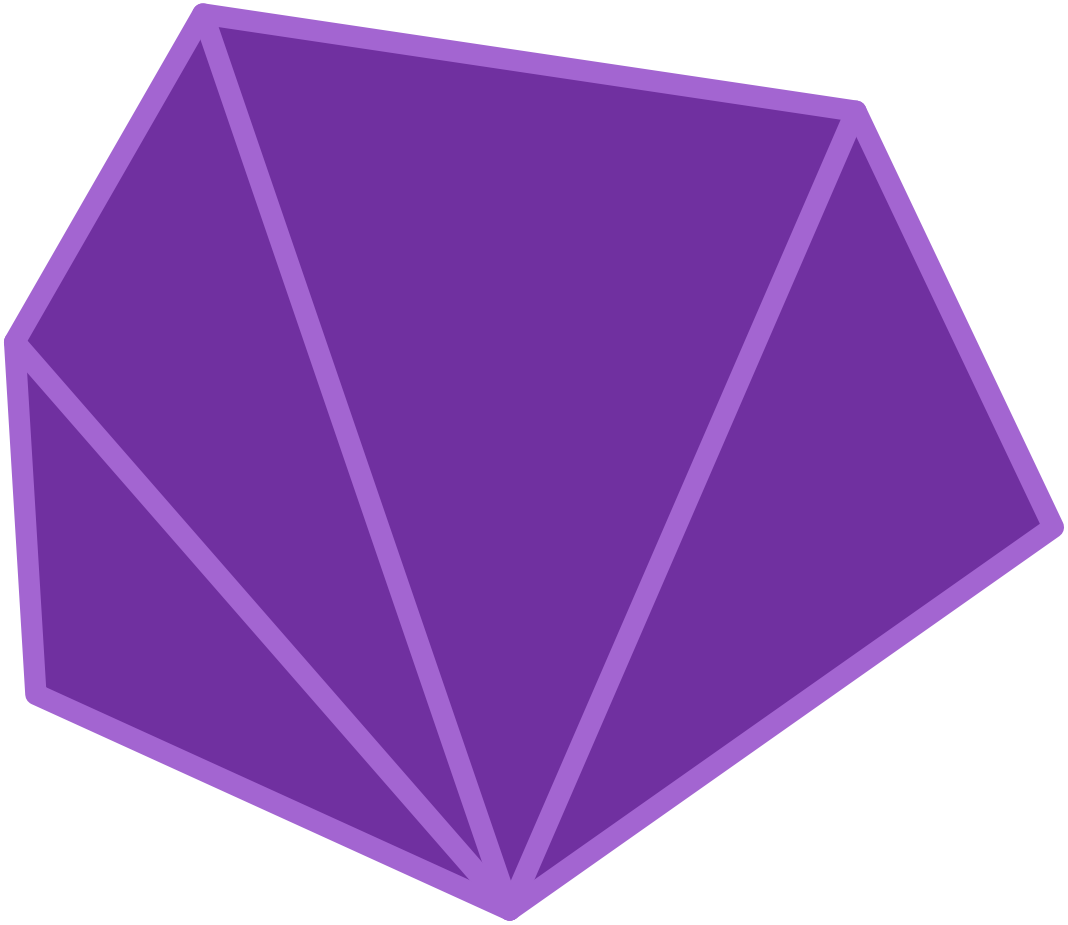
http://eikons.nl/wp-content/uploads/2012/02/car_studiolighting3.jpeg

Geometric Modeling

CS 148, Summer 2012
Introduction to Computer Graphics and Imaging

Justin Solomon

Primitive for Rendering



Triangles

Primitive for Rendering

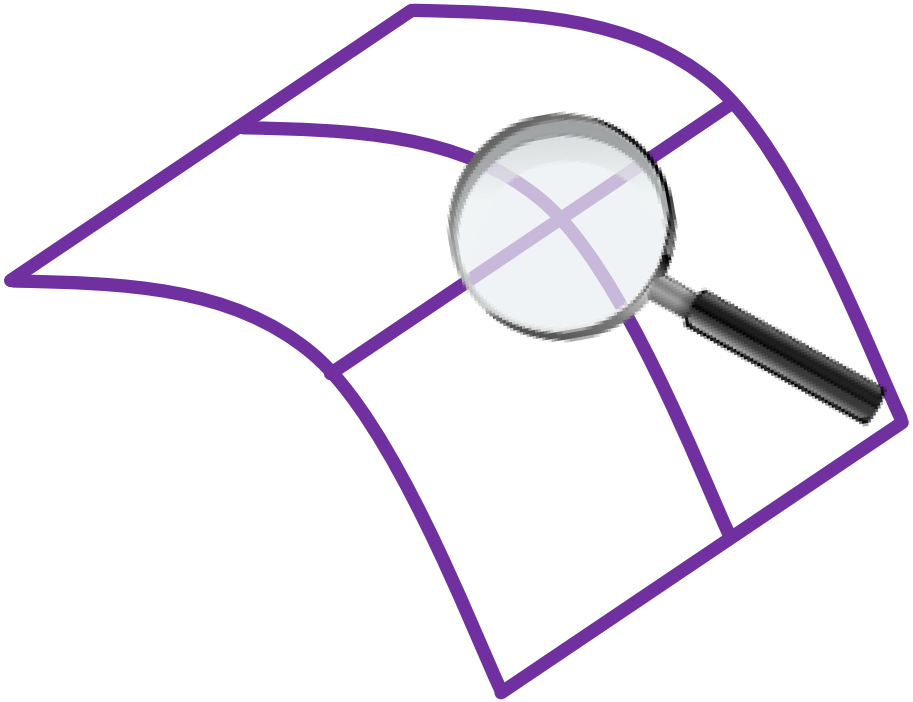
Triangles

What Do We Want?

Higher-level representation of curves and surfaces that can generate triangles easily.

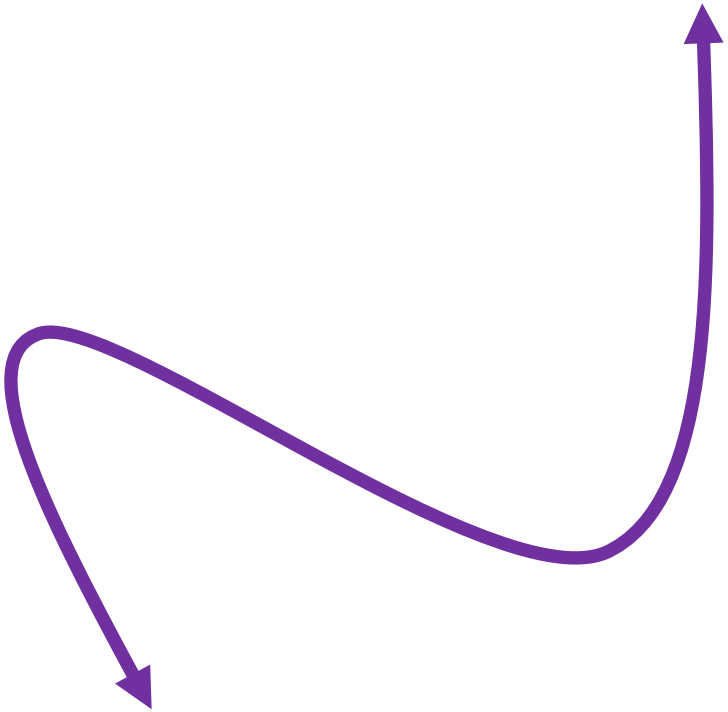
Differential Geometric Standpoint

Locally looks like a line

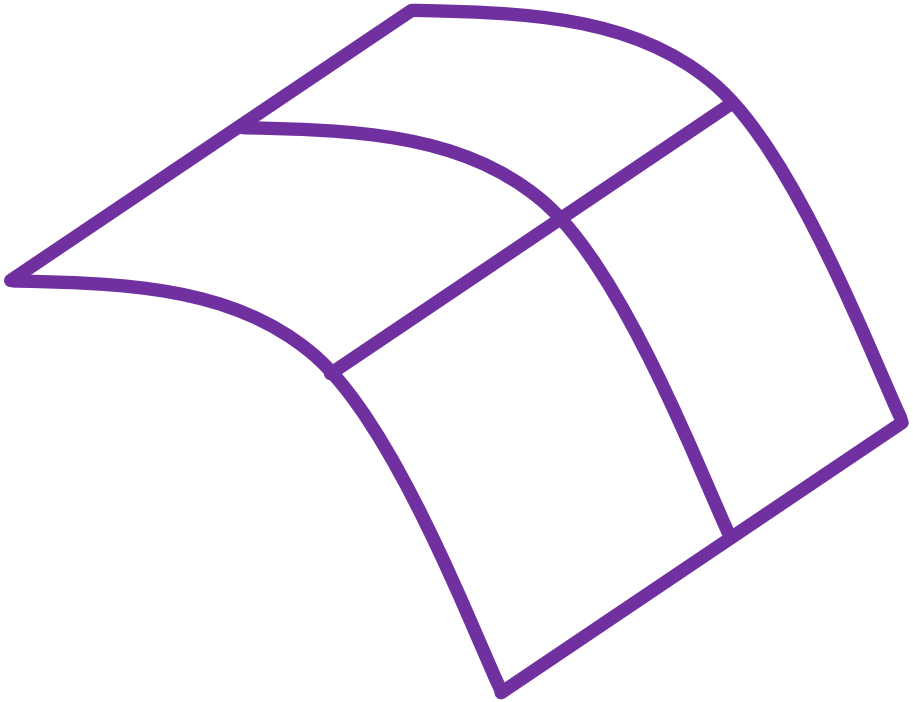


Locally looks like a plane

Parametric Geometry

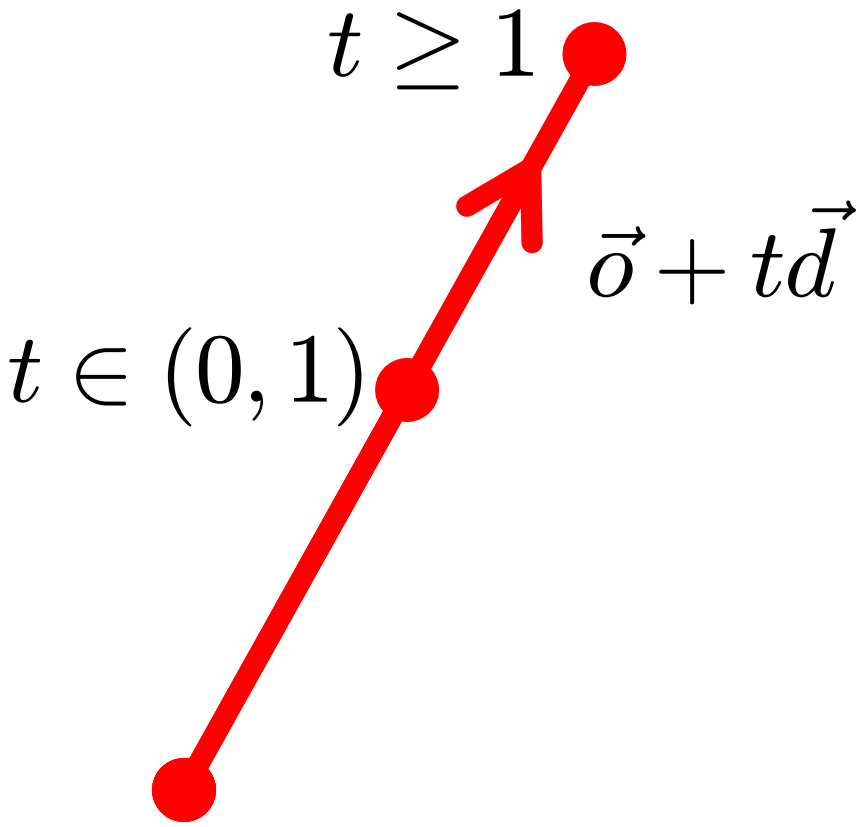


$\vec{\gamma}(t)$



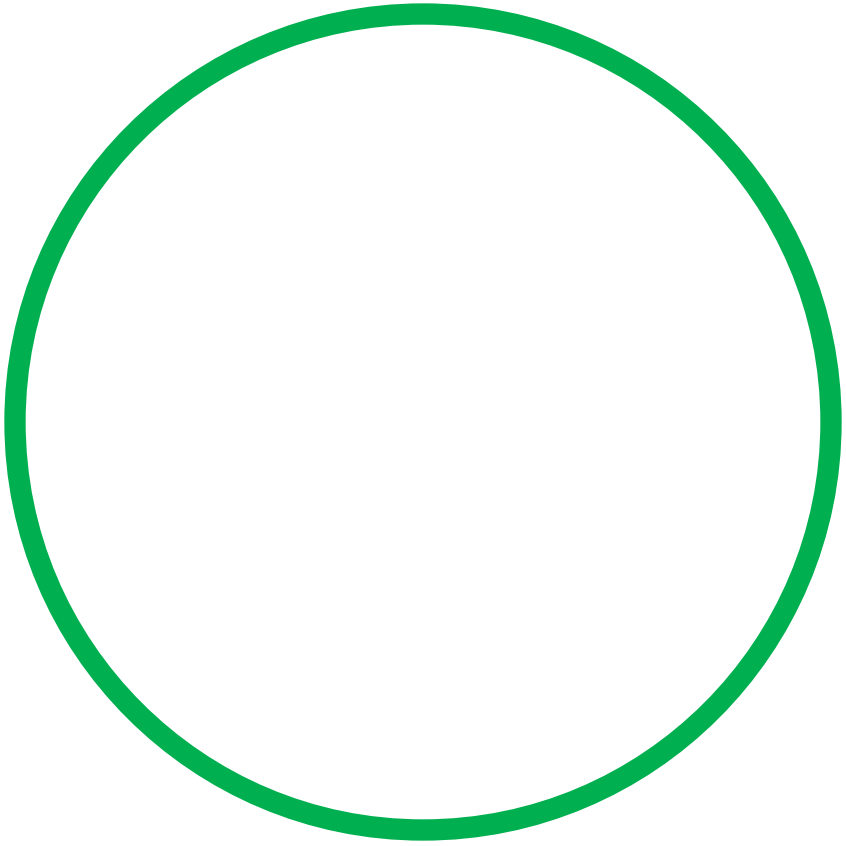
$\vec{\beta}(u, v)$

Parameterized Line



We've seen it before!

