygons with no points in their interiors
- A simple polygon w/ boundary edge set W can be optimally triangulated in $O(W^3)$ time
- Can compute optimal triangulation of R consistent with H′ and H″ in $O(|P \cap R|^3)$



The Approximation Scheme, Attempt 1

- Given P, |P|=n
- Assume the diameter of P $\geq$ sn, for fixed s $\geq$ 0
- Let Q_0 be a square of length $\leq$ 6sn that contains P

- Decompose Q_0, and for every rectangle R compute the set of all possible local hulls, H[R].
- Arbitrarily triangulate every local hull of the smallest rectangles
- For every larger rectangle, triangulate each of its local hulls using optimal triangulation of children's local hulls

The Approximation Scheme, Attempt 1: Correctness

- Define:
  - T*[R,H] is the optimal triangulation of R with local hull H
  - n[R] is the number of points of P in R

- Claim (stated without proof):
  - For every local hull H of every rectangle R, the triangulation T[R,H] created by the scheme satisfies:

    $$\text{len(T[R,H])} \leq \text{len(T*[R,H])} + 6\text{n[R]}$$

  - By assumption, len(T*) ≥ sn, so

    $$\text{len(T)} \leq \text{len(T*)} + 6n \leq (1 + 6/s)\ \text{len(T*)}$$

  - The resulting triangulation is an ε-approximation for s ≥ 6/ε

The Approximation Scheme, Attempt 1: Time Complexity

- Requires polynomial time in $\max_R|H[R]|$
- $\max_R|H[R]|$ may be exponential in P
- Scheme, as stated, is TOO SLOW

- But, we can do better if we don’t consider all local hulls for each rectangle…

The Approximation Scheme, Attempt 2

- Assume we have an algorithm L, which selects a set of “good” local hulls, $H_L[R]$
- Specifically, $\max_R |H_L[R]| \geq f(n)$ for some $f(n) \geq 1$

- Claim (again, unproven):
  - Let T^*_L be the optimal triangulation that uses only the local hulls generated by L
  - In time $O(\mathrm{poly}(n) \cdot f(n)^2)$, we can compute a triangulation T such that

$$\mathrm{len}(T) \leq (1+6/s)\ \mathrm{len}(T^*_L)$$

The Approximation Scheme, Attempt 2: The Key Observation

- A local hull generation algorithm L exists with the following properties:
  - Runs in polynomial time
  - $|H_L[R]| = n^{O((\log n)^8)}$
  - len(T^*_L) $\leq (1+O(1/s))$ len(T^*) (for some Q_0)
- The approximation scheme can find an ε-approximation to T^*_L (and hence, T^*) in time $n^{O((\log n)^8)}$, so it is a QPTAS
- Local hulls generated by L are called “smooth hulls” (more later)

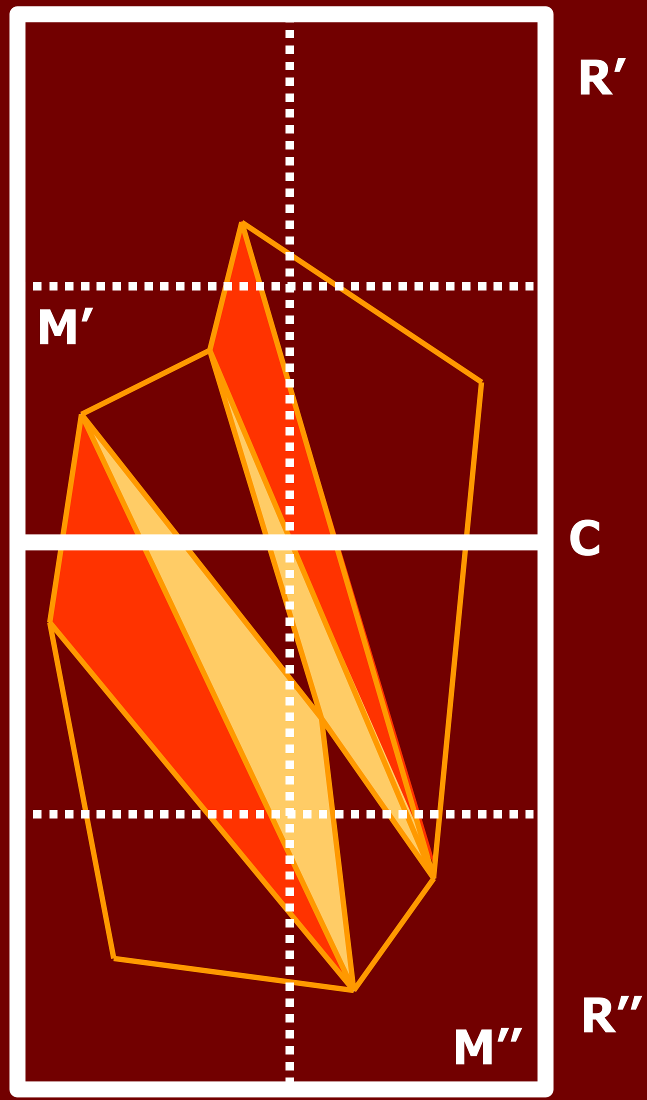
Smooth Hull Preliminaries: Characteristic Triangles I

