Approximating the Minimum Weight Steiner Triangulation

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Let P be a set of points in $\mathbb{R}^2$.

Definition: A **triangulation** of P is a plane graph such that

- the set of vertices is P
- the planar embedding uses only straight lines
- all faces, **except perhaps the outer face**, are triangles

Definition: The **weight** of a triangulation T, denoted by $|T|$, is the sum of the Euclidean lengths of its edges.

Definition: A **minimum weight triangulation** of P (MWT(P)) is a triangulation of P of minimum weight.

Definition: A **minimum weight Steiner triangulation** of P (MWST(P)), is a triangulation of $P \cup S$ of minimum weight over all point sets S.

Definition: The points in S are called **Steiner points**.

Definition: An α-**approximate** MWT(P) (resp. MWST(P)) is a triangulation of P such that $|T| \leq \alpha|\text{MWT}(P)|$ (resp. $|T| \leq \alpha|\text{MWST}(P)|$).

Quadtree Triangulation Algorithm

Definition: A **balanced quadtree** is a quadtree in which the side lengths of any two horizontally or vertically adjacent leaf squares differ by at most a factor of two.

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(1) Compute a balanced quadtree on P

(2) Compute a convex hull of P, $H(P)$

(3) Let S be the union of the following two sets

- the set of quadtree vertices inside $H(P)$
- the quadtree points and the boundary of $H(P)$

(4) For each leaf square in the quadtree, compute the MWT of the subset of $P \cup S$ contained in its interior or on its boundary.

Observation: In (4), the size of the subset is $O(1)$, because

- there is at most one point in each square (interior and the boundary)

- the balance condition (at most one subdivision point on each size of the square)
- the boundary of $H(P)$ can intersect the square at most four times

Consequence: In (4), can compute the MWT of the subset in $O(1)$ time.

Definition: The triangulation constructed from P by the above algorithm is called the **quadtree triangulation** of P (QT(P)).

Note: $QT(P)$ always includes

- the boundary of $H(P)$, because $H(P \cup S) = H(P)$
- parts of quadtree edges inside $H(P)$

Observation: Running time is inherently super-polynomial, since the size of the quadtree is not bounded in $|P|$.

Proof of Approximation Ratio

Roadmap:

(1) $|\,\mathrm{QT}(P)| \leq O(1) \cdot |\,\mathrm{MWT}(P)|$.

(2) The weight of $\mathrm{QT}(P \cup S)$ is non-increasing in S.

(3) $|\,\mathrm{QT}(P)| \leq O(1) \cdot |\,\mathrm{MWST}(P)|$.

Proof (1) $\wedge$ (2) $\Rightarrow$ (3)

Let S be the optimal set of Steiner points.

By (2), $|\,\mathrm{QT}(P)| \leq |\,\mathrm{QT}(P \cup S)|$.

By (1), $|\,\mathrm{QT}(P \cup S)| \leq O(1) \cdot |\,\mathrm{MWT}(P \cup S)|$.

By the choice of S, $|\,\mathrm{QT}(P \cup S)| \leq O(1) \cdot |\,\mathrm{MWST}(P)|$. ■

Consequence: In general, any $O(1)$-approximate MWT(P) which is non-increasing under addition of points inside $H(P)$ is also an $O(1)$-approximate MWST(P).

Proof (2)

Inserting new points inside $H(P)$ can only create more subdivisions in the quadtree.

The claim follows because the total weight of the per-square MWTs is non-decreasing in the number of points (proof?).

Proof (1) $|\,\mathrm{QT}(P)| \leq O(1) \cdot |\,\mathrm{MWT}(P)|$

Lemma 1: The cost of $QT(P)$ within a leaf square of side l is $O(l)$.

Proof:

A leaf square contains $O(1)$ points (in the interior and on the boundary).

The length of any edge in the $\mathrm{QT}(P)$ restricted to the square is clearly $O(l)$.

The claim follows. ■

Assume: All input points are in squares of side length $O(\epsilon/n \cdot \mathrm{diam}(P))$ (otherwise, further subdivision only increases the cost of $\mathrm{QT}(P)$).

Observation: By the above Lemma, the weight of the $\mathrm{QT}(P)$ in the all squares that contain input points is $O(\epsilon)\,\mathrm{diam}(P)$.

Observation: The weight of $\mathrm{MWT}(P) \geq \mathrm{diam}(P)$.

Consequence: The cost of the quadtree triangulation in the squares that contain the input points is at most $O(\epsilon)\,\mathrm{MWT}(P)$, and thus can be disregarded in the following analysis.