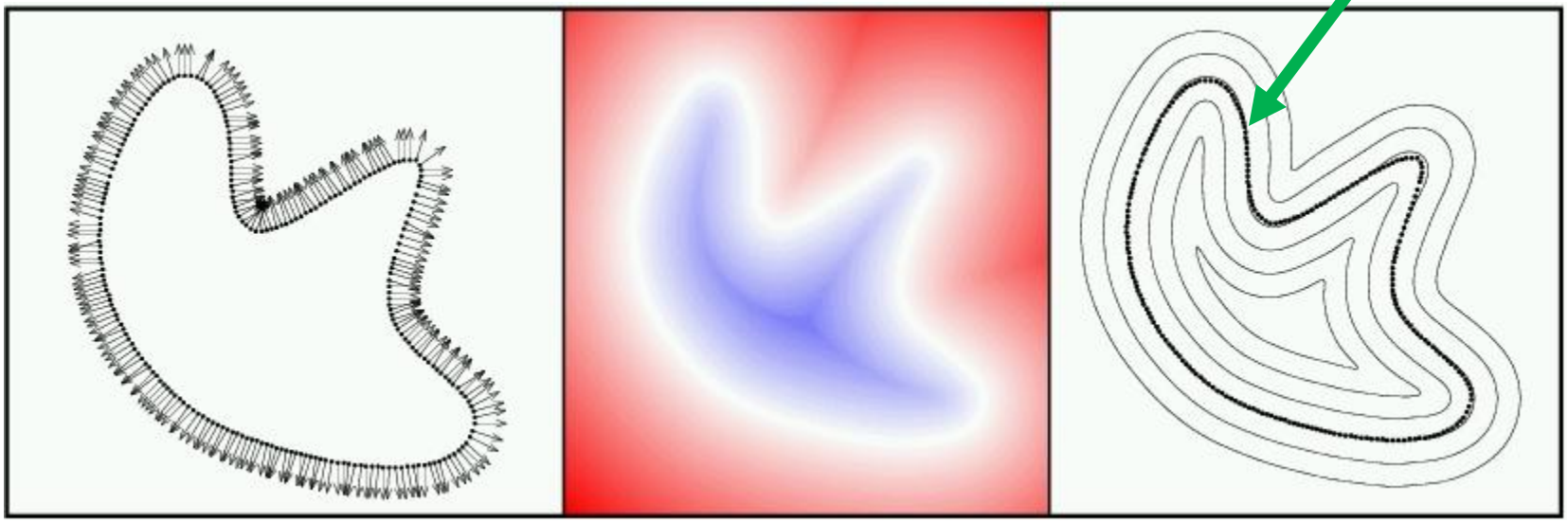
Parameterized Circle



$$\vec{\gamma}(t) = (\cos t, \sin t)$$

Implicit Modeling

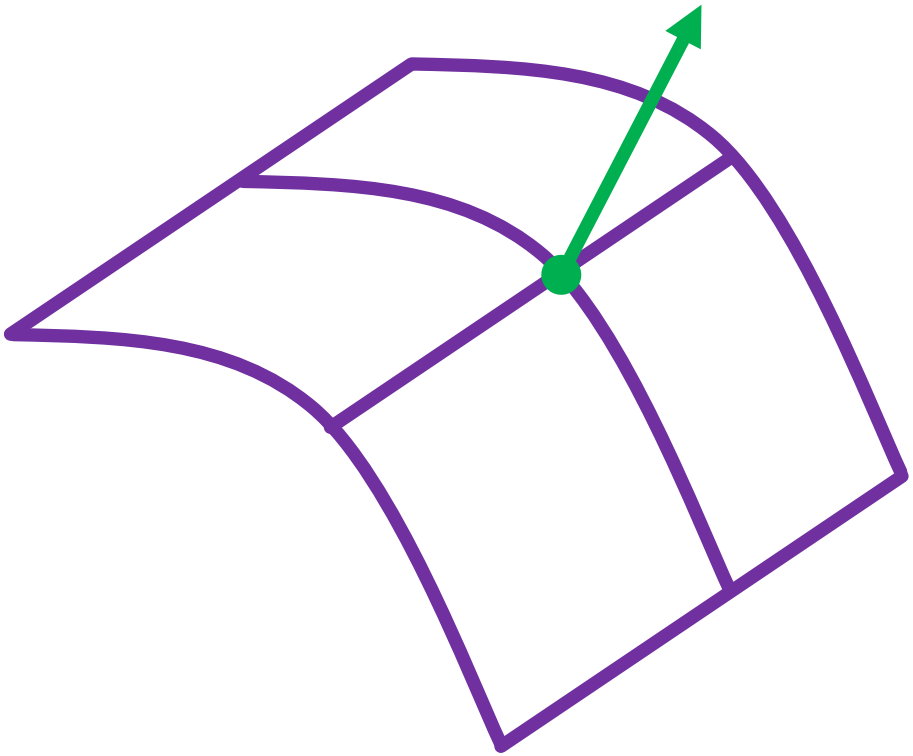
Level set
(isocontour)



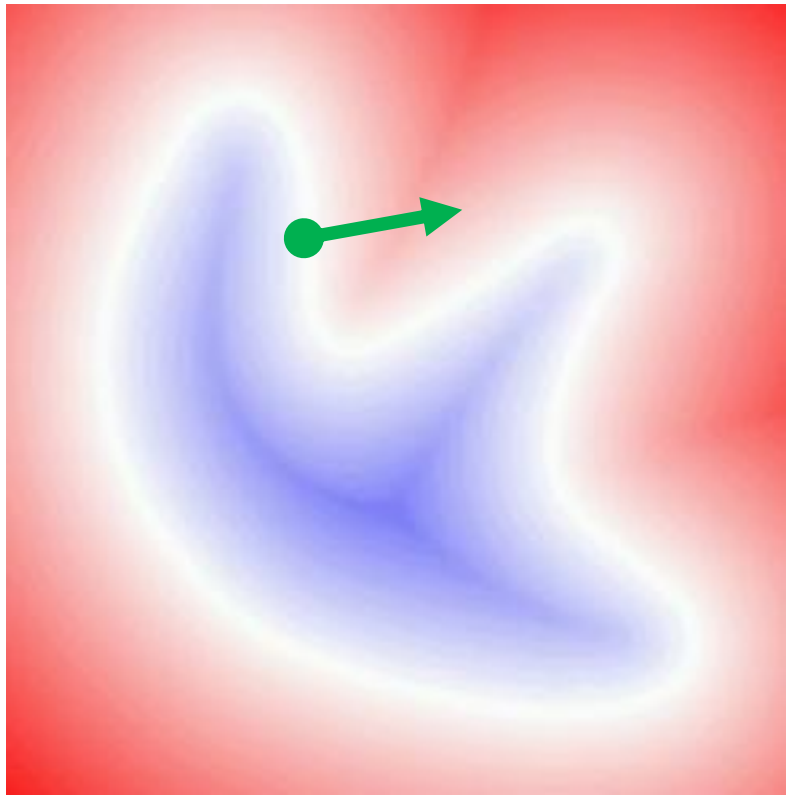
$$f(x, y) = 0$$

https://buffy.eecs.berkeley.edu/PHP/resabs/images/2005/100895-0.jpg

(Non-Unit) Normals

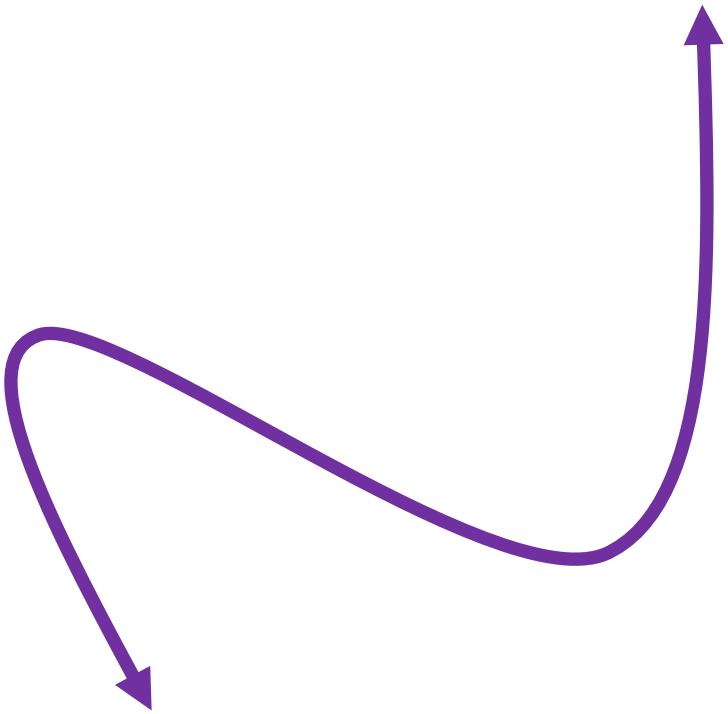


$$\frac{\partial \vec{\beta}}{\partial u} \times \frac{\partial \vec{\beta}}{\partial v}$$