- Let R′ & R″ be sibling rectangles with parent R
- Let t be the depth of the subdivision of Q_0
- Let $m = 2^{\lceil \log(st) \rceil}$
- Note: $m = O(st) = O(s \cdot \log s + s \cdot \log n)$
- Divide R′ & R″ into $O(m^2)$ regular, square cells
- Call the resulting sets of cells M[R′] and M[R″]

Smooth Hull Preliminaries: Characteristic Triangles II

- Given a fixed T and Q_0, consider sibling pair R′ and R″, divided by separator C
- Let M′ ∈ M[R′] and M″ ∈ M[R″]

- This defines a set of border triangles with at least one endpoint in M′ and M″ (filled)
- The first and last of these triangles to appear along the separator are *characteristic triangles* (filled orange)



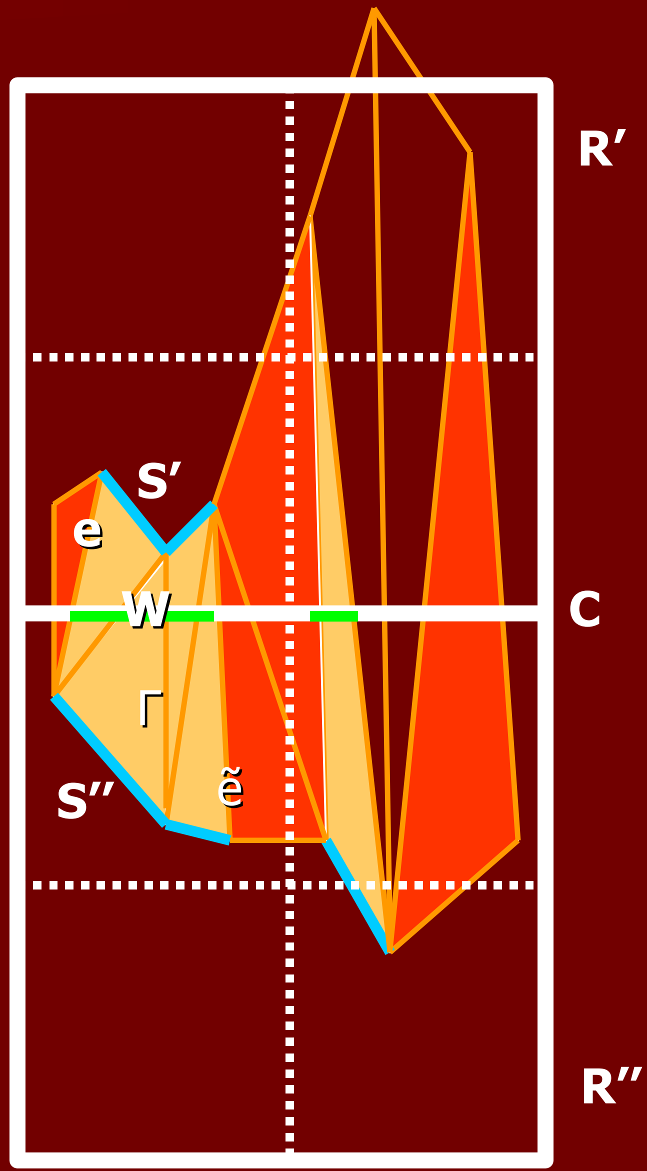
Smooth Hull Preliminaries: Characteristic Triangles III

