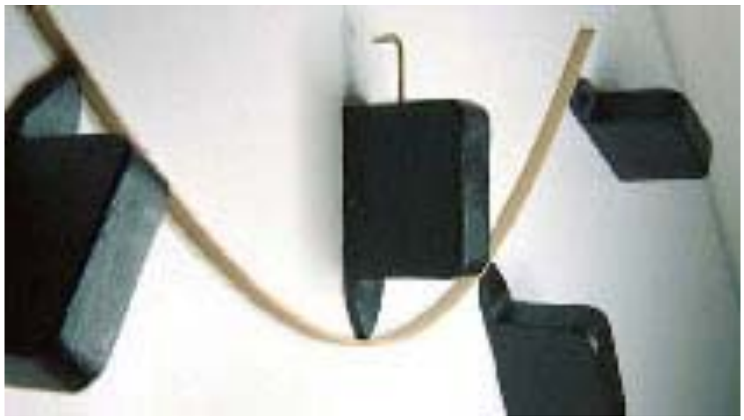
CS148: Introduction to Computer Graphics and Imaging

# Splines and Curves



## Topics

Splines

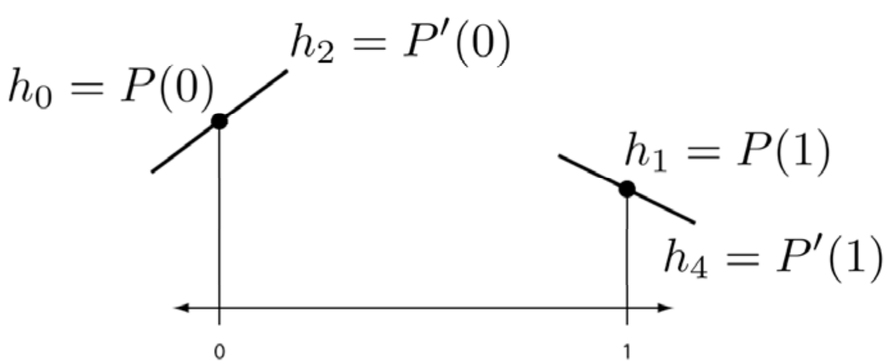
- Cubic Hermite interpolation
- Matrix representation of cubic polynomials
- Catmull-Rom interpolation

Curves

- Bezier curve
- Chaiken's subdivision algorithm
- Properties of Bezier curves

# Cubic Hermite Interpolation

## Cubic Hermite Interpolation




**Given: values and derivatives at 2 points**

## Cubic Hermite Interpolation

**Assume cubic polynomial**

$$P(t) = a\,t^3 + b\,t^2 + c\,t + d$$
$$P'(t) = 3a\,t^2 + 2b\,t + c$$

**Solve for coefficients:**

$$P(0) = h_0 = d$$
$$P(1) = h_1 = a + b + c + d$$
$$P'(0) = h_2 = c$$
$$P'(1) = h_3 = 3a + 2b + c$$



## Matrix Representation

$$h_0 = d$$
$$h_1 = a + b + c + d$$
$$h_2 = c$$
$$h_3 = 3a + 2b + c$$

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

## Matrix Representation

**Inverse**

$$\mathbf{a} = \mathbf{M}\mathbf{h}$$

$$\mathbf{h} = \mathbf{M}^{-1}\mathbf{a}$$

$$\mathbf{I} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{M}\mathbf{M}^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



## Matrix Representation

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

## Matrix Representation of Polynomials

$$P(t) = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$



## Matrix Representation of Polynomials

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

## Matrix Representation of Polynomials

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} h_0 & h_1 & h_2 & h_3 \end{bmatrix}}$$



## Matrix Representation

**Transpose** $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

$$\left( \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right)^T = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

## Matrix Representation of Polynomials

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} h_0 & h_1 & h_2 & h_3 \end{bmatrix}} \underbrace{\begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}}_{\begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_3(t) \end{bmatrix}}$$