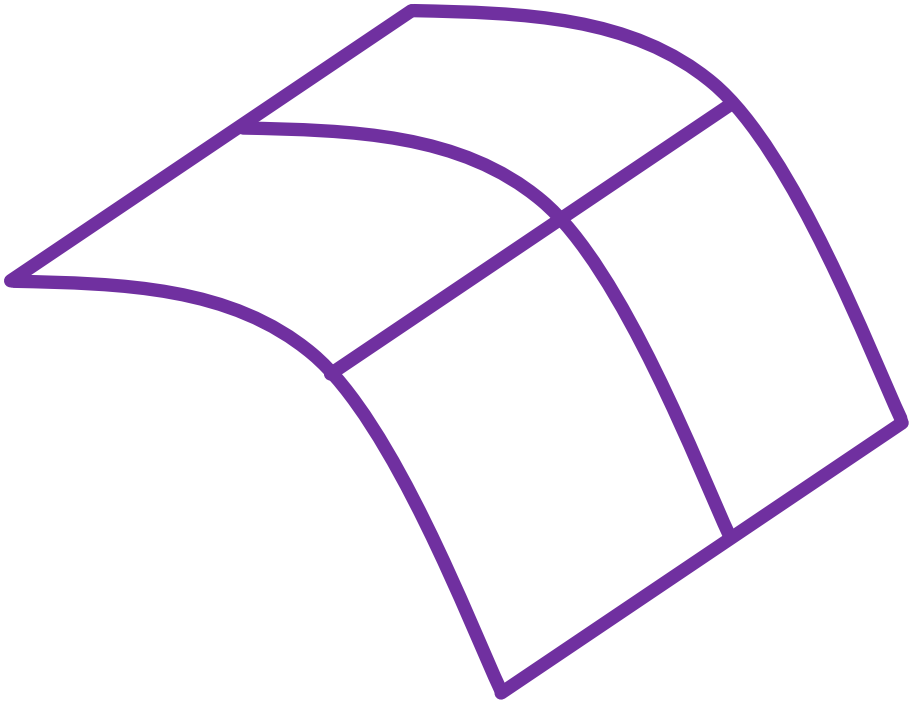


$$\nabla f$$

Today: Parameterized Geometry

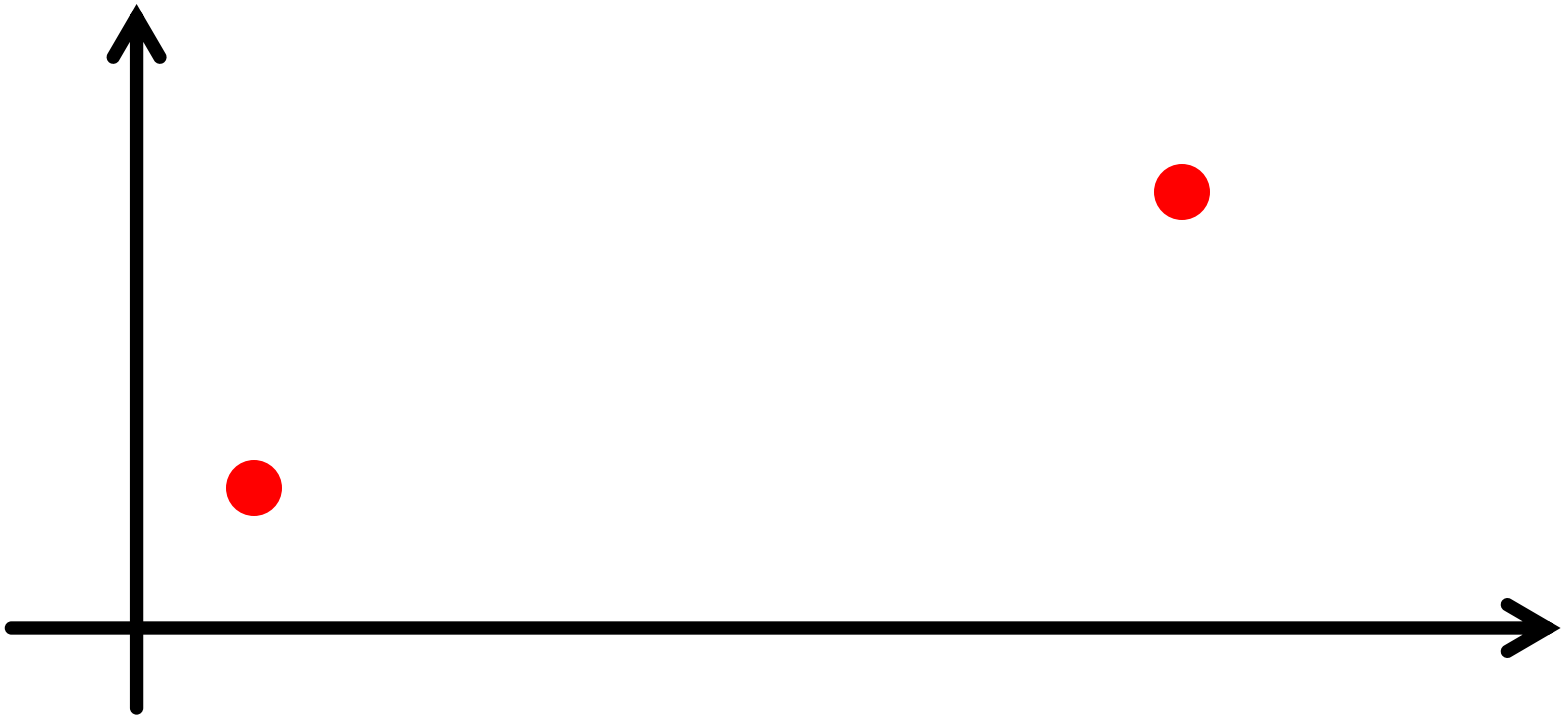


$\vec{\gamma}(t)$



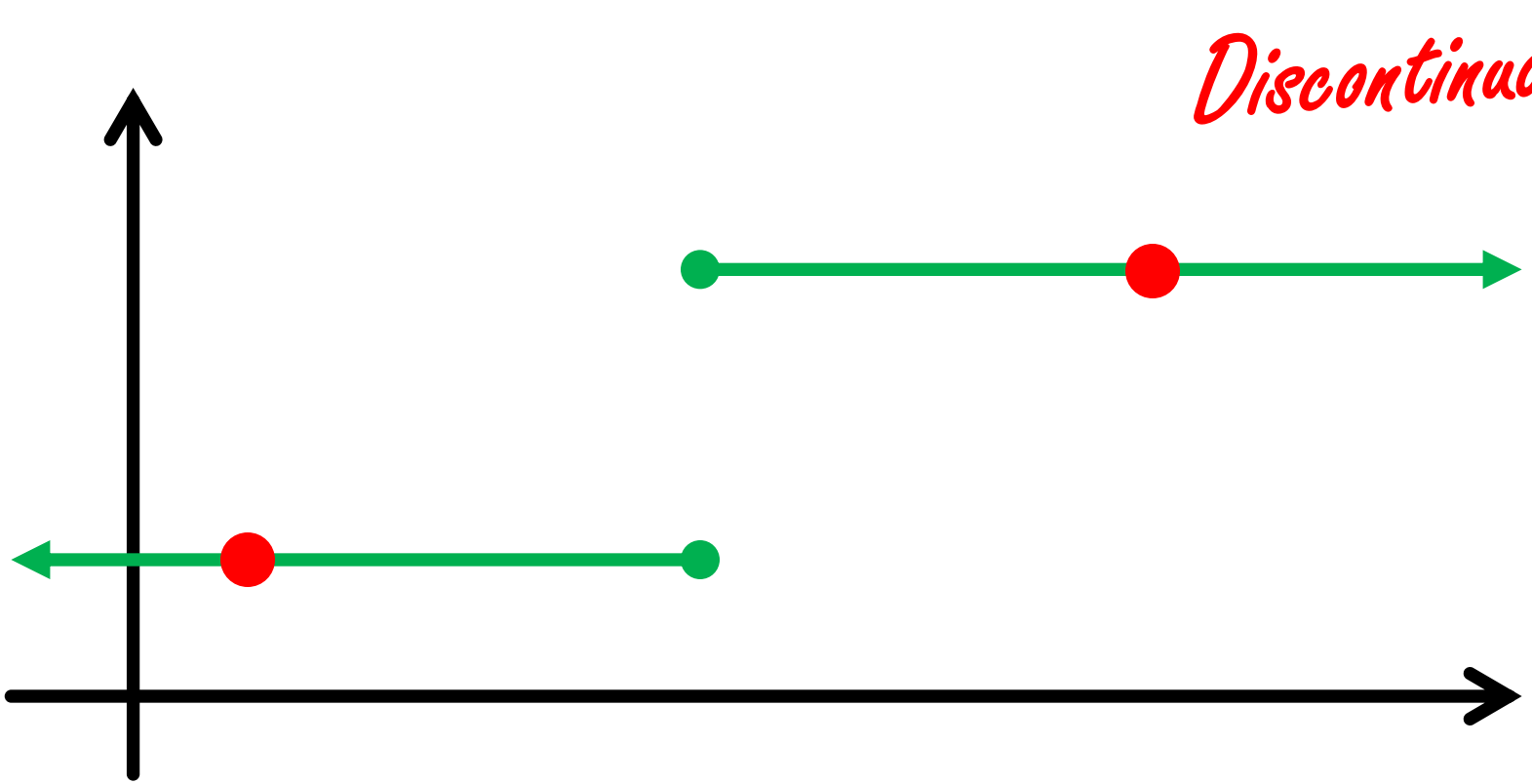
$\vec{\beta}(u, v)$

1D Problem: Interpolation



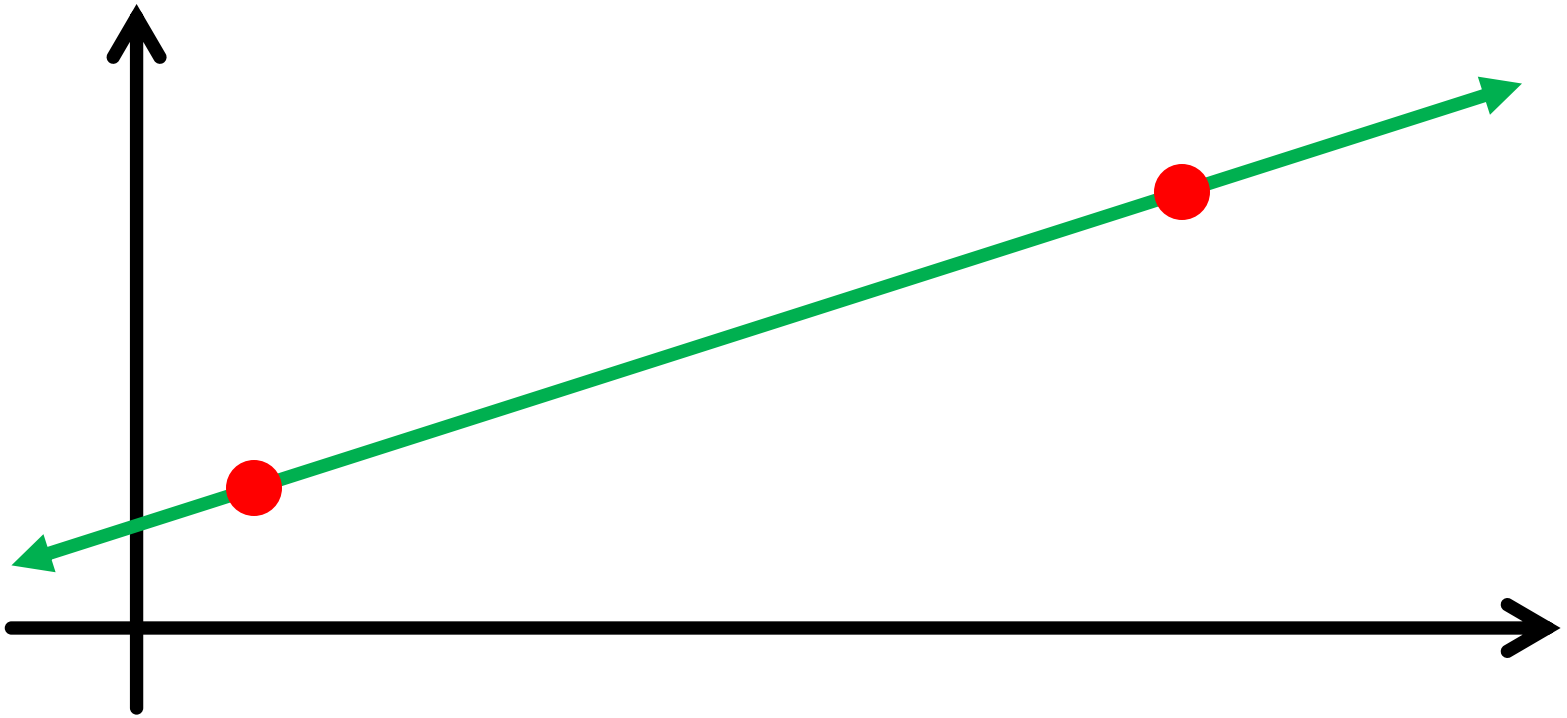
How do you fill the space?

1D Problem: Interpolation



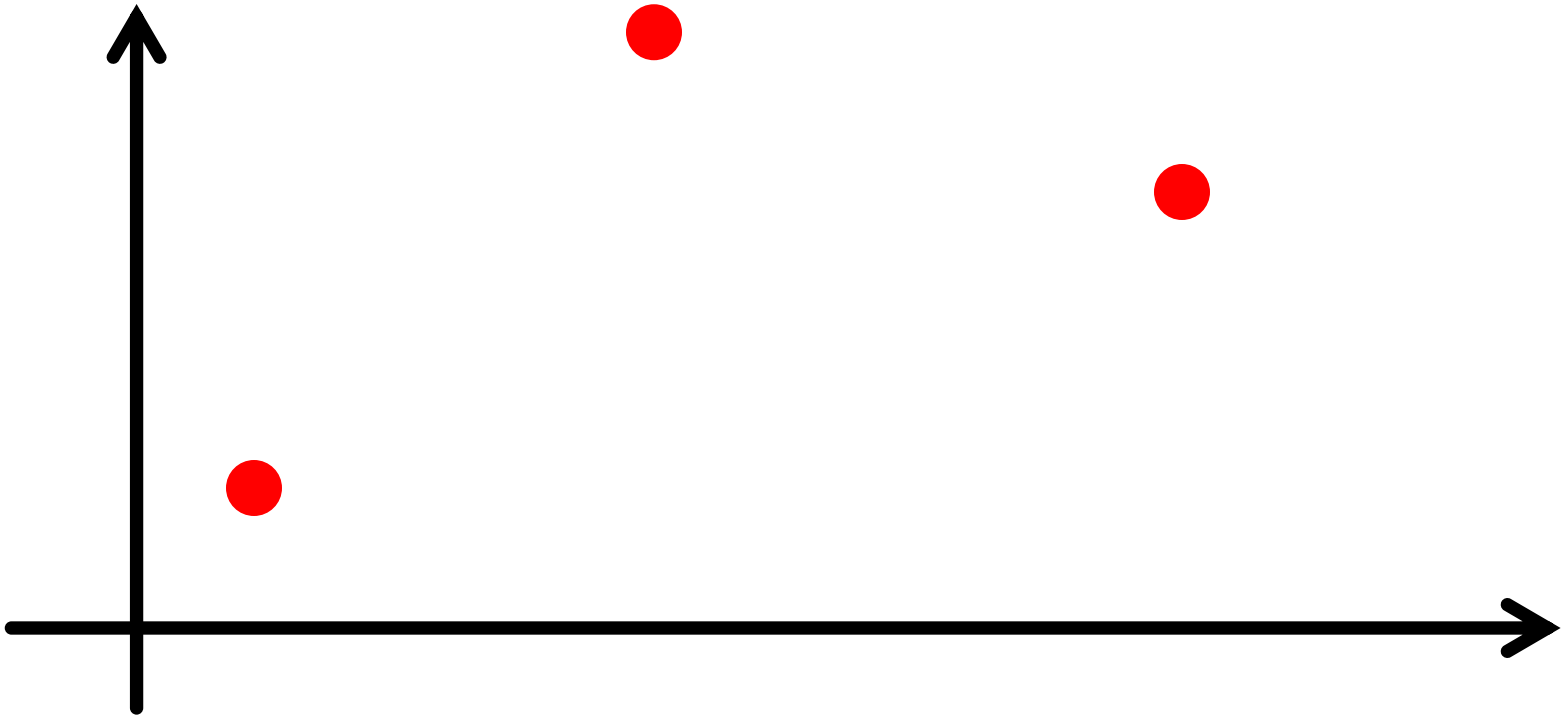
Nearest-neighbor interpolation

1D Problem: Interpolation



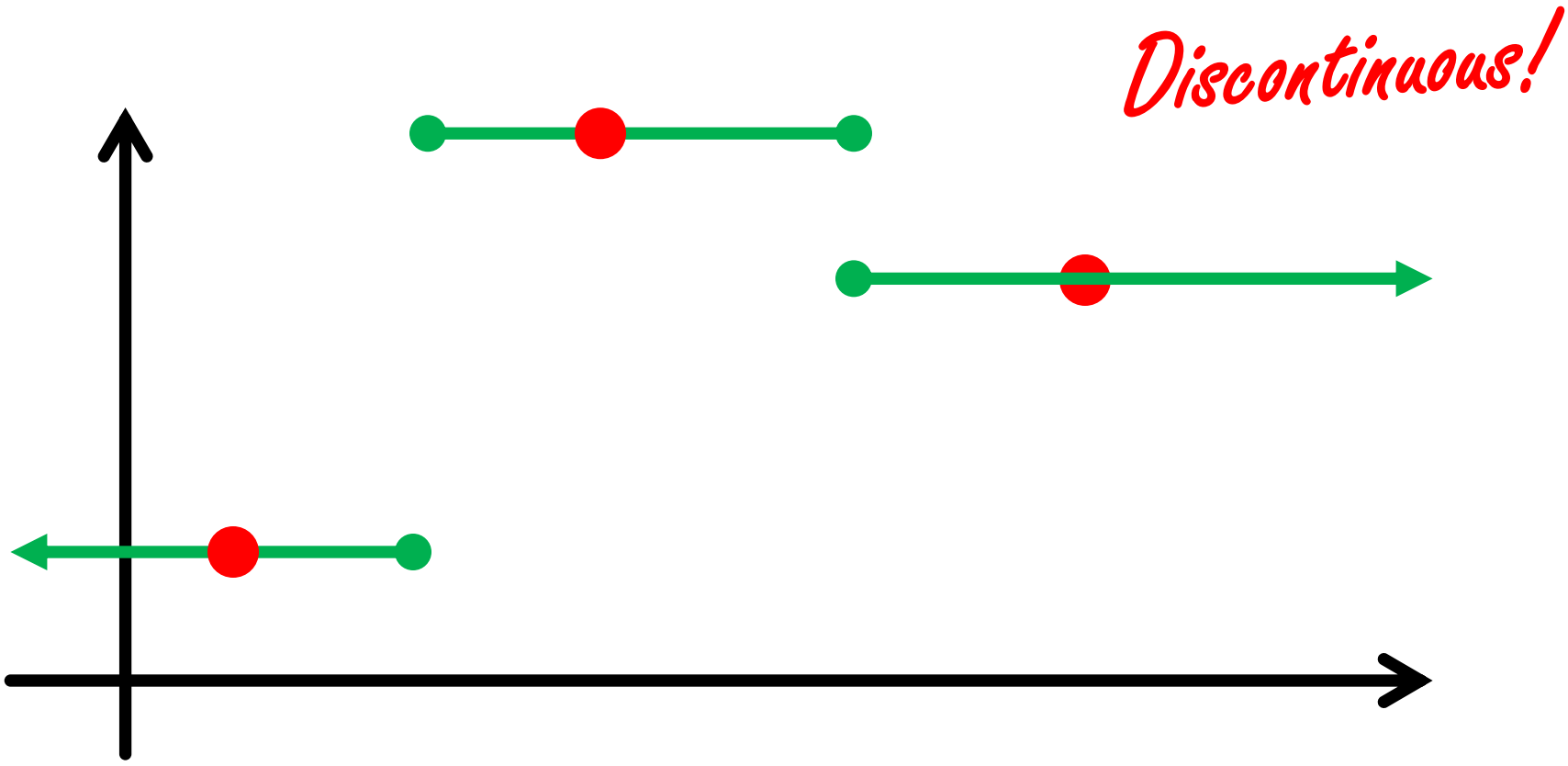
Linear interpolation

1D Problem: Interpolation



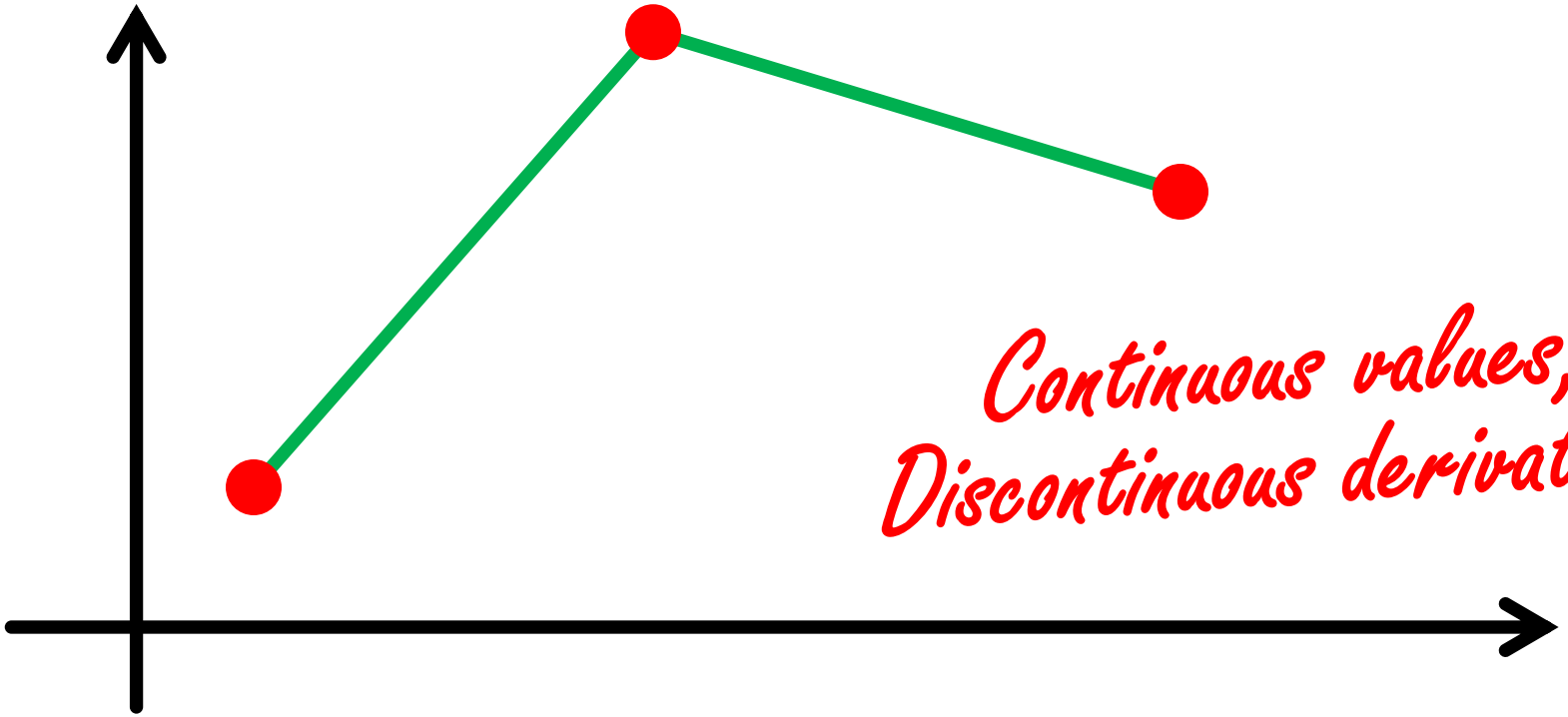
Three points?