- If we collect all characteristic triangles across all cells $M' \in M[R']$ and $M'' \in M[R'']$, we define a set $T_{char}(C,T)$

- Note that $m = O(\log n)$
- Thus, $|T_{char}(C,T)| = O(\log^4 n)$
- $T_{char}(C,T)$ is a sparse subset of all border triangles for a separator C

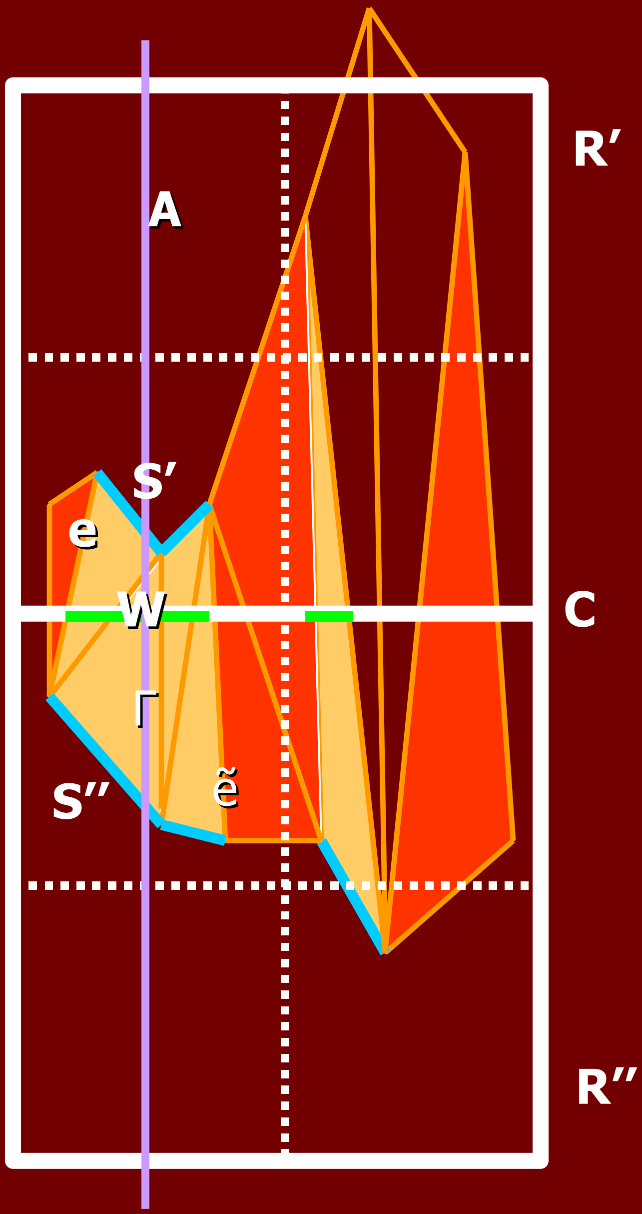
Smooth Hull Preliminaries: Gaps

- Let E_{cut} be the set of cut edges bounding H or some $\Delta \in T_{char}(C,T)$
- Let e,ẽ be subsequent edges of E_{cut}
- A set of noncharacteristic triangles between e,ẽ is called a "gap", Γ(e,ẽ) (golden triangles)
- The intersection of a gap with the separator is called a "window", W(Γ) (green segments)
- Triangles through a particular window define border strips, S(Γ)′ and S(Γ)″ (blue edges)



Smooth Hull Preliminaries: Border Strips are Thin

- Claim: For every gap Γ, there exists a line A such that:
  - Either:
    - A is parallel to C, and runs through the center of W(Γ), or
    - A intersects two points of P and intersects W(Γ)
  - No point on S′ or S″ is more than 2c away from A, where c is the width of a cell in M[R′], M[R″]
- No proof, but believable given that characteristic triangles are densely sampled