## Hermite Basis Matrix

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_3(t) \end{bmatrix}$$

$$P(t) = \sum_{i=0}^{3} h_i \, H_i(t)$$

## Hermite Basis Matrix

$$\begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_3(t) \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$H_0(t) = 2t^3 - 3t^2 + 1$$

$$H_1(t) = -2t^3 + 3t^2$$

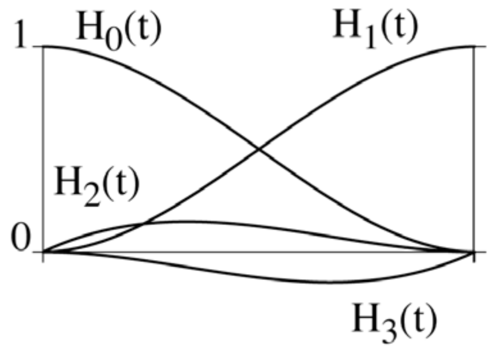
$$H_2(t) = t^3 - 2t^2 + t$$

$$H_3(t) = t^3 - t^2$$

$$\mathbf{M_H} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$



## Hermite Basis Functions




$$H_0(t) = 2t^3 - 3t^2 + 1$$

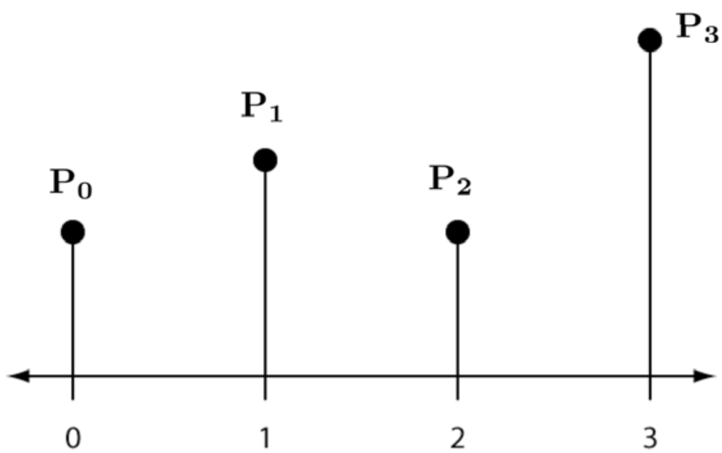
$$H_1(t) = -2t^3 + 3t^2$$

$$H_2(t) = t^3 - 2t^2 + t$$

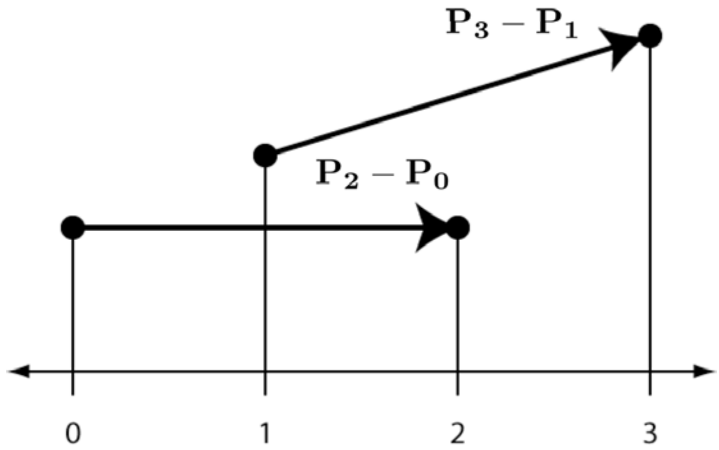
$$H_3(t) = t^3 - t^2$$

## Catmull-Rom Interpolation

## Catmull-Rom Interpolation



## Catmull-Rom Interpolation

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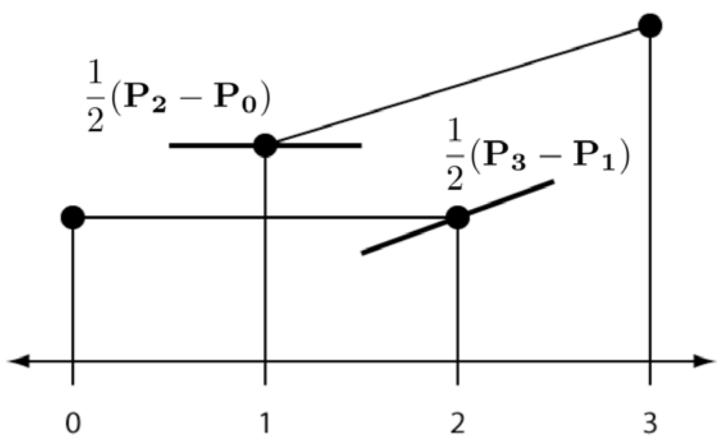
## Catmull-Rom Interpolation