1D Problem: Interpolation



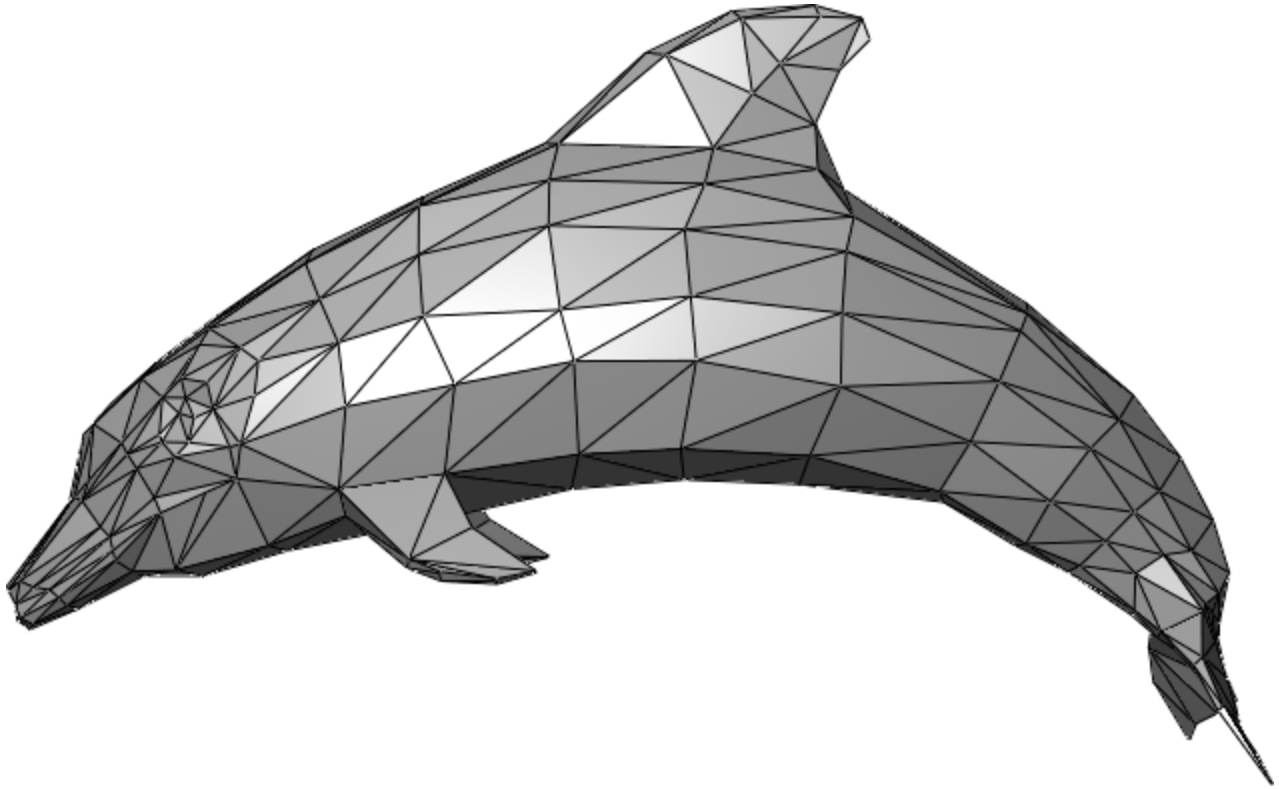
Nearest-neighbor

1D Problem: Interpolation



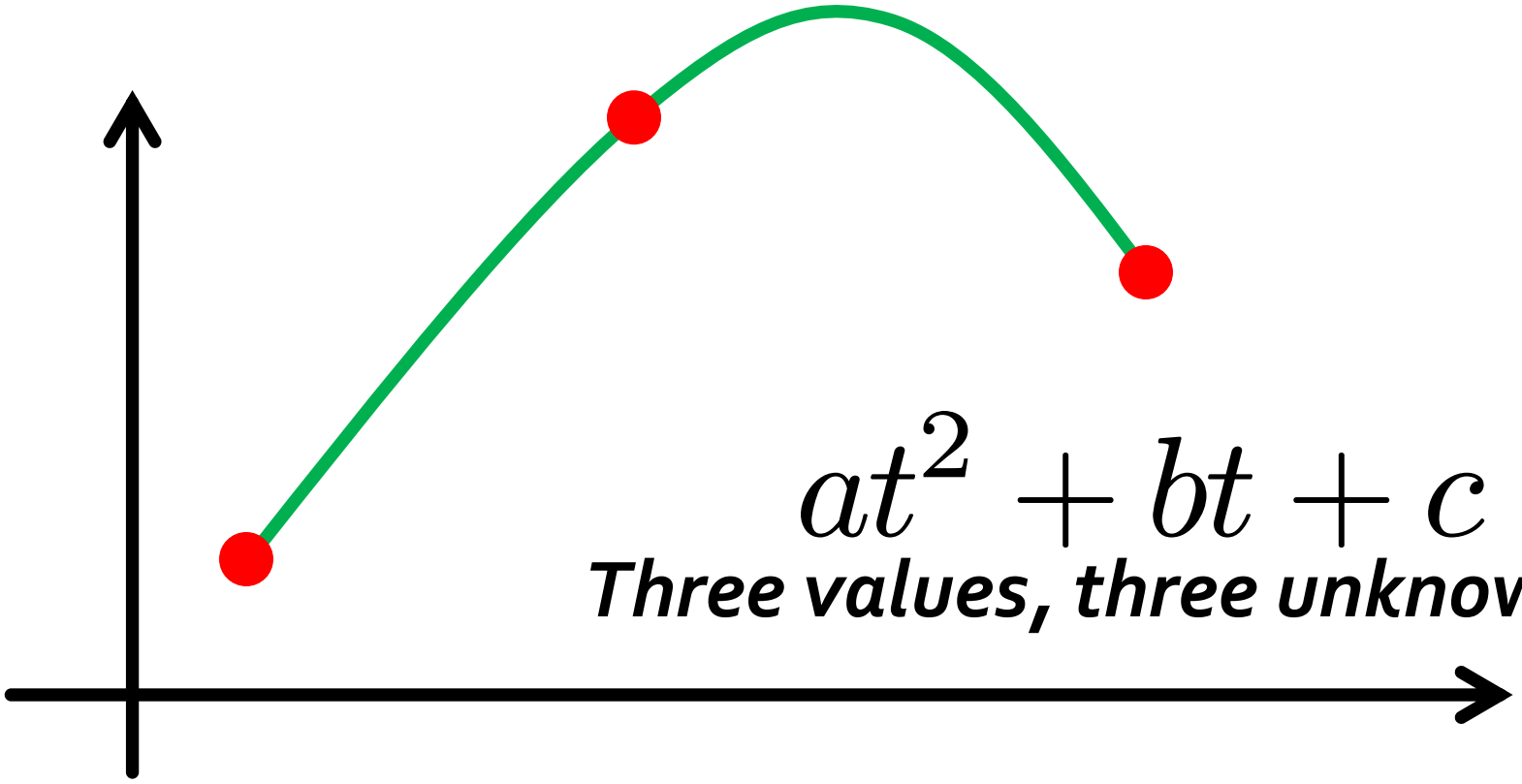
Piecewise linear

Piecewise Linear: Used Often!



http://upload.wikimedia.org/wikipedia/commons/f/fb/Dolphin_triangle_mesh.png

1D Problem: Interpolation

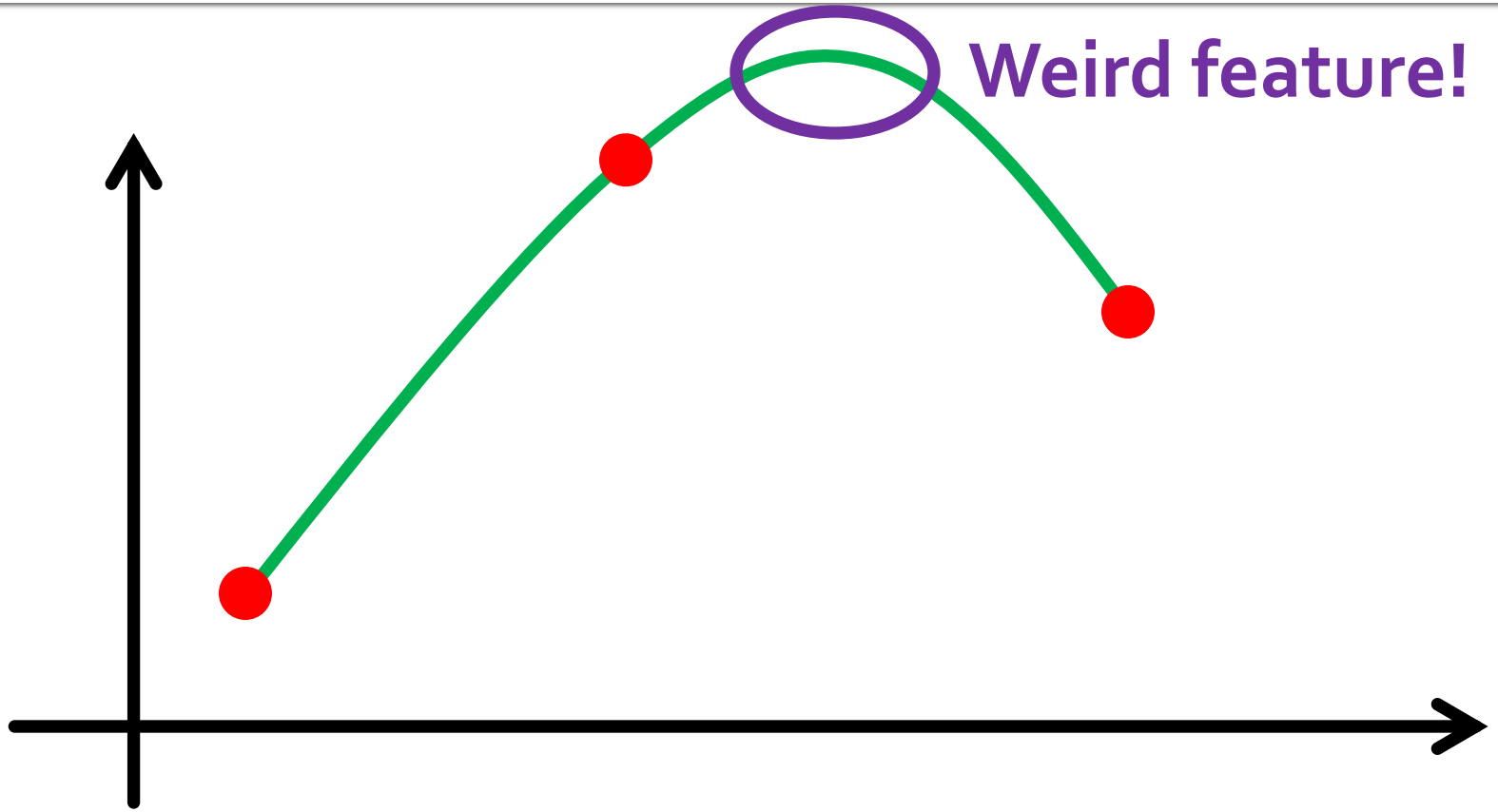


$$at^2 + bt + c$$

Three values, three unknowns!