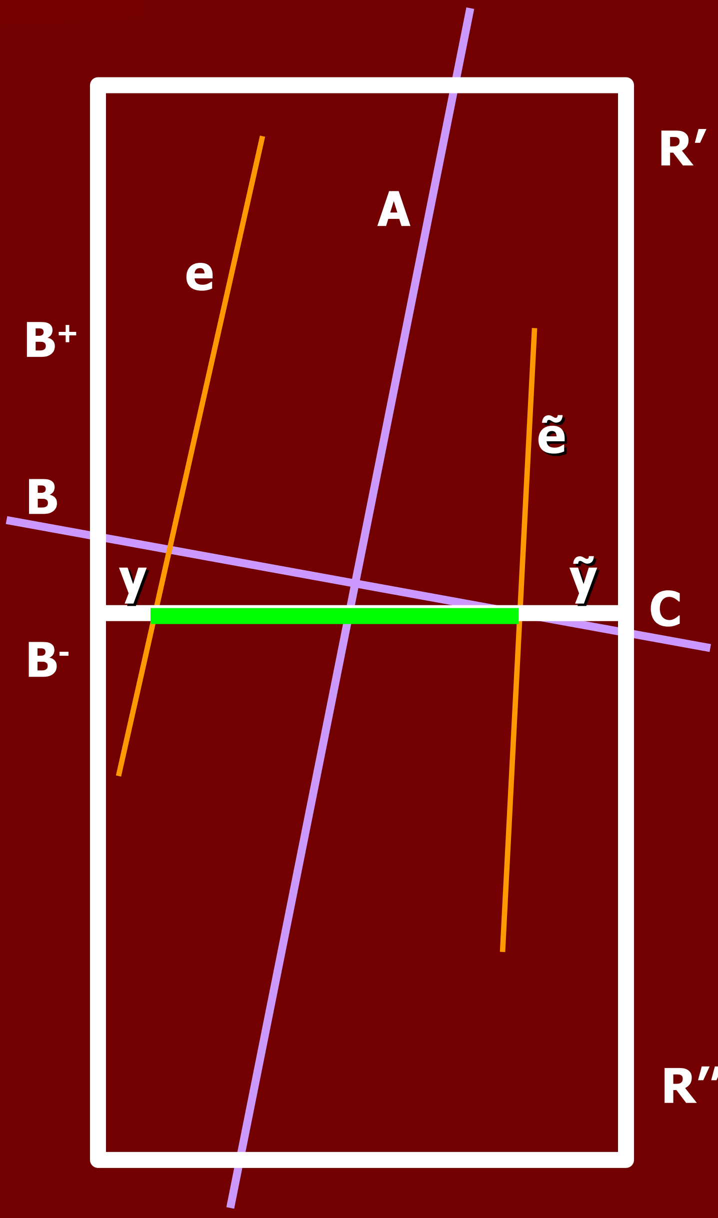


Smooth Hull Preliminaries: Skeletons

- The previous analysis identified border strips, S(Γ), from a triangulation
- Given a point set P, we would like to identify good border strips, and use them to build up a triangulation
- Given cut edges e, ẽ (and the corresponding gap, we will create these border strips by:
  - selecting an arbitrary A
  - selecting a subset of points close to A (known as a "skeleton")
  - finding a good way to connect the skeleton points