A charging scheme – distributing the weight of $\mathrm{QT}(P)$ onto some $\mathrm{MWT}(P)$.

Separate treatment for the squares entirely inside $H(P)$ and those that are cut by the boundary of $H(P)$.

Squares inside $H(P)$

Will eventually be charged to various edges of the $\mathrm{MWT}(P)$, but through a two-stage process.

First charging to the quadtree inside the convex hull.

Lemma: The weight of $QT(P)$ restricted to the quadtree squares inside $H(P)$ is at most $O(1)$ times the weight of the edges of those squares.

Proof: Consider such a square.

By Lemma 1, the weight of QT(P) inside the square is at most an $O(1)$ times its side length.

Charge this weight to the bottom and right edges of the square $\Rightarrow$ $O(1)$ charge per unit length.

Leaf nodes disjoint $\Rightarrow$ sets of bottom and right edges disjoint $\Rightarrow$ each edge charged at most once.

Charging the quadtree to MWT(P)

Lemma: The weight of the quadtree squares entirely inside $H(P)$ is at most $O(1) \cdot |\,\text{MWT}(P)|$.

Proof: Consider a square s of side length l which is entirely inside $H(P)$.

Intuition: Charge s to pair of "long" edges of the MWT whose common endpoint is "close" to s.

One key insight of the paper: such two edges always exist.

Claim: For any square of size l, there is a point in P within $O(l)$ from the square which is incident on two edges in the MWT(P) whose total length is $\Omega(l)$.

Proof [Claim $\Rightarrow$ Lemma]:

Charge the weight of s to the two edges proportional to their length. A MWT edge of length λ is charged an $O(l/\lambda)$ amount by s.

The Claim implies that the only size l squares that may charge some edge of MWT(P) are those that are within $O(l)$ of any of the two endpoints. Clearly, there is $O(1)$ of such squares.

An edge of length λ can be charged by squares of size $l = C\lambda$ for a constant C (also by Claim), and smaller.

The number of squares of any size is bounded by the same constant.

The total charge $O(1)[O(l/\lambda)+O(l/(2\lambda))+\cdots] = O(l/\lambda) = O(C) = O(1)$. ■

Proof [Claim]: There is a node within $O(l)$...

Induction on the size l.

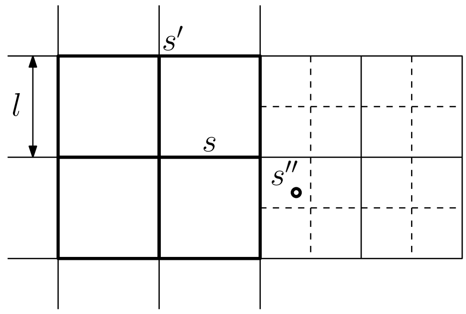
Base case: smallest squares contain points ⇒ holds trivially.

Assume that the claim holds for all sizes smaller than l, consider a square s of size l.

If there is a point in one of the siblings of s, we are done.

Otherwise, the parent square s' was subdivided because of the balance condition.

There must be at least one sibling of s' which is subdivided strictly one level more (see the figure below).



Let s'' be a leaf square in the subtree of this sibling.

All squares involved in the above argument are within $O(l)$ from s.

The claim for s now follows from the induction hypothesis applied to s''.

Let c be the center of s.

Let x be some point within $O(l)$ from s, whose existence is guaranteed by the first part.

Let s' be the smallest square that contains s and whose two sides are parallel to the line through c and x.

Consider the segment cx and all the edges of the MWT that intersect it, ordered according to the intersection points from c to x.

Any two consecutive edges share exactly one vertex.

Number the endpoints $t_1, t_2, \ldots, x$, and add t_0, the third vertex of the MWT triangle containing c.

Clearly, there must be one of the points in the sequence within s'. Let t be the first such point.

Case 1: $t \in \{t_0, t_1, t_2\}$ (figure below, left)

Then t satisfies the conditions of our Claim.

It is inside s', thus within $O(l)$ from s.

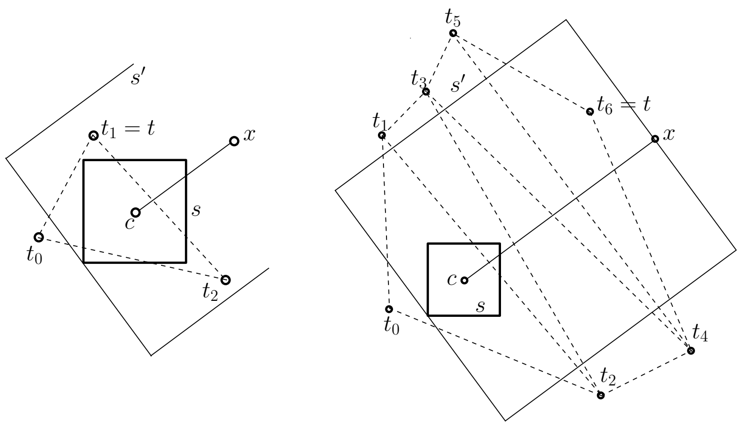
Consider the segments ct_0, ct_1, ct_2.