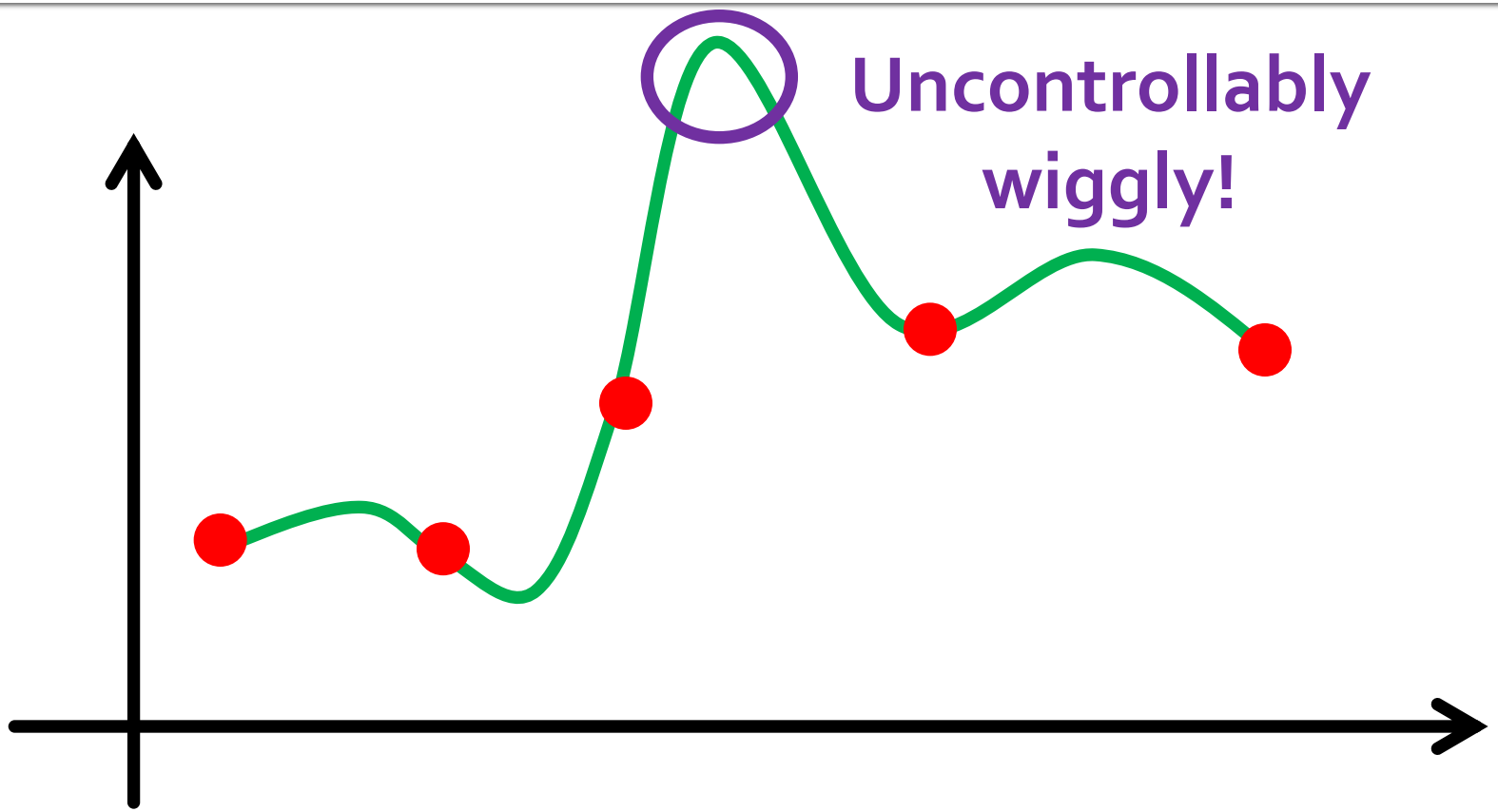
Parabola

1D Problem: Interpolation



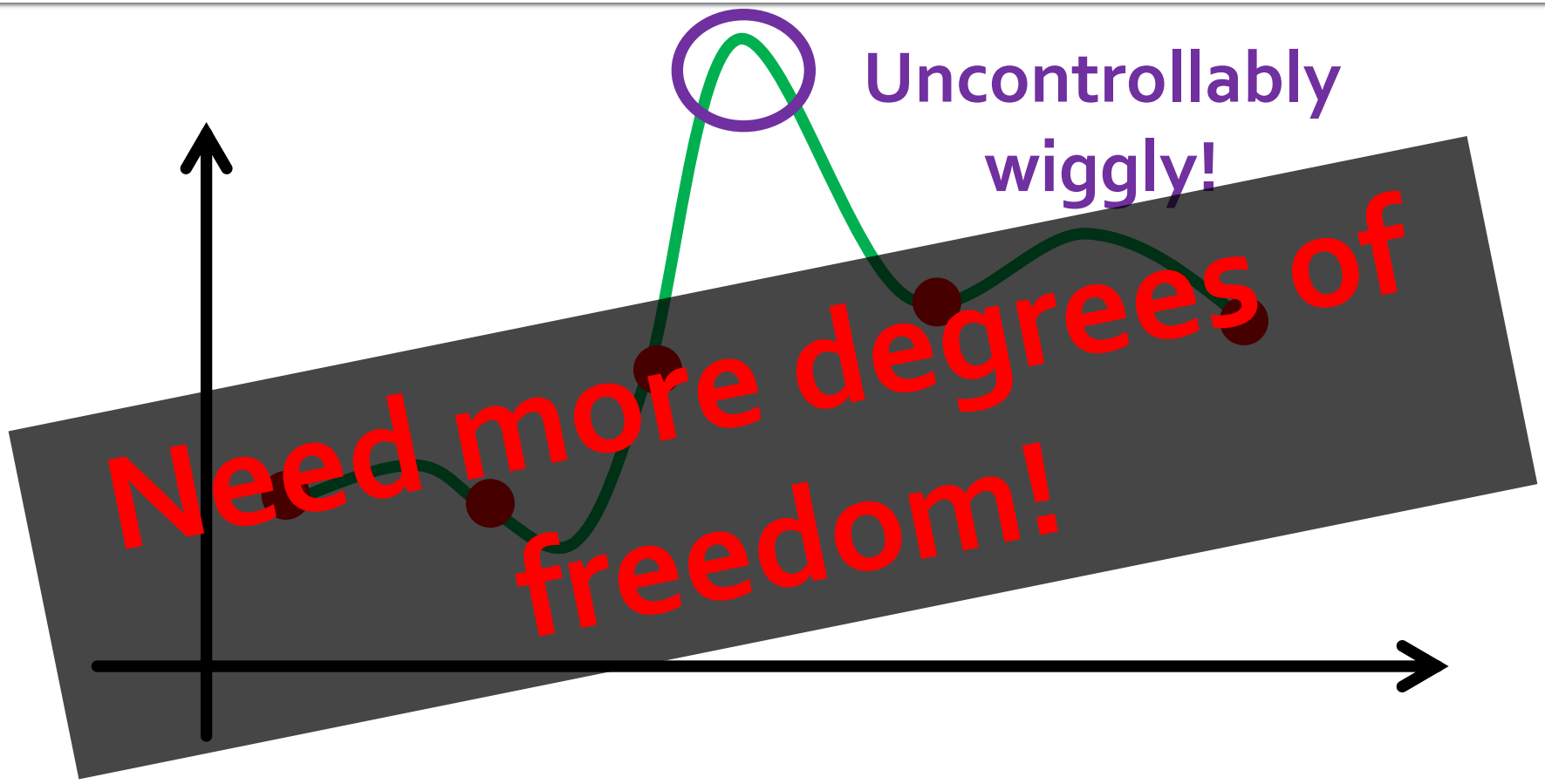
Parabola

Problems with Polynomials



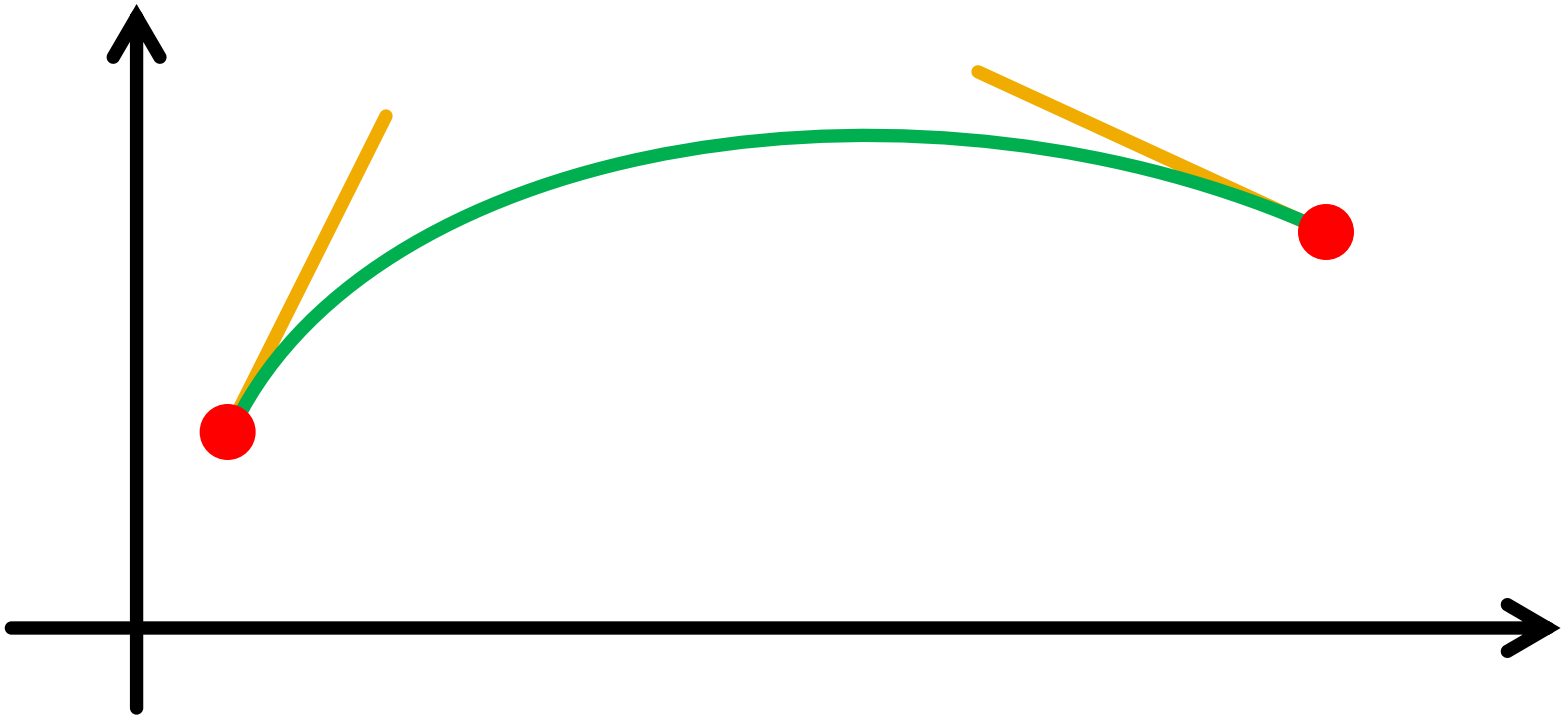
k points yields degree *k*

Problems with Polynomials



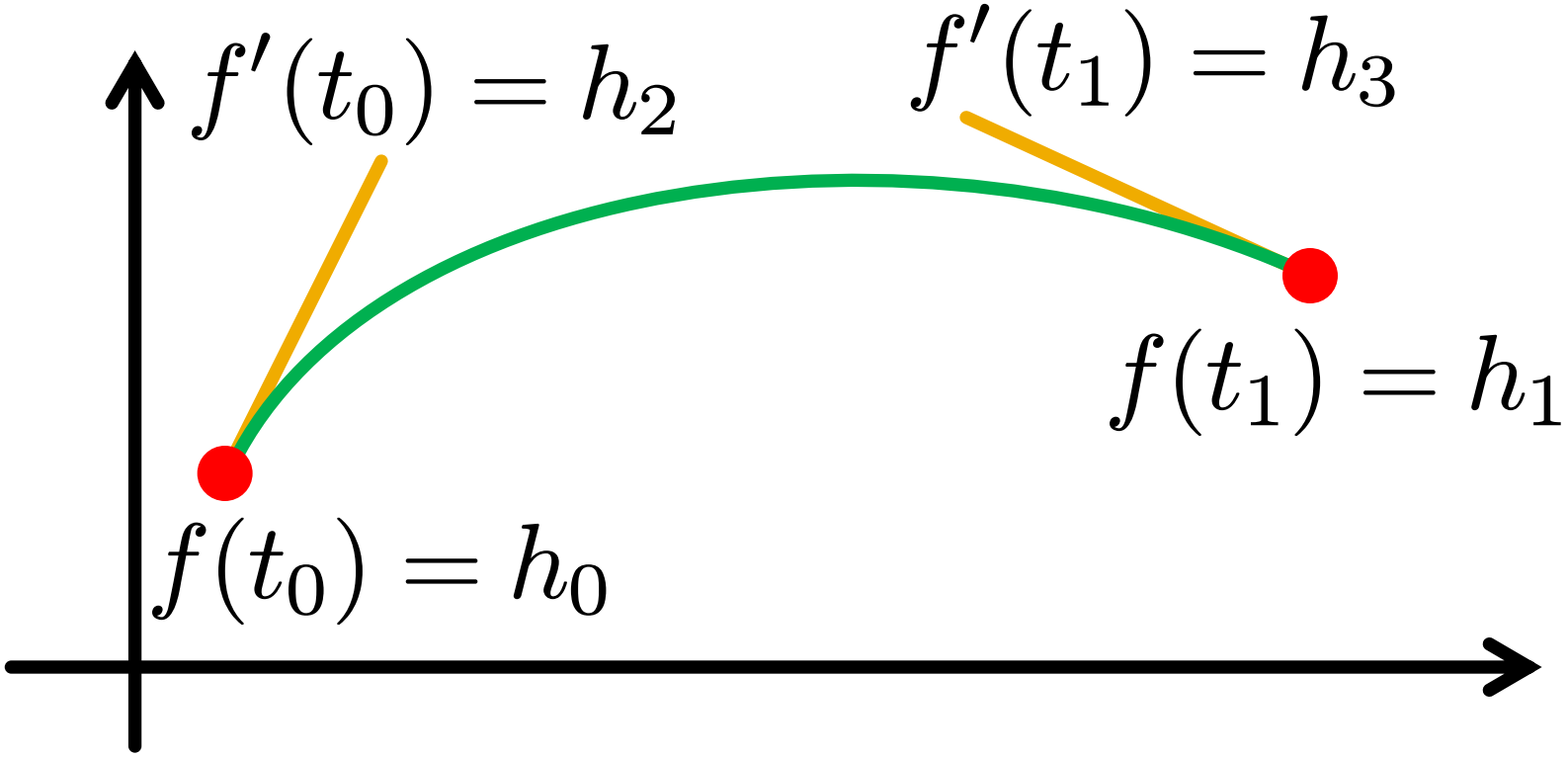
k points yields degree *k*

New Idea

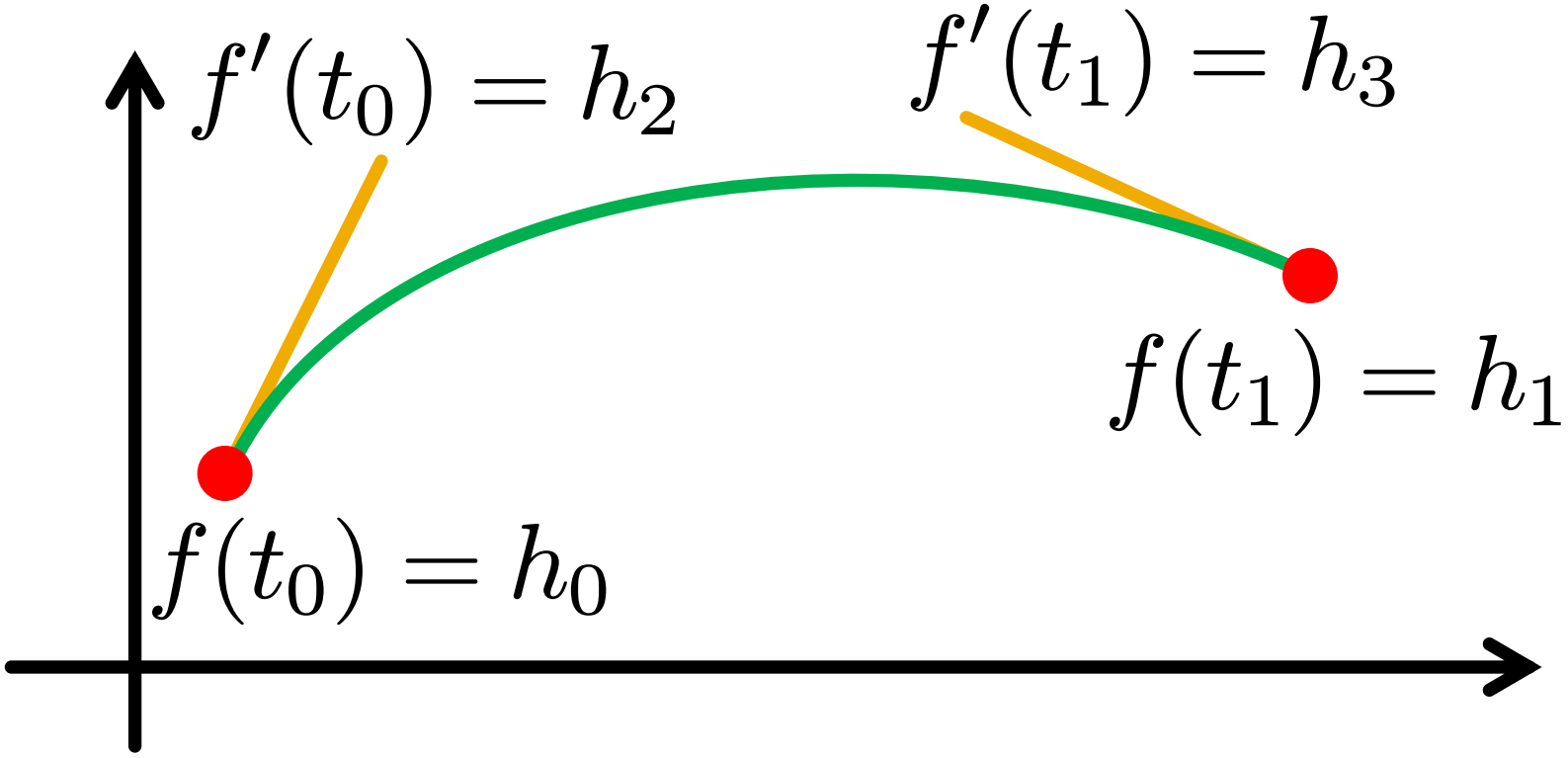


Specify points and slopes

Degrees of Freedom?