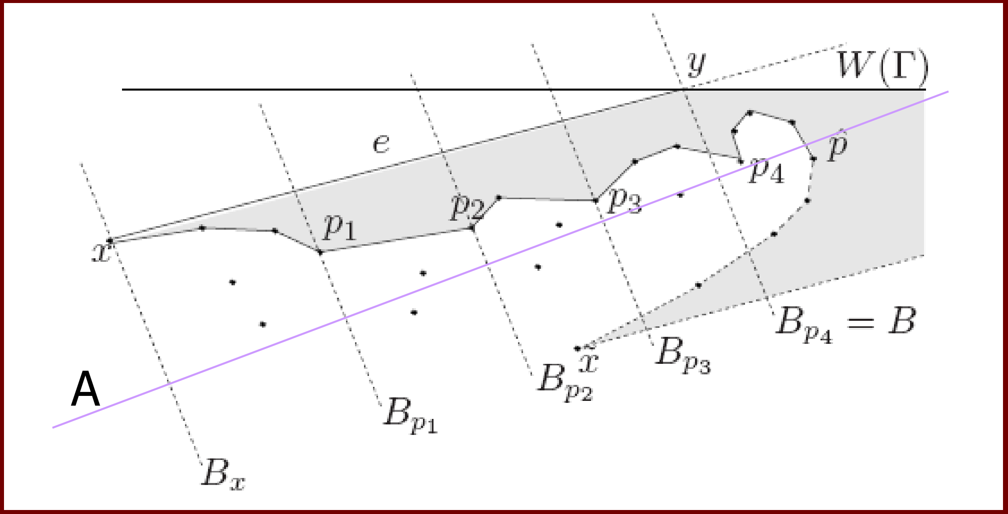
Smooth Hull Preliminaries: Skeleton Preliminaries

- Given:
  - Rectangle R with separator C
  - Cut edges e,ẽ intersecting C at y,ỹ
  - A line A through window W(Γ)
- For building S′~ for R′, we define:
  - Line B ⊥ A intersecting y and/or ỹ, so Γ does not appear on the R′ side of B
  - B^+ and B^- are halfspaces defined by B, oriented towards R′ and R″, respectively
  - Line B_p // B:
    - intersects a point p if $P \in B^+$
    - else $B_p = B$



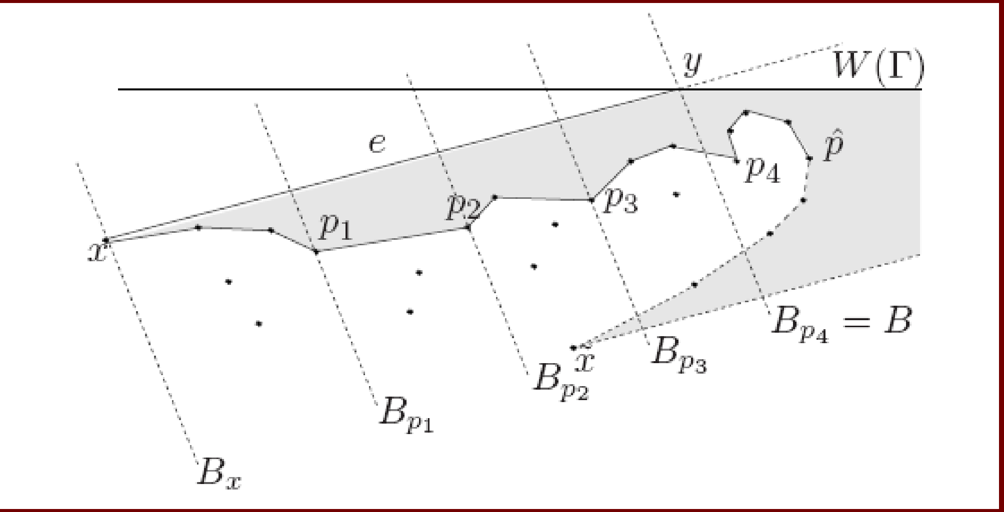
Smooth Hull Preliminaries: Skeleton Definition

- A skeleton is a subset of points K in R′, about the line A.
- K can be decomposed into K+, K-, and p^
- K has the following properties:
  - p^ is the point nearest to W(Γ), along the axis of A
  - Endpoint of e is in K^+ and farthest from W(Γ) in K^+, along the axis of A
  - Endpoint of ẽ is in K^- and farthest from W(Γ) in K^-, along the axis of A
  - K^+ and K^- each have at most one point in B^-
  - Any two points in K^+ (or K^-) are separated by at least c_l along A