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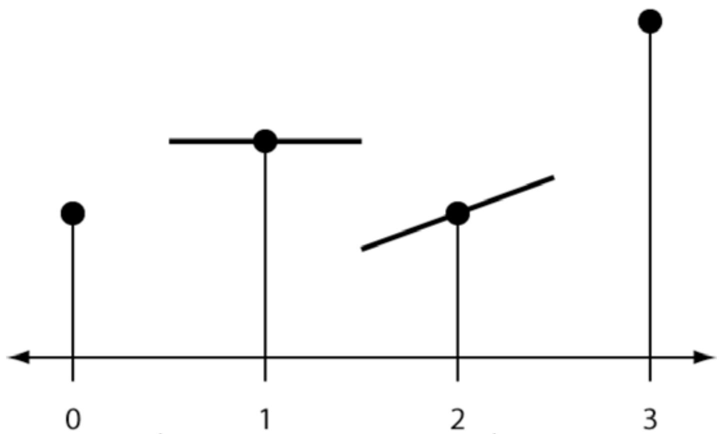
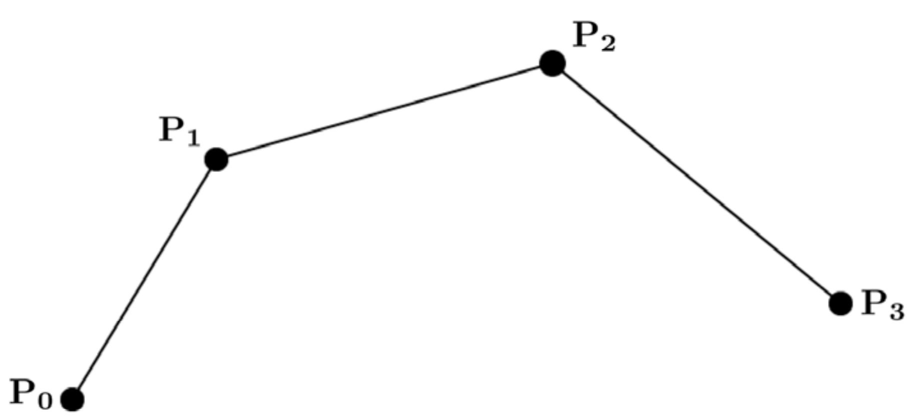
## Catmull-Rom Interpolation



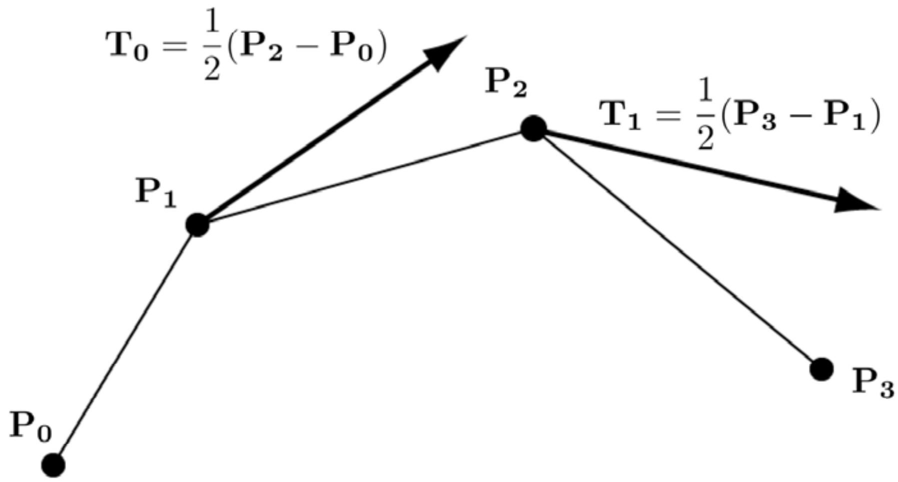

$$\mathbf{T_0} = \frac{1}{2}(\mathbf{P_2} - \mathbf{P_0}) \quad \mathbf{T_1} = \frac{1}{2}(\mathbf{P_3} - \mathbf{P_1})$$



## Catmull-Rom Interpolation



## Catmull-Rom Interpolation



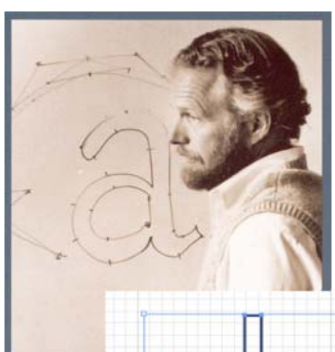



## Catmull-Rom To Hermite Interpolation

$$\mathbf{P_0} = \mathbf{P_1}$$
$$\mathbf{P_1} = \mathbf{P_2}$$
$$\mathbf{T_0} = \frac{1}{2}(\mathbf{P_2} - \mathbf{P_0})$$
$$\mathbf{T_1} = \frac{1}{2}(\mathbf{P_3} - \mathbf{P_1})$$

