Differential Geometry of Curves

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Motivation

• Applications
  From “Discrete Elastic Rods” by Bergou et al.

• Good intro to differential geometry on surfaces

• Nice theorems
Parameterized Curves

Intuition

A particle is moving in space

At time $t$ its position is given by

$$\alpha(t) = (x(t), y(t), z(t))$$
A parameterized differentiable curve is a differentiable map $\alpha: I \to R^3$ of an interval $I = (a,b)$ of the real line $R$ into $R^3$.

$\alpha$ maps $t \in I$ into a point $\alpha(t) = (x(t), y(t), z(t)) \in R^3$ such that $x(t), y(t), z(t)$ are differentiable.

A function is differentiable if it has, at all points, derivatives of all orders.
Parameterized Curves

A Simple Example

\[ \alpha_1(t) = (a \cos(t), a \sin(t)) \]
\[ t \in [0, 2\pi] = I \]

\[ \alpha_2(t) = (a \cos(2t), a \sin(2t)) \]
\[ t \in [0, \pi] = I \]

\( \alpha(I) \subset \mathbb{R}^3 \) is the trace of \( \alpha \)

→ Different curves can have same trace
More Examples

\[ \alpha(t) = (a \cos(t), a \sin(t), bt), \quad t \in R \]

\[ b = \varnothing \]
More Examples

\[ \alpha(t) = (t^3, t^2), \quad t \in \mathbb{R} \]

Is this “OK”?
The Tangent Vector

Let

$$\alpha(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$$

Then

$$\alpha'(t) = (x'(t), y'(t), z'(t)) \in \mathbb{R}^3$$

is called the tangent vector (or velocity vector) of the curve $\alpha$ at $t$
Back to the Circle

\[ \alpha(t) = (\cos(t), \sin(t)) \]

\[ \alpha'(t) = (-\sin(t), \cos(t)) \]

\[ \alpha'(t) \] - direction of movement

\[ |\alpha'(t)| \] - speed of movement
Back to the Circle

$\alpha_1(t) = (\cos(t), \sin(t))$

$\alpha_2(t) = (\cos(2t), \sin(2t))$

Same direction, different speed
The Tangent Line

Let $\alpha: I \rightarrow \mathbb{R}^3$ be a parameterized differentiable curve.

For each $t \in I$ s.t. $\alpha'(t) \neq 0$ the tangent line to $\alpha$ at $t$ is the line which contains the point $\alpha(t)$ and the vector $\alpha'(t)$.
Regular Curves

If $\alpha'(t) = 0$, then $t$ is a singular point of $\alpha$.

$$\alpha(t) = (t^3, t^2), \ t \in \mathbb{R}$$

A parameterized differentiable curve $\alpha: I \rightarrow \mathbb{R}^3$ is regular if $\alpha'(t) \neq 0$ for all $t \in I$.
Spot the Difference

\[ \alpha_1(t) = (t^3, t^2) \]
Differentiable
Not regular

\[ \alpha_2(t) = (t, |t|) \]
Not differentiable

Which differentiable curve has the same trace as \( \alpha_2 \) ?
Arc Length of a Curve

How long is this curve?

Approximate with straight lines

Sum lengths of lines: \[ \Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} \]
Arc Length

Let $\alpha: I \rightarrow \mathbb{R}^3$ be a parameterized differentiable curve. The *arc length* of $\alpha$ from the point $t_0$ is:

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| \, dt$$

$$= \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

The arc length is an *intrinsic* property of the curve – does not depend on choice of parameterization.
Examples

\[ \alpha(t) = (a \cos(t), a \sin(t)), \ t \in [0,2\pi] \]
\[ \alpha'(t) = (-a \sin(t), a \cos(t)) \]

\[
L(\alpha) = \int_0^{2\pi} |\alpha'(t)|\,dt \\
= \int_0^{2\pi} \sqrt{a^2 \sin^2(t) + a^2 \cos^2(t)}\,dt \\
= a \int_0^{2\pi} \,dt = 2\pi a
\]
Examples

\[ \alpha(t) = (a \cos(t), b \sin(t)), \quad t \in [0, 2\pi] \]
\[ \alpha'(t) = (-a \sin(t), b \cos(t)) \]

\[ L(\alpha) = \int_{0}^{2\pi} |\alpha'(t)| \, dt \]
\[ = \int_{0}^{2\pi} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)} \, dt \]
\[ = ?? \]

No closed form expression for an ellipse
Closed-Form Arc Length Gallery

Cycloid
\[ \alpha(t) = (at - a \sin(t), a - a \cos(t)) \]
\[ L(\alpha) = 8a \]

Logarithmic Spiral
\[ \alpha(t) = (ae^{bt} \cos(t), ae^{bt} \sin(t)) \]

Catenary
\[ \alpha(t) = (t, a/2 \left( e^{t/a} + e^{-t/a} \right)) \]
Curves with Infinite Length

The integral \( s(t) = \int_{t_0}^{t} |\alpha'(t)| \, dt \) does not always converge.