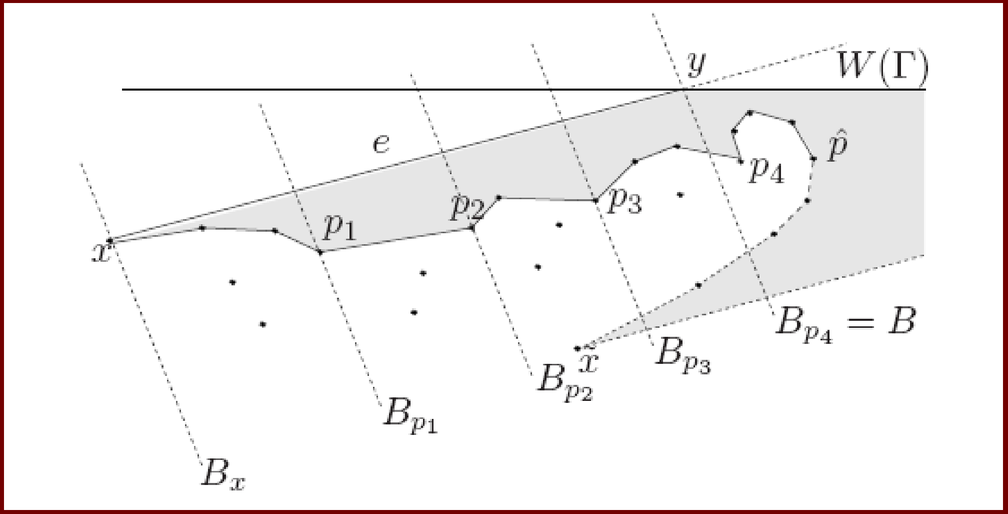
Smooth Hull Preliminaries: Convex Continuation S~ of K

- S~ is a piecewise-linear path connecting K
- To join adjacent points p,q in K^+
  - Consider the trapezoid D_{pq} bounded by pq, B_p, B_q, & e
  - Determine the convex hull of points $P \cap D_{pq}$ (including p,q)
  - Join p,q using the convex hull path that runs nearer to e
- Analogous method used to connect K^- and points about p^



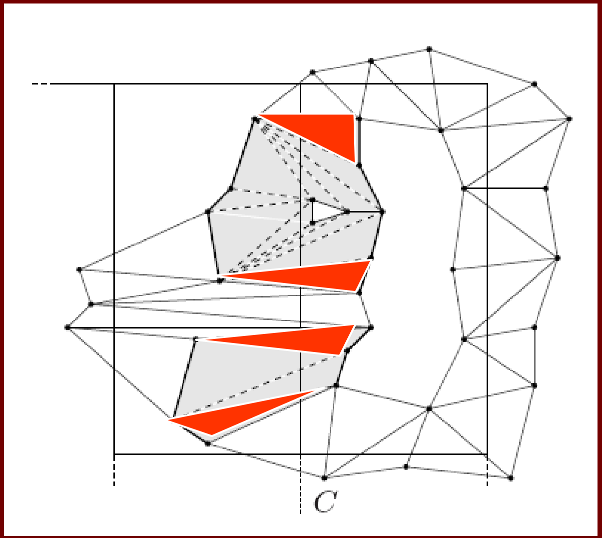
Smooth Hull Preliminaries: Feasible Skeletons

- A skeleton is considered "feasible" if:
  - S is not self-intersecting nor intersects with e & ẽ
  - Area bounded by S~, e, ẽ, and W(Γ) contains no points
  - Every vertex in S~ has a clear line of sight to W(Γ)