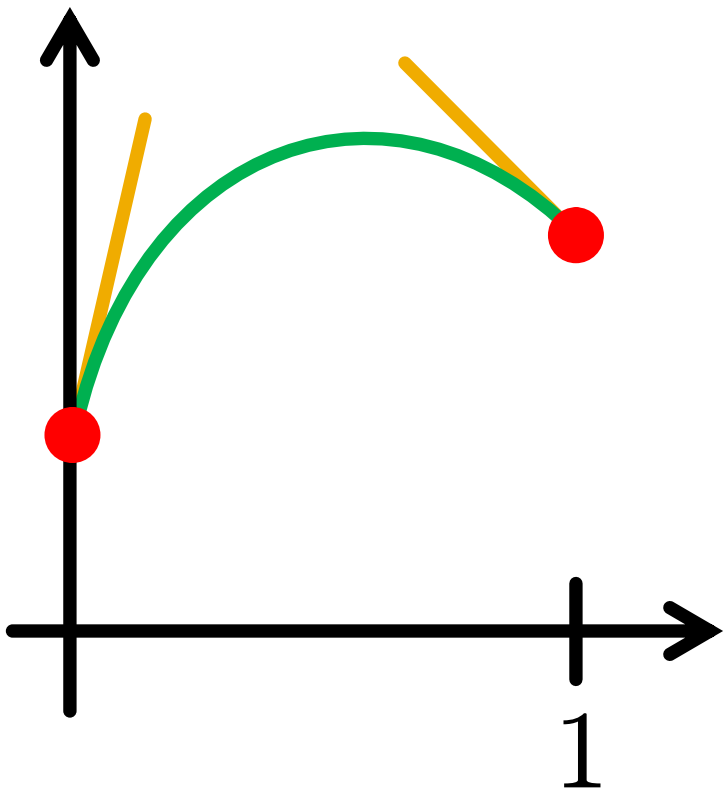


Degrees of Freedom?



Four: Two points, two slopes

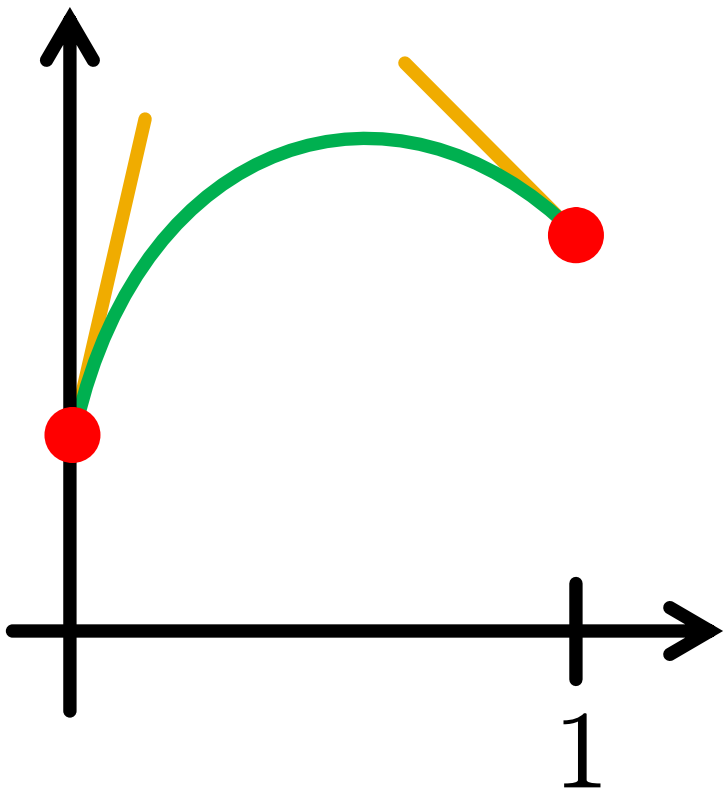
Cubic Hermite Interpolation



$$P(t) = at^3 + bt^2 + ct + d$$

$$P'(t) = 3at^2 + 2bt + c$$

Cubic Hermite Interpolation



$$P(t) = at^3 + bt^2 + ct + d$$

$$P'(t) = 3at^2 + 2bt + c$$

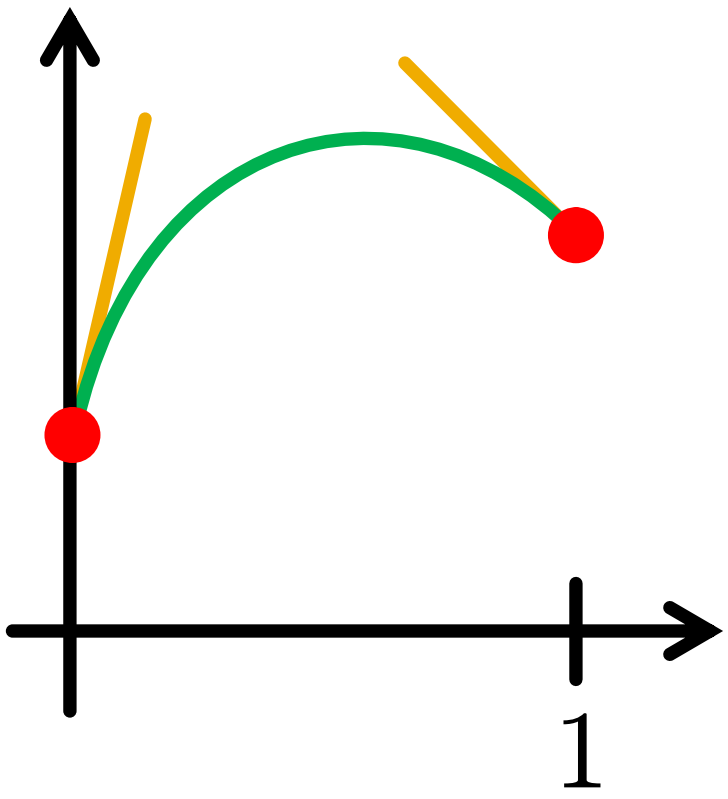
$$P(0) = d$$

$$P(1) = a + b + c + d$$

$$P'(0) = c$$

$$P'(1) = 3a + 2b + c$$

Cubic Hermite Interpolation



$$P(t) = at^3 + bt^2 + ct + d$$

$$P'(t) = 3at^2 + 2bt + c$$

$$P(0) = d = h_0$$

$$P(1) = a + b + c + d = h_1$$

$$P'(0) = c = h_2$$

$$P'(1) = 3a + 2b + c = h_3$$

Matrix Representation

$$
\begin{aligned}
P(0) &= d = h_0 \\
P(1) &= a + b + c + d = h_1 \\
P'(0) &= c = h_2 \\
P'(1) &= 3a + 2b + c = h_3
\end{aligned}
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
a \\ b \\ c \\ d
\end{pmatrix}
=
\begin{pmatrix}
h_0 \\ h_1 \\ h_2 \\ h_3
\end{pmatrix}
$$

Matrix Representation

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

Invert matrix

$$\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Matrix Representation

$$\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$a = 2h_0 - 2h_1 + h_2 + h_3$$

$$b = -3h_0 + 3h_1 - 2h_2 - h_3$$

$$c = h_2$$

$$d = h_0$$

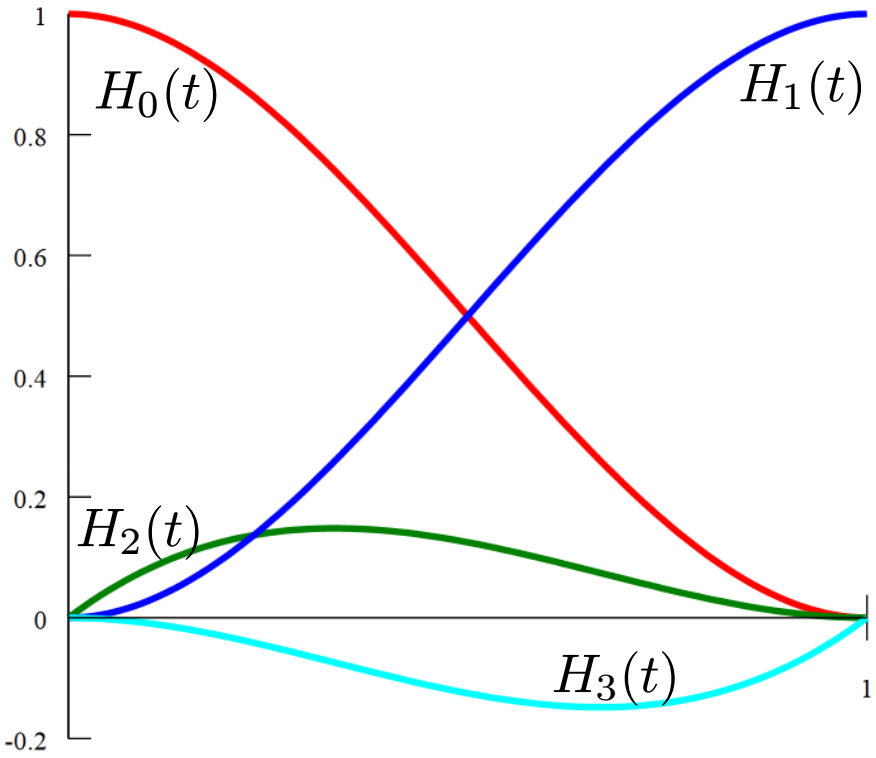
Back to the Curve

$$
\begin{aligned}
P(t) =& at^3 + bt^2 + ct + d \\
=& (2h_0 - 2h_1 + h_2 + h_3)t^3 + \\
& (-3h_0 + 3h_1 - 2h_2 - h_3)t^2 + \\
& h_2 t + h_0
\end{aligned}
$$

Back to the Curve

$$
\begin{aligned}
P(t) =& at^3 + bt^2 + ct + d \\
=& (2h_0 - 2h_1 + h_2 + h_3)t^3 + \\
& (-3h_0 + 3h_1 - 2h_2 - h_3)t^2 + \\
& h_2 t + h_0 \\
=& h_0(2t^3 - 3t^2 + 1) + h_1(-2t^3 + 3t^2) + \\
& h_2(t^3 - 2t^2 + t) + h_3(t^3 - t^2)
\end{aligned}
$$

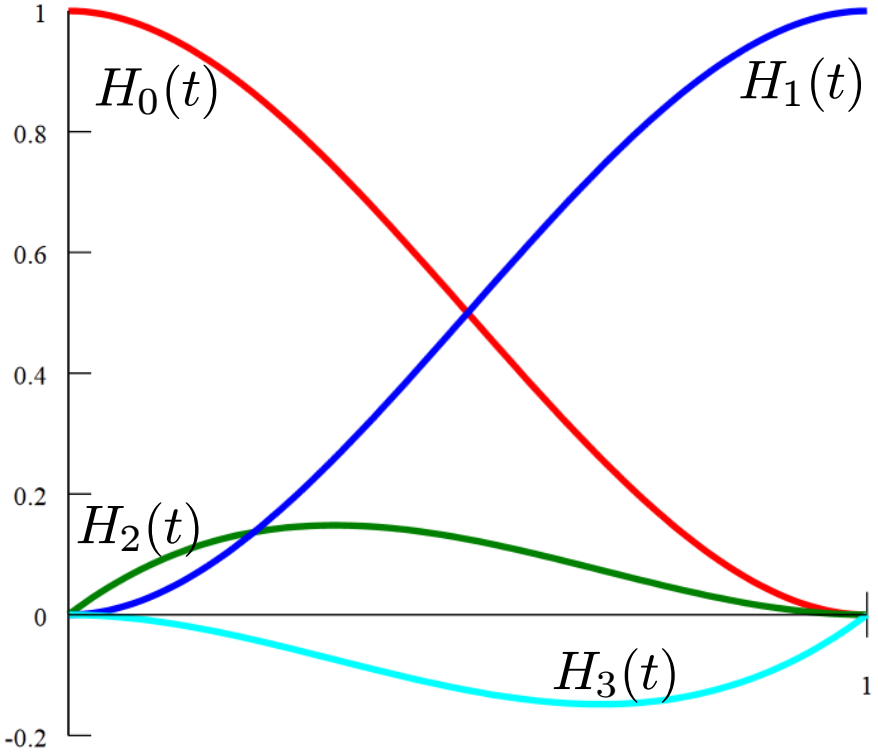
Hermite Basis



http://en.wikipedia.org/wiki/Cubic_Hermite_spline

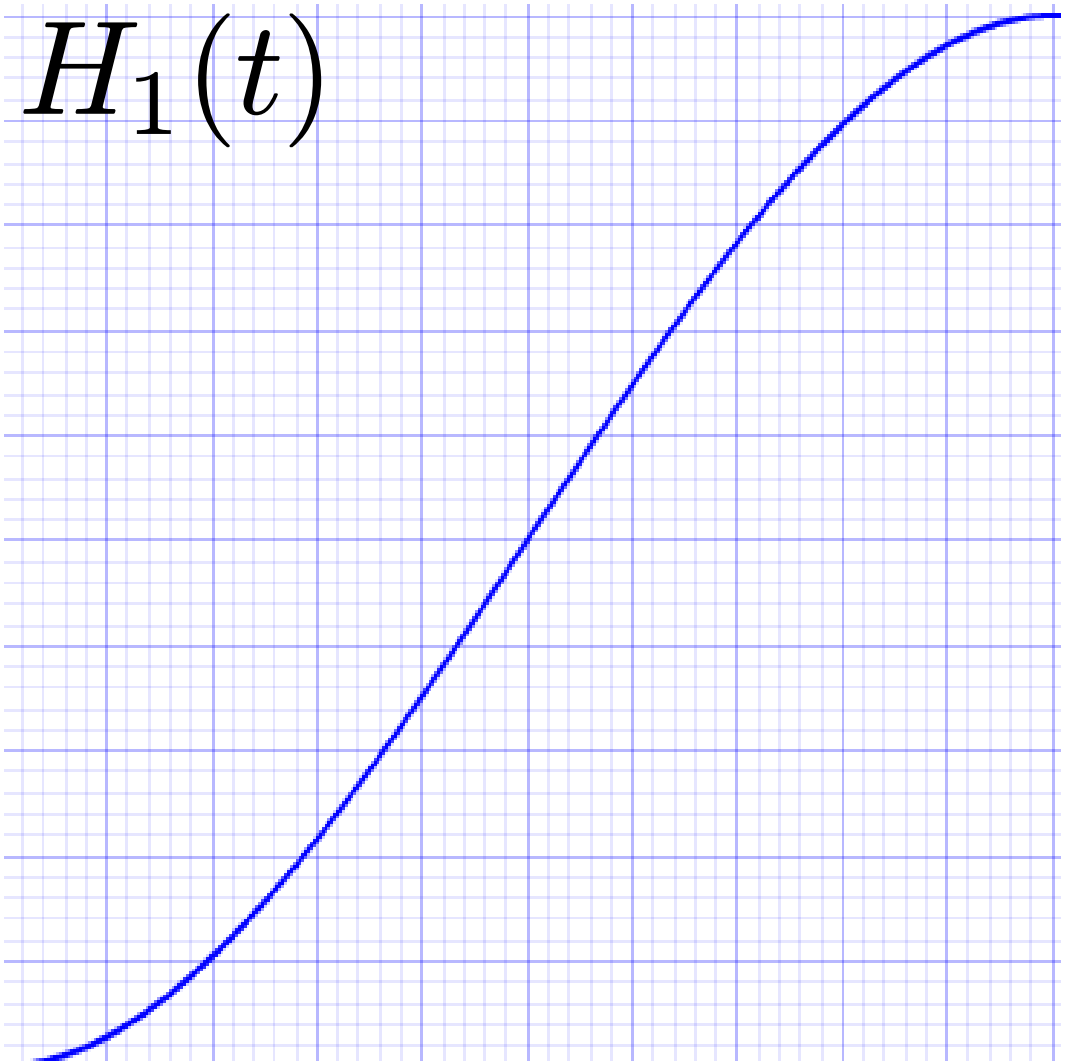
$$=h_0(2t^3-3t^2+1)+h_1(-2t^3+3t^2)+$$
$$h_2(t^3-2t^2+t)+h_3(t^3-t^2)$$