→ Some curves have infinite length

Koch Snowflake
Arc Length Parameterization

A curve $\alpha: I \rightarrow \mathbb{R}^3$ is parameterized by arc length if $|\alpha'(t)| = 1$, for all $t$

For such curves we have

$$s(t) = \int_{t_0}^{t} dt = t - t_0$$
Arc Length Re-Parameterization

Let $\alpha: I \to \mathbb{R}^3$ be a regular parameterized curve, and $s(t)$ its arc length.
Then the inverse function $t(s)$ exists, and
\[ \beta(s) = \alpha(t(s)) \]
is parameterized by arc length.

**Proof:**
$\alpha$ is regular $\Rightarrow s'(t) = |\alpha'(t)| > 0$
$\Rightarrow s(t)$ is a monotonic increasing function
$\Rightarrow$ the inverse function $t(s)$ exists
$\Rightarrow \beta'(s) = \alpha'(t(s))t'(s) = \alpha'(t(s))/s'(t(s)) = \alpha'(t(s))/\left|\alpha'(t(s))\right|$
$\Rightarrow |\beta'(s)| = 1$
The Local Theory of Curves

Defines local properties of curves

Local = properties which depend only on behavior in neighborhood of point

We will consider only curves parameterized by arc length
Curvature

Let \( \alpha: I \rightarrow \mathbb{R}^3 \) be a curve parameterized by arc length \( s \). The curvature of \( \alpha \) at \( s \) is defined by:

\[
|\alpha''(s)| = \kappa(s)
\]

\( \alpha'(s) \) – the tangent vector at \( s \)

\( \alpha''(s) \) – the change in the tangent vector at \( s \)

\( R(s) = 1/\kappa(s) \) is called the radius of curvature at \( s \).
Examples

**Straight line**

\[ \alpha(s) = us + v, \; u,v \in \mathbb{R}^2 \]
\[ \alpha'(s) = u \]
\[ \alpha''(s) = 0 \quad \rightarrow \quad |\alpha''(s)| = 0 \]

**Circle**

\[ \alpha(s) = (a \cos(s/a), a \sin(s/a)), \; s \in [0,2\pi a] \]
\[ \alpha'(s) = (-\sin(s/a), \cos(s/a)) \]
\[ \alpha''(s) = (-\cos(s/a)/a, -\sin(s/a)/a) \quad \rightarrow \quad |\alpha''(s)| = 1/a \]
Examples

Cornu Spiral
A curve for which $\kappa(s) = s$

Generalized Cornu Spiral
A curve for which $\kappa(s)$ is a polynomial function of $s$
The Normal Vector

$|\alpha'(s)|$ is the arc length
$\alpha'(s)$ is the tangent vector

$|\alpha''(s)|$ is the curvature
$\alpha''(s)$ is ?
Detour to Vector Calculus

Lemma:
Let \( f, g : I \to \mathbb{R}^3 \) be differentiable maps which satisfy \( f(t) \cdot g(t) = \text{const} \) for all \( t \).

Then:
\[
f'(t) \cdot g(t) = -f(t) \cdot g'(t)
\]

And in particular:
\[
|f(t)| = \text{const} \text{ if and only if } f(t) \cdot f'(t) = 0 \text{ for all } t
\]
Detour to Vector Calculus

Proof:
If \( f \cdot g \) is constant for all \( t \), then \((f \cdot g)' = 0\).

From the product rule we have:

\[
(f \cdot g)'(t) = f(t)' \cdot g(t) + f(t) \cdot g'(t) = 0
\]

\[\rightarrow\]

\[
f'(t) \cdot g(t) = -f(t) \cdot g'(t)
\]

Taking \( f = g \) we get:

\[
f'(t) \cdot f(t) = -f(t) \cdot f'(t)
\]

\[\rightarrow\]

\[
f'(t) \cdot f(t) = 0
\]
Back to Curves

\( \alpha \) is parameterized by arc length
\[
\alpha'(s) \cdot \alpha'(s) = 1
\]

Applying the Lemma
\[
\alpha''(s) \cdot \alpha'(s) = 0
\]