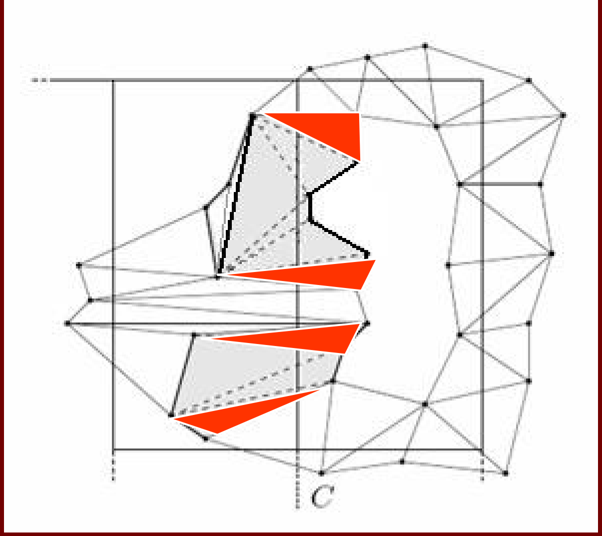


Smooth Hull Definition

- Given a triangulation T and a square Q_0, T is said to "patch Γ smoothly" if there exist A and K that induce $S^{\sim}(K) = S(\Gamma,T)$
- A local hull H′ of R′ is considered "smooth" if all gaps that induce its edges are smoothly patched or if it is identical to conv(P)
- A triangulation is said to be smooth if all of its local hulls are smooth



NOT SMOOTH



SMOOTH

Smooth Hull Generation Algorithm, L

- Initialize $H_L[Q_0] = \{ conv(P) \}$
- Top-down algorithm computes $H_L[R']$ from $H_L[R]$ as follows:
  - For each $H \in H_L[R]$, enumerate all feasible sets T_{char}
  - For each pair (H, T_{char}), compute set of gaps $\{\Gamma_1,\ldots,\Gamma_r\}$
  - For gap set, enumerate all possible sets of feasible skeletons, $\{K_1,\ldots,K_r\}$
  - For every skeleton set, if the corresponding S are disjoint, combine it with edges from H and T_{char} lying entirely in R' to get a local hull H' which we add to $H_L[R']$

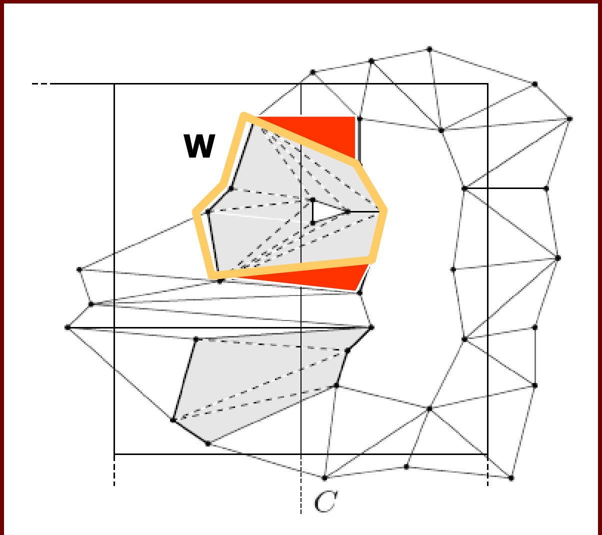
Smooth Hull Generation Algorithm, L

- Claims:
  - $H_L[R]$ contains all smooth hulls of R.
  - For fixed s, $|H_L[R]| \leq n^{O((\log n)^8)}$
  - The family $(H_L[R])$ can be computed in time $n^{O((\log n)^8)}$
- Proof is "far beyond the scope of this extended abstract"

Smoothing Lemma

- Let W(T,e,ẽ) be the outer boundary of a set of gap triangles of T

- Smoothing Lemma:
  - Given
    - a triangulation T and
    - a separator C at level l with gap Γ(e,ẽ)
  - There exist feasible skeletons K & K′ and a triangulation T~ such that
    - T and T~ only differ inside W(T,e,ẽ)
    - T~ patches Γ(e,ẽ) smoothly with respect to K and K′
    - len(T~)-len(T) ≤ $18c_l \cdot h$, where h is # of edges of T inside W(T,e,ẽ)



Choosing Q_0

- T_L* will not always generally be a good approximation to T*
- There exists a set of squares Q = { Q_1,..., Q_k } of size poly(n), which by must contain at least one good square Q_0
- We can run the algorithm once on each square in the set, and take the smallest T_L*

Conclusion

- This paper claims to provide a QPTAS for minimum weight triangulation, but without proofs, cannot confirm
- Framework suggests that if a polynomial number of good local hulls could be generated, a PTAS would exist