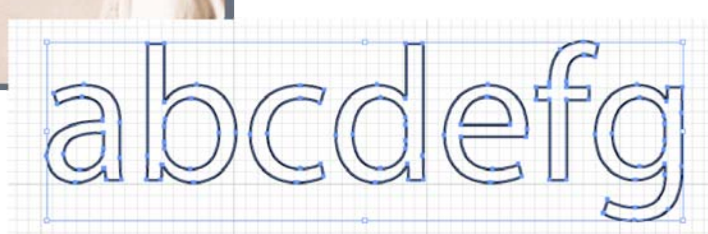
# Bezier Curves

## Paths

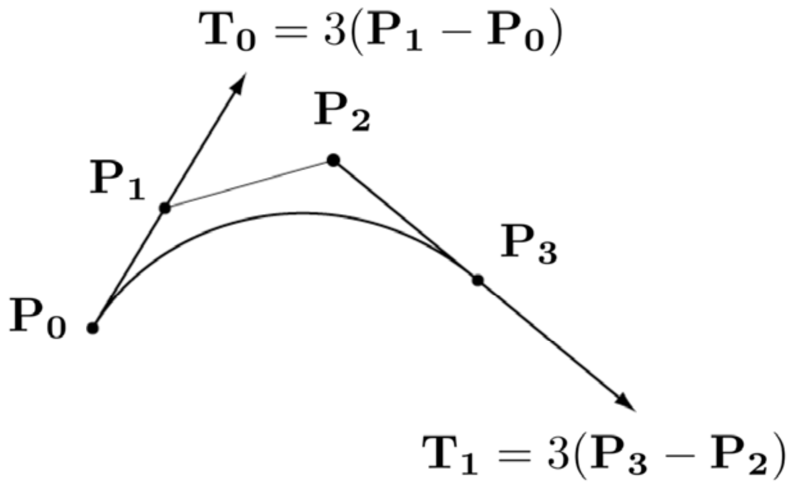


**Capabilities**

1. **Smooth curves**
2. **Line and curve segments**
3. **Kinks**



## Bezier Curve

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## Bezier To Hermite Interpolation

$$\mathbf{P_0} = \mathbf{P_0}$$

$$\mathbf{P_1} = \mathbf{P_3}$$

$$\mathbf{T_0} = 3(\mathbf{P_1} - \mathbf{P_0})$$

$$\mathbf{T_1} = 3(\mathbf{P_3} - \mathbf{P_2})$$

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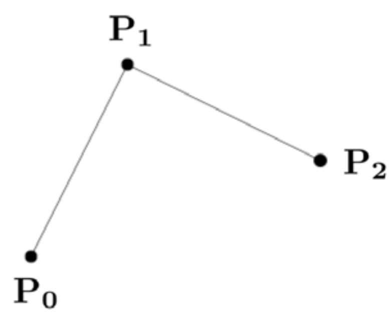
## Demo of Bezier Curves

## Subdivision

## Chaiken's Algorithm

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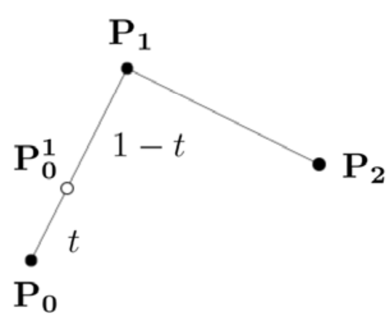
## Chaiken's Algorithm

$$\mathbf{P_0^1} = (1-t)\mathbf{P_0} + t\mathbf{P_1}$$

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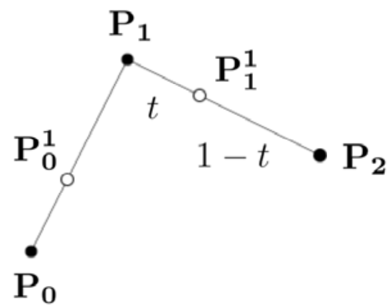
## Chaiken's Algorithm

$$\mathbf{P_0^1} = (1-t)\mathbf{P_0} + t\mathbf{P_1}$$
$$\mathbf{P_1^1} = (1-t)\mathbf{P_1} + t\mathbf{P_2}$$