\( \rightarrow \) The tangent vector is orthogonal to \( \alpha''(s) \)
The Normal Vector

\[ \alpha'(s) = T(s) \] - tangent vector

|\alpha'(s)| - arc length

\[ \alpha''(s) = T'(s) \] - normal direction

|\alpha''(s)| - curvature

If |\alpha''(s)| \neq 0, define \( N(s) = T'(s)/|T'(s)| \)

Then \( \alpha''(s) = T'(s) = \kappa(s)N(s) \)
The Osculating Plane

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the osculating plane at $s$. 
The Binormal Vector

For points $s$, s.t. $\kappa(s) \neq 0$, the binormal vector $B(s)$ is defined as:

$$B(s) = T(s) \times N(s)$$

The binormal vector defines the osculating plane.
The Frenet Frame

$$\{T(s), N(s), B(s)\}$$ form an orthonormal basis for $R^3$ called the Frenet frame.

How does the frame change when the particle moves?

What are $T', N', B'$ in terms of $T, N, B$?
\[ T'(s) \]

Already used it to define the curvature:

\[ T'(s) = \kappa(s)N(s) \]

Since in the direction of the normal, its orthogonal to \( B \) and \( T \)
What is $N'(s)$ as a combination of $N, T, B$?

We know: $N(s) \cdot N(s) = 1$

From the lemma $\rightarrow N'(s) \cdot N(s) = 0$

We know: $N(s) \cdot T(s) = 0$

From the lemma $\rightarrow N'(s) \cdot T(s) = -N(s) \cdot T'(s)$

From the definition $\rightarrow \kappa(s) = N(s) \cdot T'(s)$

$\rightarrow N'(s) \cdot T(s) = -\kappa(s)$
The Torsion

Let $\alpha: I \rightarrow \mathbb{R}^3$ be a curve parameterized by arc length $s$. The torsion of $\alpha$ at $s$ is defined by:

$$\tau(s) = N'(s) \cdot B(s)$$

Now we can express $N'(s)$ as:

$$N'(s) = -\kappa(s) \mathbf{T}(s) + \tau(s) \mathbf{B}(s)$$
The curvature indicates how much the normal changes, in the direction tangent to the curve.

The torsion indicates how much the normal changes, in the direction orthogonal to the osculating plane of the curve.

The curvature is always positive, the torsion can be negative.

Both properties do not depend on the choice of parameterization.
What is $B'(s)$ as a combination of $N,T,B$?

We know: $B(s) \cdot B(s) = 1$

From the lemma $\Rightarrow B'(s) \cdot B(s) = 0$

We know: $B(s) \cdot T(s) = 0, B(s) \cdot N(s) = 0$

From the lemma $\Rightarrow$

\[ B'(s) \cdot T(s) = -B(s) \cdot T'(s) = -B(s) \cdot \kappa(s)N(s) = 0 \]

From the lemma $\Rightarrow$

\[ B'(s) \cdot N(s) = -B(s) \cdot N'(s) = -\tau(s) \]

Now we can express $B'(s)$ as:

$$B'(s) = -\tau(s) N(s)$$
The Frenet Formulas

\[ T'(s) = \kappa(s)N(s) \]
\[ N'(s) = -\kappa(s)T(s) + \tau(s)B(s) \]
\[ B'(s) = -\tau(s)N(s) \]

In matrix form:

\[
\begin{bmatrix}
T'(s) & N'(s) & B'(s)
\end{bmatrix} =
\begin{bmatrix}
T(s) & N(s) & B(s)
\end{bmatrix}
\begin{bmatrix}
0 & -\kappa(s) & 0 \\
\kappa(s) & 0 & -\tau(s) \\
0 & \tau(s) & 0
\end{bmatrix}
\]
An Example – The Helix

\[ \alpha(t) = (a \cos(t), a \sin(t), bt) \]

In arc length parameterization:
\[ \alpha(s) = (a \cos(s/c), a \sin(s/c), bs/c), \text{ where } c = \sqrt{a^2 + b^2} \]

Curvature: \[ \kappa(s) = \frac{a}{a^2 + b^2} \]
Torsion: \[ \tau(s) = \frac{b}{a^2 + b^2} \]

Note that both the curvature and torsion are constants
A Thought Experiment

Take a straight line
Bend it to add curvature
Twist it to add torsion
→ You got a curve in $R^3$

Can we define a curve in $R^3$ by specifying its curvature and torsion at every point?
The Fundamental Theorem of the Local Theory of Curves

Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, $s \in I$, there exists a regular parameterized curve $\alpha: I \to R^3$ such that $s$ is the arc length, $\kappa(s)$ is the curvature, and $\tau(s)$ is the torsion of $\alpha$. Moreover, any other curve $\beta$, satisfying the same conditions, differs from $\alpha$ only by a rigid motion.