The maximum angle formed by them is between 120° (averaging argument) and 180° (containment).

The vertices corresponding to these two segments are the endpoints of the desired edge.

By triangle inequality, the sum of the lengths of any two edges is at least l.

Case 2: $t = t_i$ for $i \geq 3$ (figure below, right)



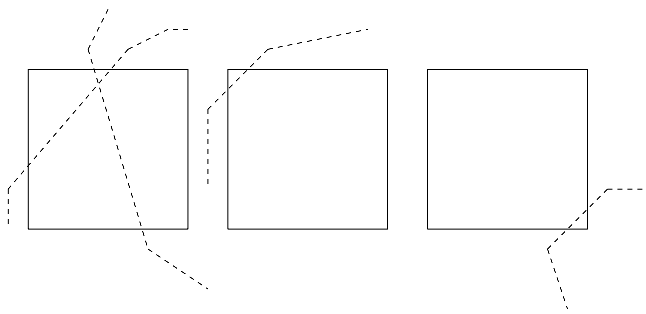
By choice of t, it follows that the length of (t_{i-1}, t_{i-2}) is $O(l)$.

By triangle inequality, t_i satisfies the conditions of the Claim. ■

Squares crossed by the boundary of $H(P)$

Charging directly to the boundary of $H(P)$, without intermediate charging to the quadtree edges.

Note: No vertices of $H(P)$ inside any square, because we are only considering empty squares.



Case 1: If s is cut at least $l/3$ away from any vertex, then the intersecting segment is at least $l/3$ long

Charge the weight of the triangulation to the segment, $O(1)$ charge per unit length. Each intersecting segment charged only by one square across all scales (the one that it intersects).

Case 2: If s is cut near a corner (both segments are less than $l/3$ from some corner), and c is outside $H(P)$, the triangulation is trivial.

Charge to the intersecting segment, $2 = O(1)$ charge per unit length. Each segment charged only once across all scales, as in Case 1.

Case 3: If s is cut near a corner, and c is inside $H(P)$, then the two orthogonal neighbors closest to the corner are in Case 1 (have "long" intersecting segments).

Note: These neighbors are taken to be leaf squares. By balance condition, they are at least half the size of s. The lengths of their intersecting segments of $H(P)$ are thus $\Omega(l)$.

Charge to the two intersecting segments proportional to their length.

The new charge per unit length is again $O(1)$.

Each square can be charged by $O(1)$ squares across all scales (only by the neighboring leaf nodes, whose scale must differ by one).

That concludes the proof of the lemma. ■

Note: There is still the issue of non-polynomial running time and quadtree size.

Truncated Quadtree Triangulation Algorithm

Let r be the side length of the smallest bounding square of P.

Everything is the same, except the quadtree squares are not subdivided once their side length becomes at most r/n.

Definition: The triangulation produced by the algorithm described above is called the *truncated quadtree triangulation*.

Intuition: The weight of an arbitrary triangulation on the points inside such small square is negligible.

Lemma: The weight of any triangulation on k input points inside a square of side length l (together with the $O(1)$ corner points and subdivision points on the boundary of the square) is $O(kl)$.

Proof: Easy – any triangulation of $O(k)$ points has $O(k)$ edges (by Euler's Theorem), each one of which is of length $O(l)$. ■

Theorem: The truncated tree triangulation described above can be constructed in time $O(n \log n)$, has $O(n \log n)$ Steiner points, and is an $O(1)$-approximate MWST(P).

Proof: The number of levels is clearly $O(\log n)$.

Fix a level.

By the same reasoning as before (proof of Claim), for any square at this level on of the following holds

- The square contains a point
- One of the square's siblings contains a point
- One of the square's parent's siblings contains a point (the balance condition)

Consequence: The number of squares at each level is $O(n)$. Thus, the number of squares at all levels, and thus the construction time, is $O(n \log n)$.

The triangulation within each square can be constructed in $O(k \log k)$ time, where k is the number of points in the square.

There are n points total, thus the time for computing the triangulations, and the total running time, is $O(n \log n)$. ■