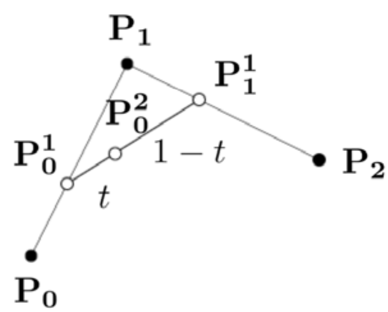





## Chaiken's Algorithm

$$\mathbf{P_0^1} = (1-t)\mathbf{P_0} + t\mathbf{P_1}$$
$$\mathbf{P_1^1} = (1-t)\mathbf{P_1} + t\mathbf{P_2}$$

$$\mathbf{P_0^2} = (1-t)\mathbf{P_0^1} + t\mathbf{P_1^1}$$

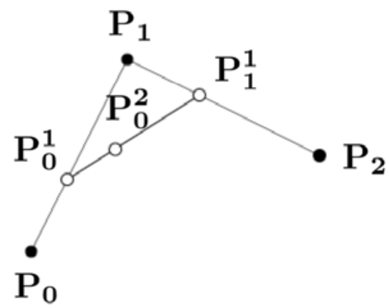


## Chaiken's Algorithm

$$\mathbf{P}_0^1 = (1-t)\mathbf{P}_0 + t\mathbf{P}_1$$
$$\mathbf{P}_1^1 = (1-t)\mathbf{P}_1 + t\mathbf{P}_2$$

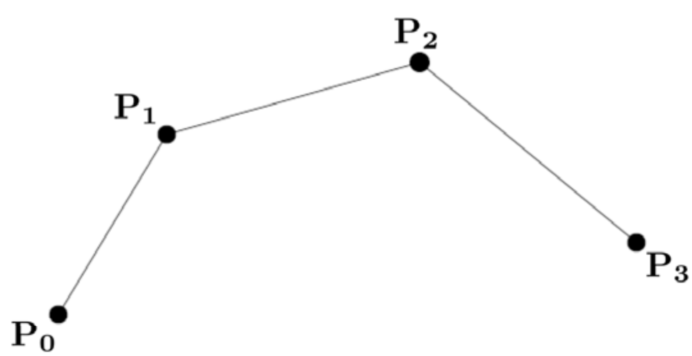
$$\mathbf{P}_0^2 = (1-t)\mathbf{P}_0^1 + t\mathbf{P}_1^1$$

$$\mathbf{P}(t) = \mathbf{P}_0^2$$

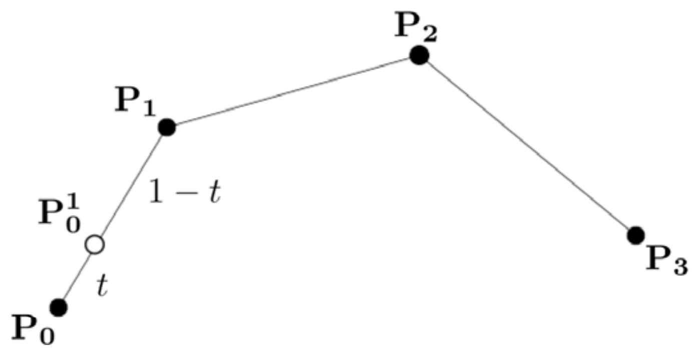





## Bezier Curve



## Bezier Curve

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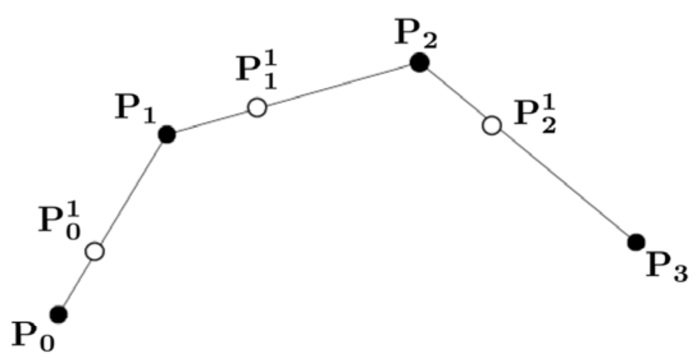
## Bezier Curve

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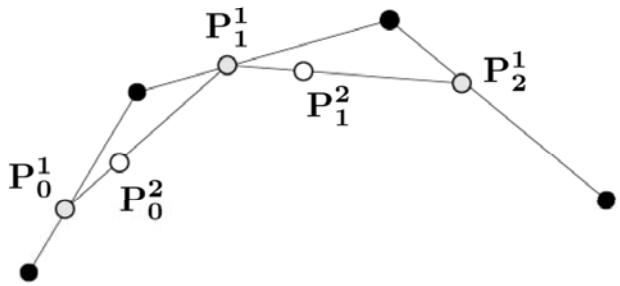
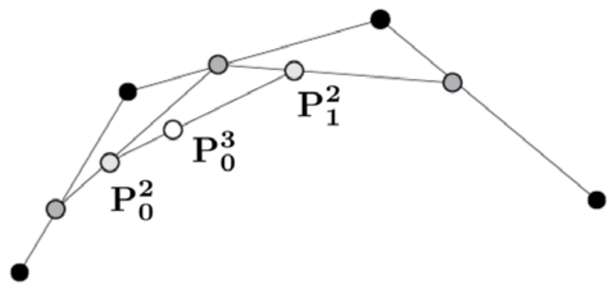
## Bezier Curve






## Bezier Curve



## Properties

**Property 1: Interpolate end points**

$$\mathbf{P}(0) = \mathbf{P_0}$$

$$\mathbf{P}(1) = \mathbf{P_3}$$

**Property 2: Tangents**

$$\mathbf{P}'(0) = 3(\mathbf{P_1} - \mathbf{P_0})$$

$$\mathbf{P}'(1) = 3(\mathbf{P_3} - \mathbf{P_2})$$

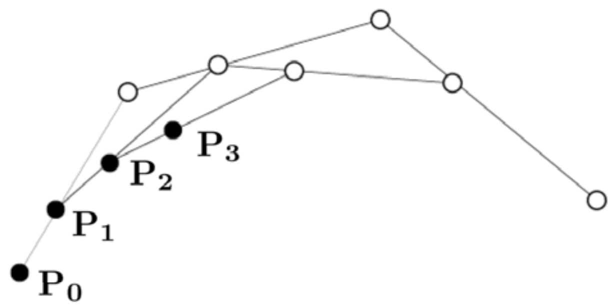
**Property 3: Convex hull property**

$$\mathbf{P}(t) \text{ inside chull}(\mathbf{P_0}, \mathbf{P_1}, \mathbf{P_2}, \mathbf{P_2})$$



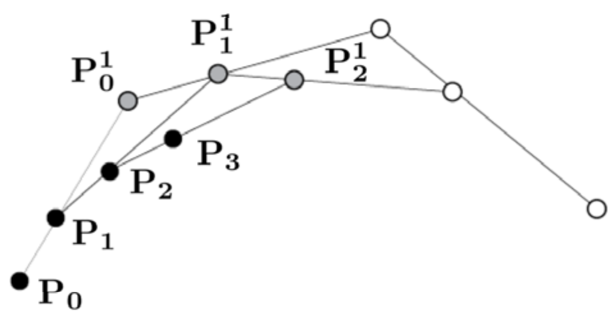
## Extrapolation

## Bezier Curve - Extrapolation

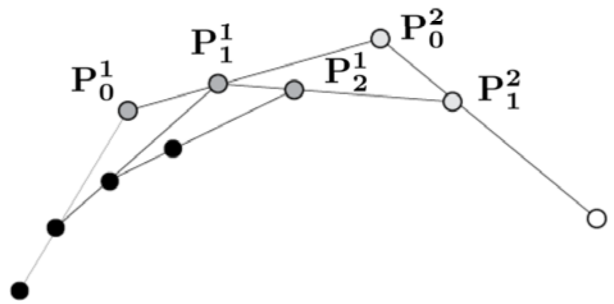





## Bezier Curve - Extrapolation

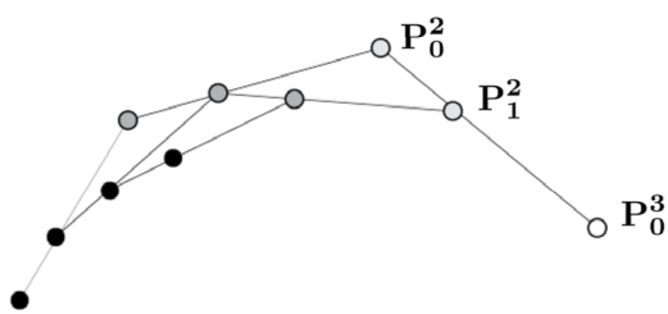


## Bezier Curve - Extrapolation

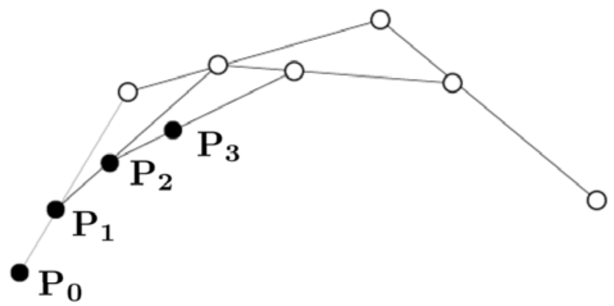





## Bezier Curve - Extrapolation



## Bezier Curve

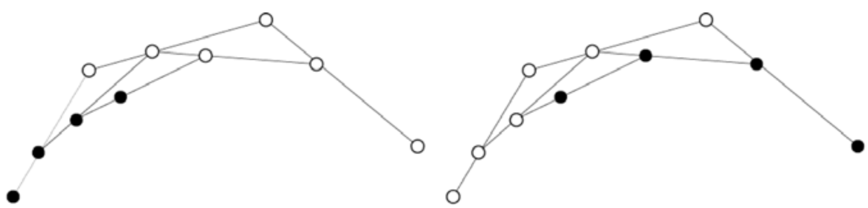



Left Bezier Curve



## Bezier Curve

**Can subdivide a Bezier curve into two pieces**



Left Bezier Curve Right Bezier Curve

## Applications of Subdivision

Drawing Bezier curve

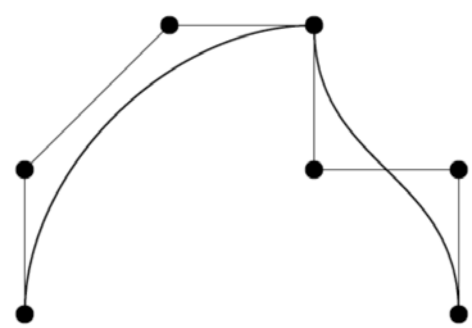
- ??

Intersect two Bezier curves

- ??

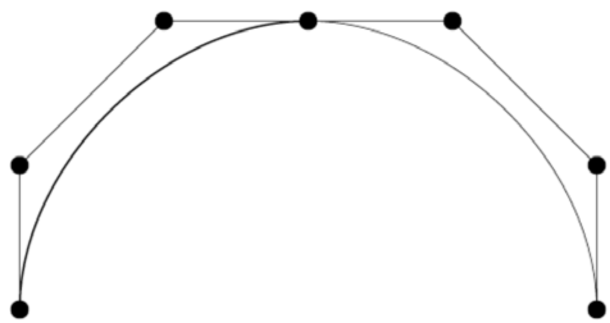


## Continuity between 2 Bezier Curves



3rd point of the 1st curve is the same as the 1st point of the 2nd curve

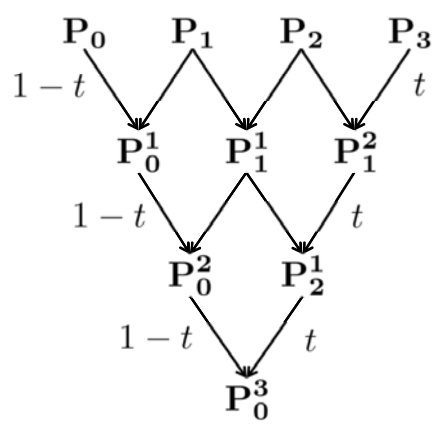
## Continuity between 2 Bezier Curves



**Tangent of the 1st curve is equal to the tangent of the 2nd curve**

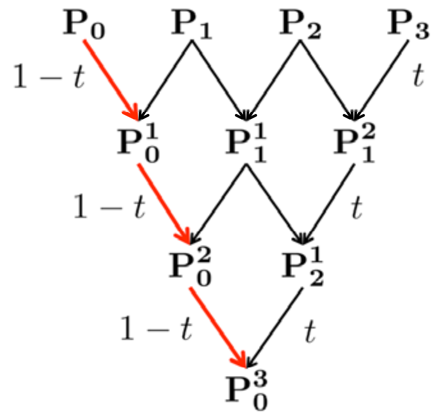


## Pyramid Algorithm




$$P(t) = \sum_i^n P_i \, B_i^n(t)$$

# Pyramid Algorithm



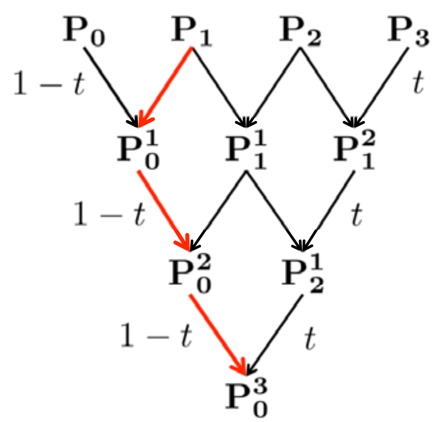

$$P(t) = \sum_i^n P_i \, B_i^n(t)$$

$$(1-t)^3 \mathbf{P_0}$$

$$B_0^3(t) = (1-t)^3$$



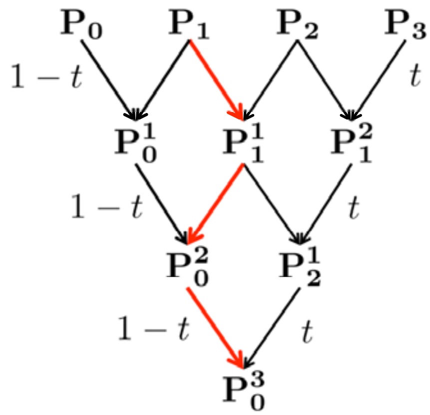
# Pyramid Algorithm




$$P(t) = \sum_i^n P_i \, B_i^n(t)$$

$$t(1-t)^2 \mathbf{P_1}$$

## Pyramid Algorithm

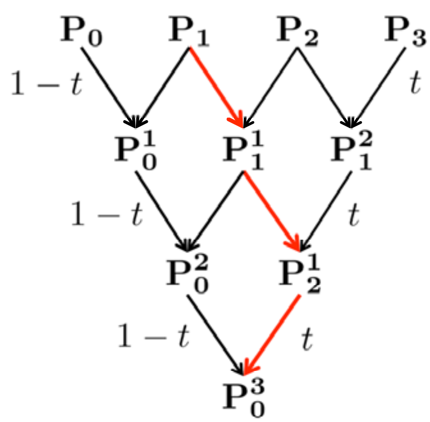



$$P(t) = \sum_i^n P_i \, B_i^n(t)$$

$$t(1-t)^2 \mathbf{P_1}$$



## Pyramid Algorithm

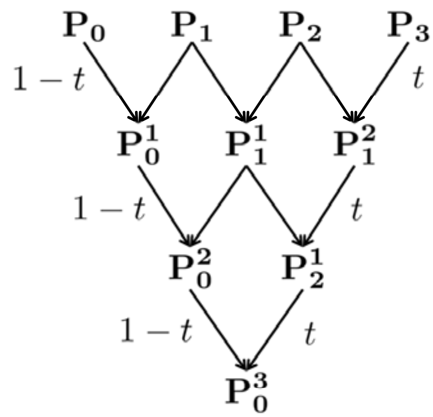



$$P(t) = \sum_i^n P_i \, B_i^n(t)$$

$$t(1-t)^2 \mathbf{P_1}$$

$$B_1^3(t) = 3t(1-t)^2$$

## Bernstein Polynomials




$$P(t) = \sum_i^n P_i \, B_i^n(t)$$

$$B_0^3(t) = (1-t)^3$$

$$B_1^3(t) = 3t(1-t)^2$$

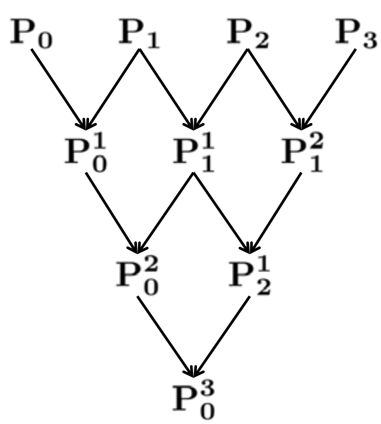
$$B_2^3(t) = 3t^2(1-t)$$

$$B_3^3(t) = t^3$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



## Pyramid Algorithm




$$P(t) = \sum_i^n P_i \, B_i^n(t)$$

## Things to Remember

Splines

- Cubic hermite interpolation
- Matrix representation of cubic polynomials
- Catmull-Rom splines
- How to think of CR in terms of Hermite spline

Curves

- Bezier curve
- How to think of BC in terms of Hermite spline
- Chaiken’s algorithm
- Subdivision algorithm including applications
- Properties of Bezier curves