How to Compute Cubic Curve



$$P(t) = P(0)H_0(t) + P(1)H_1(t) + P'(0)H_2(t) + P'(1)H_1(t)$$

"Ease" Function

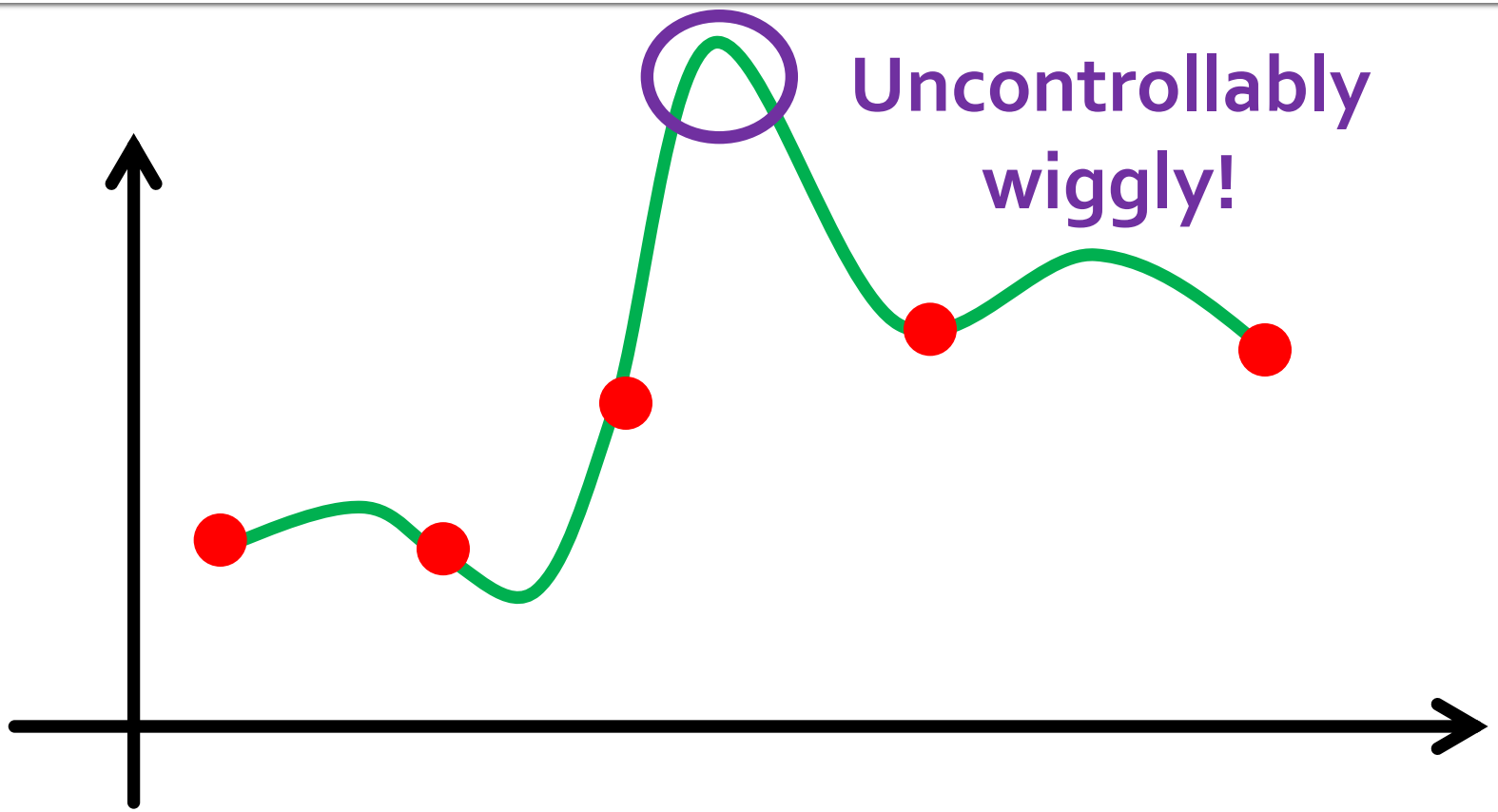


Ends slow

Starts slow

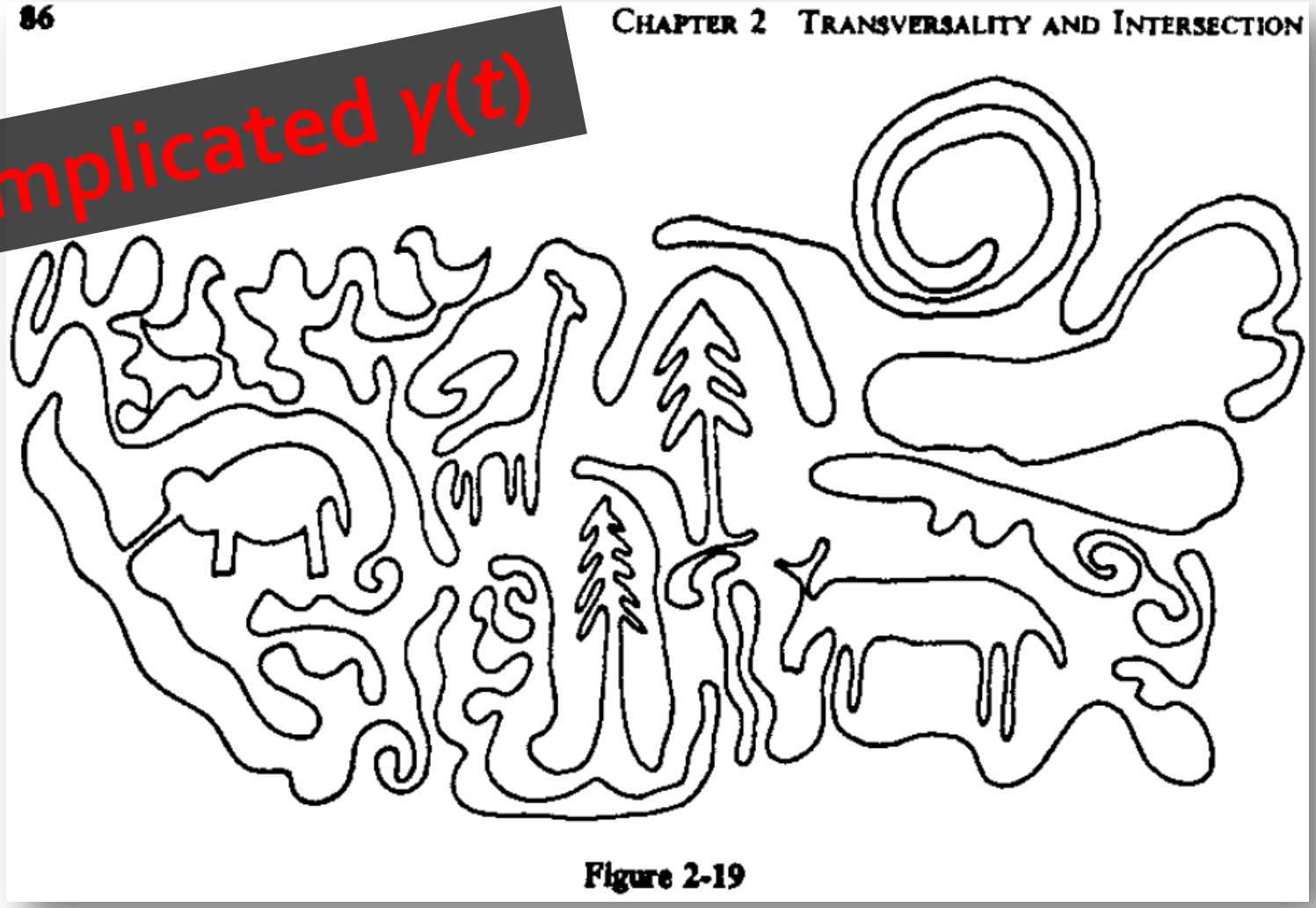
Recall:

Problems with Polynomials



k* points yields degree *k

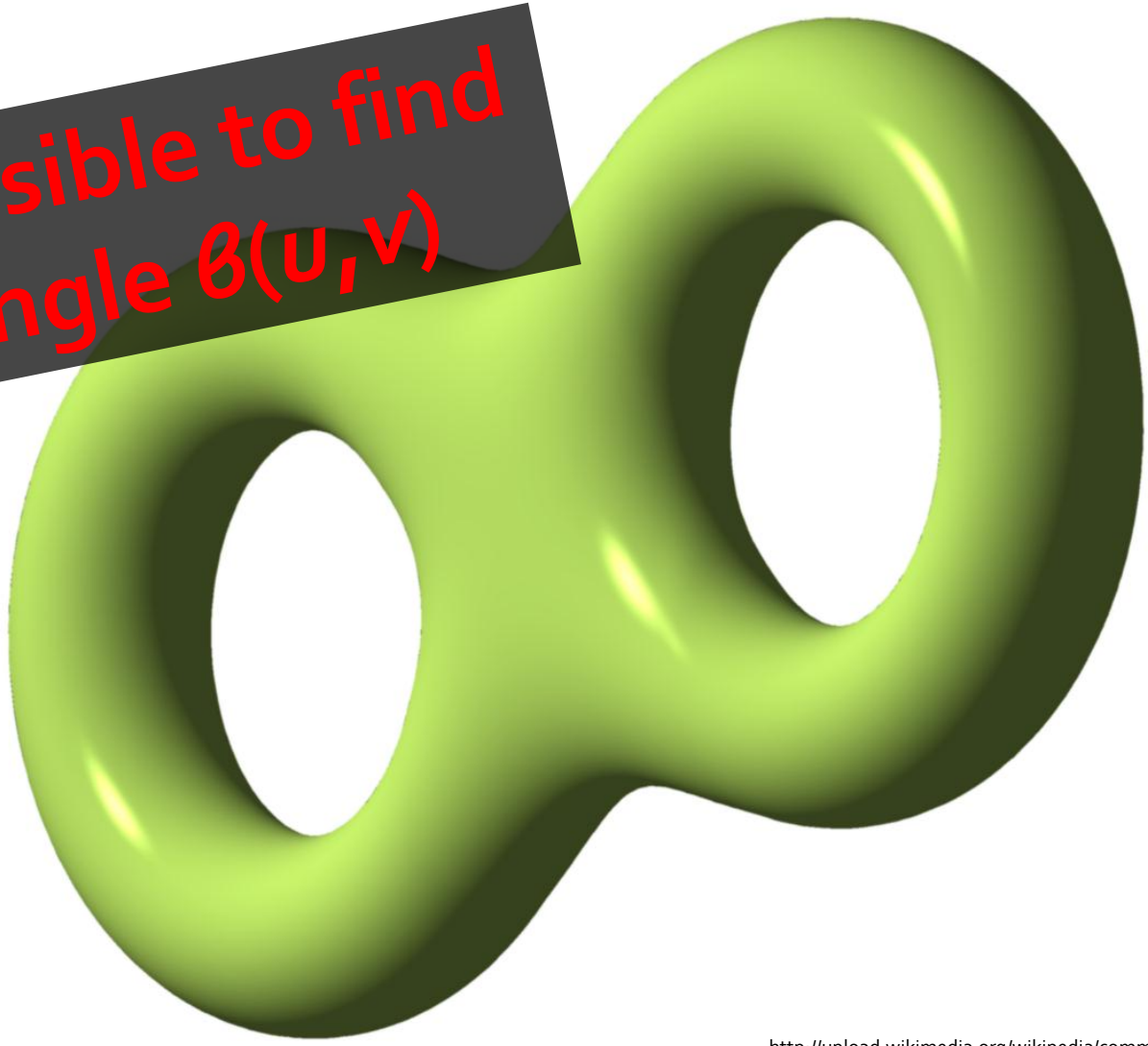
Global Parameterization



Differential Topology, Guillemin and Pollack

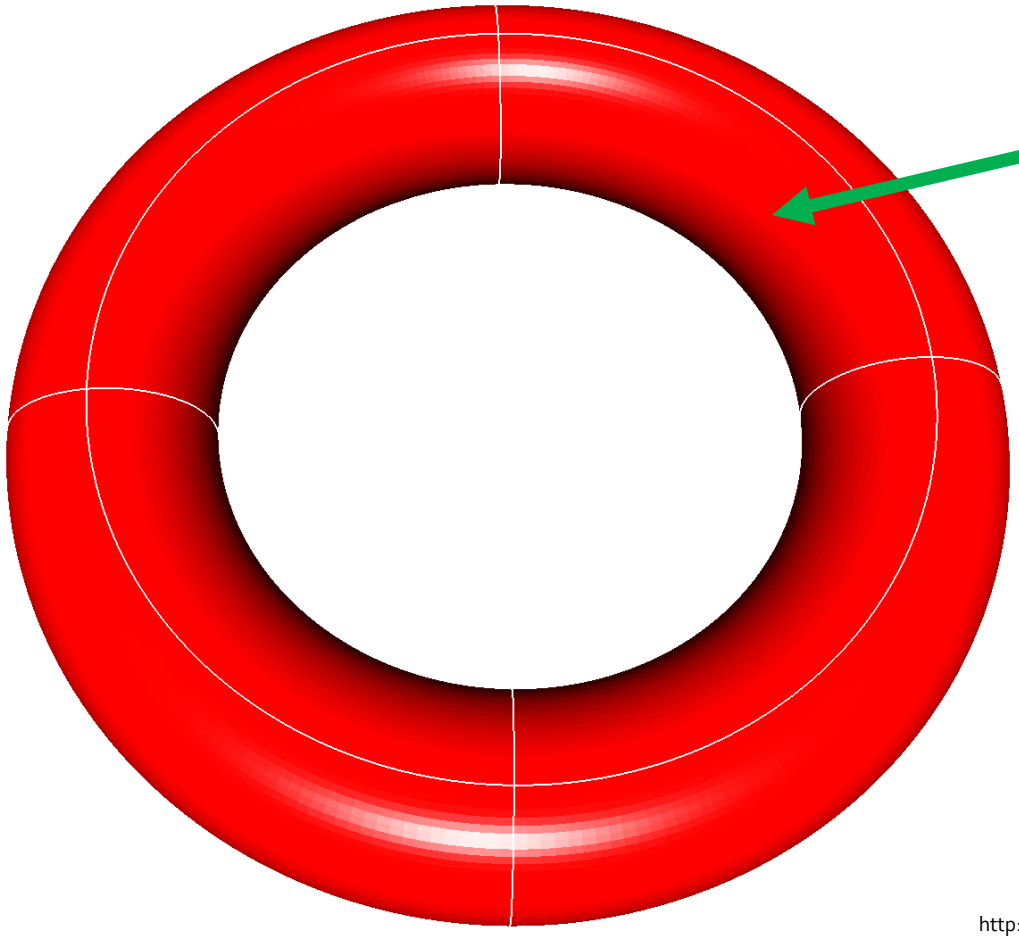
Global Parameterization

Impossible to find a single $b(u,v)$



http://upload.wikimedia.org/wikipedia/commons/b/bc/Double_torus_illustration.png

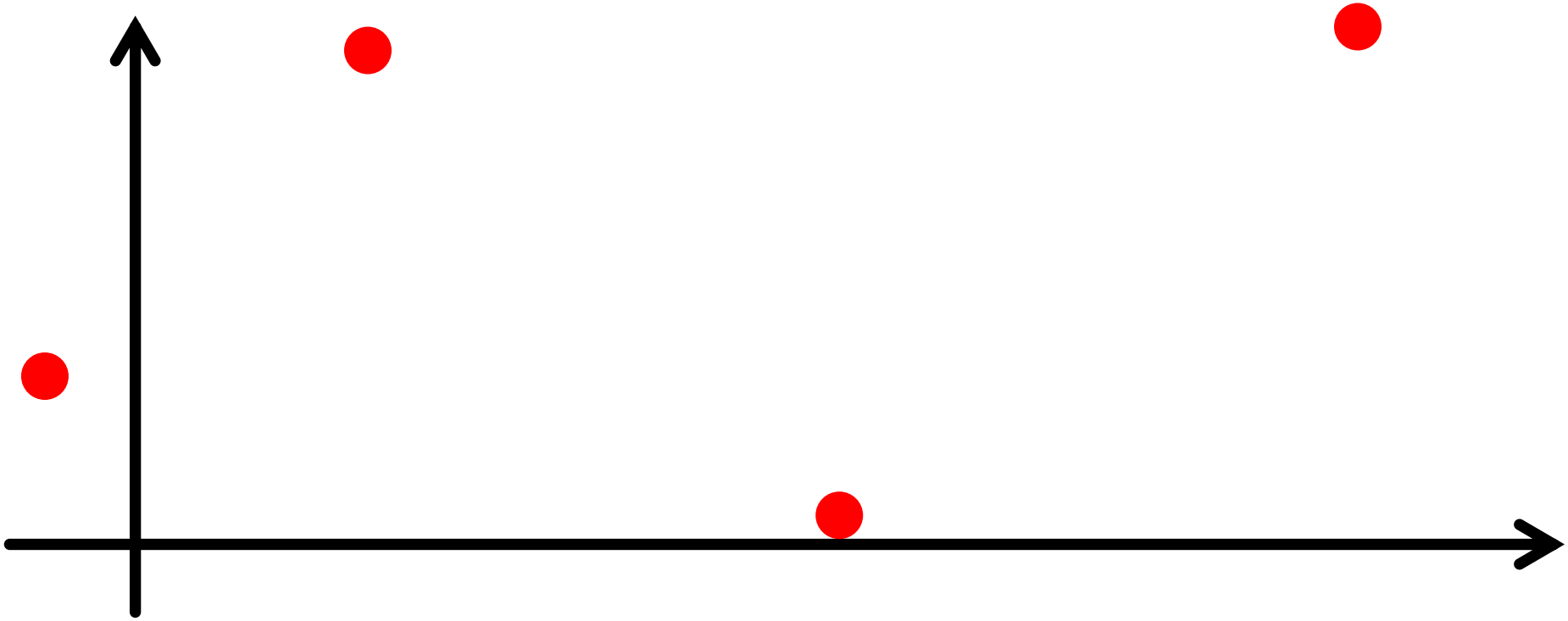
Better Idea



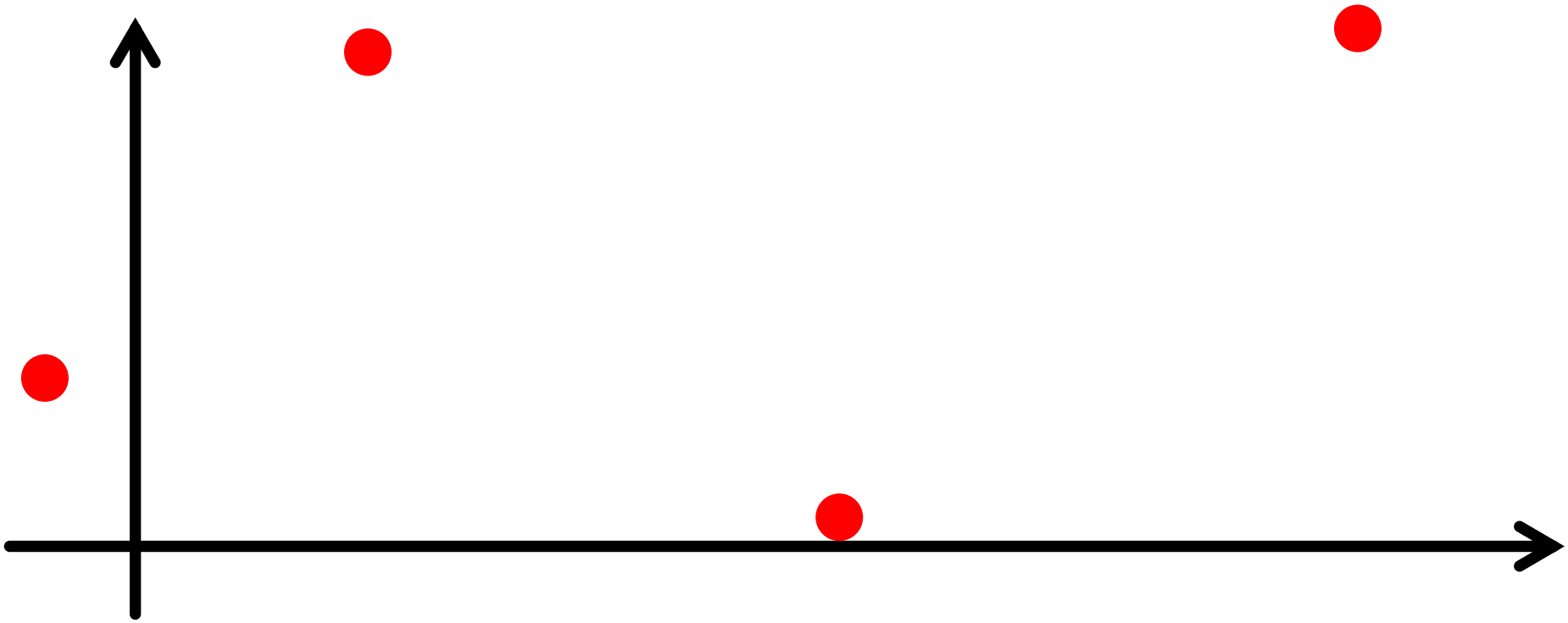
http://pages.cs.wisc.edu/~csverma/ME758/torus.png

Union of simpler building blocks

Interpolating Data Points

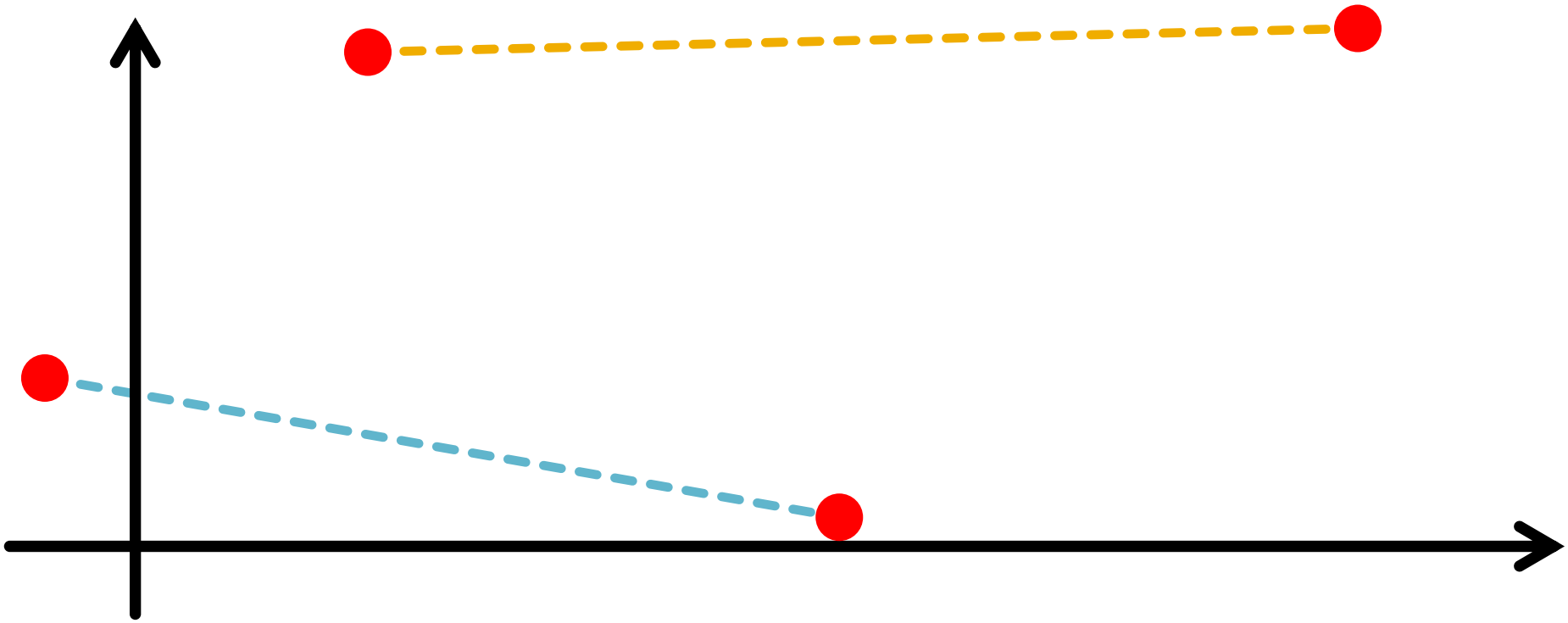


Interpolating Data Points



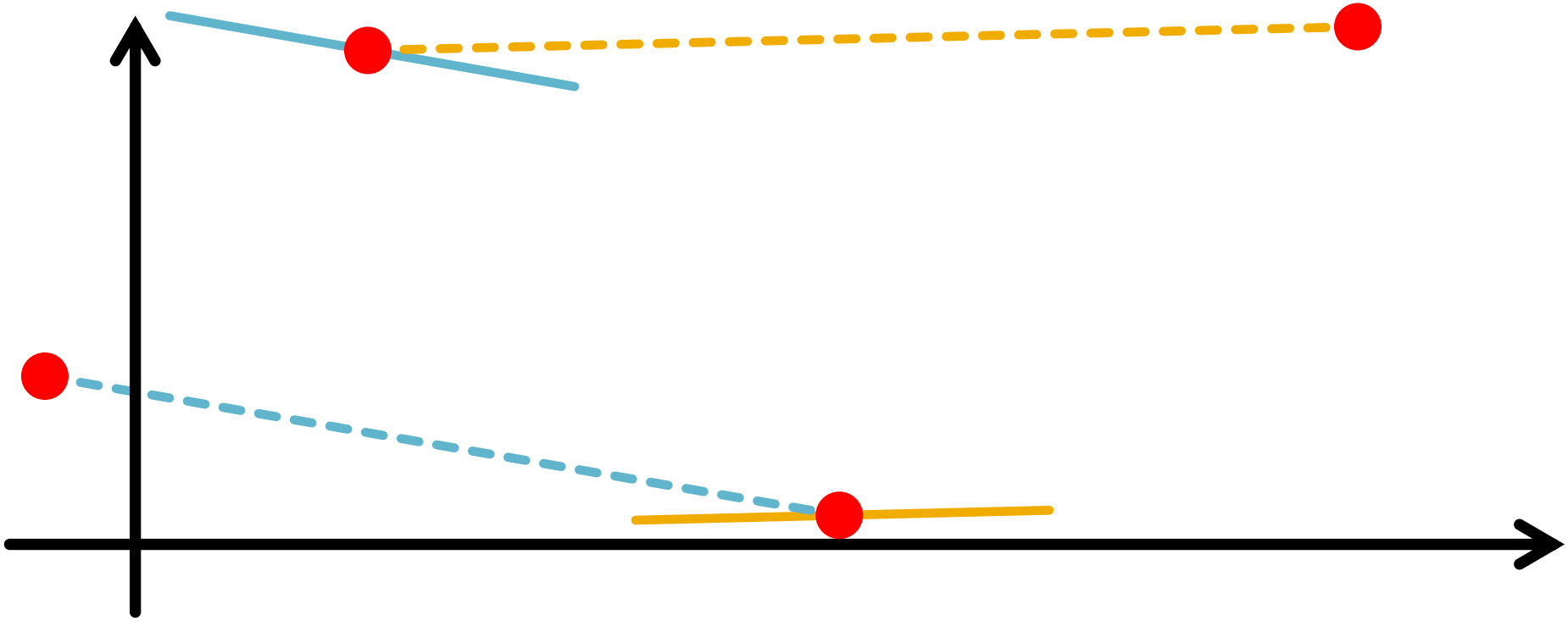
Catmull-Rom interpolation

Interpolating Data Points



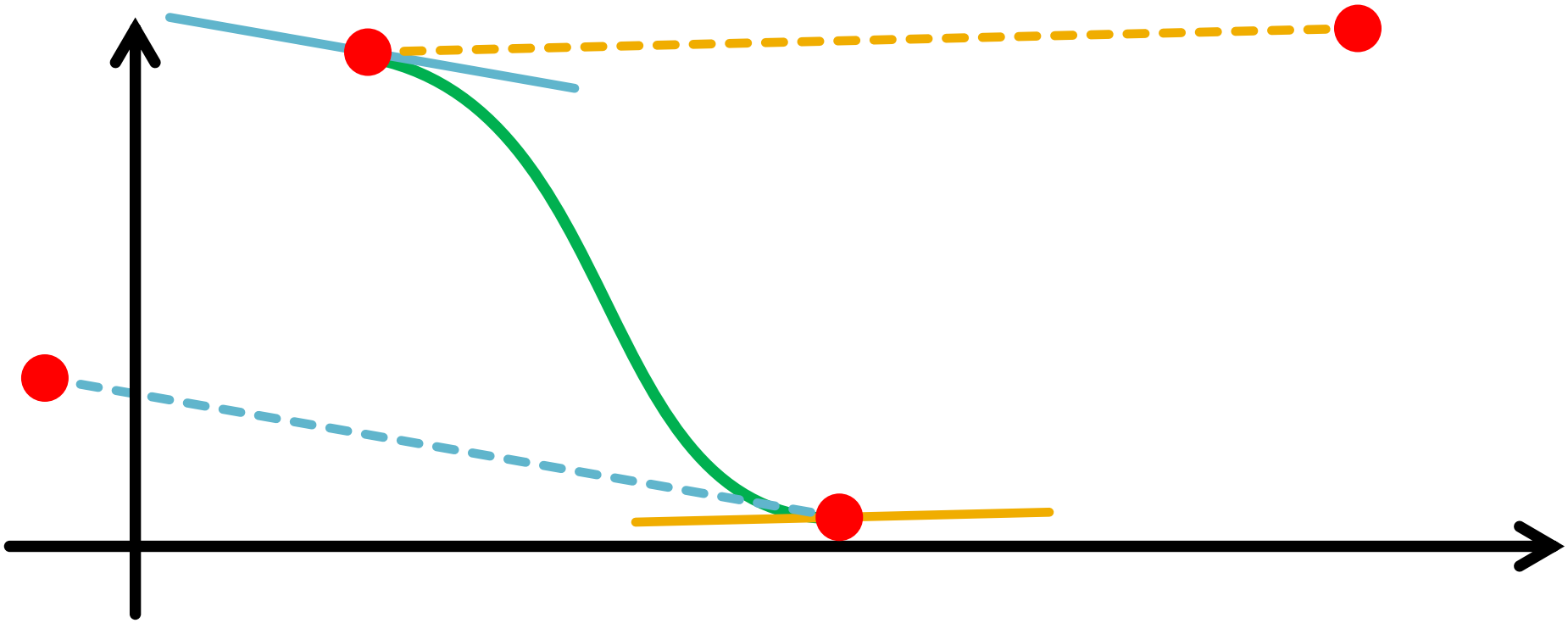
Catmull-Rom interpolation

Interpolating Data Points



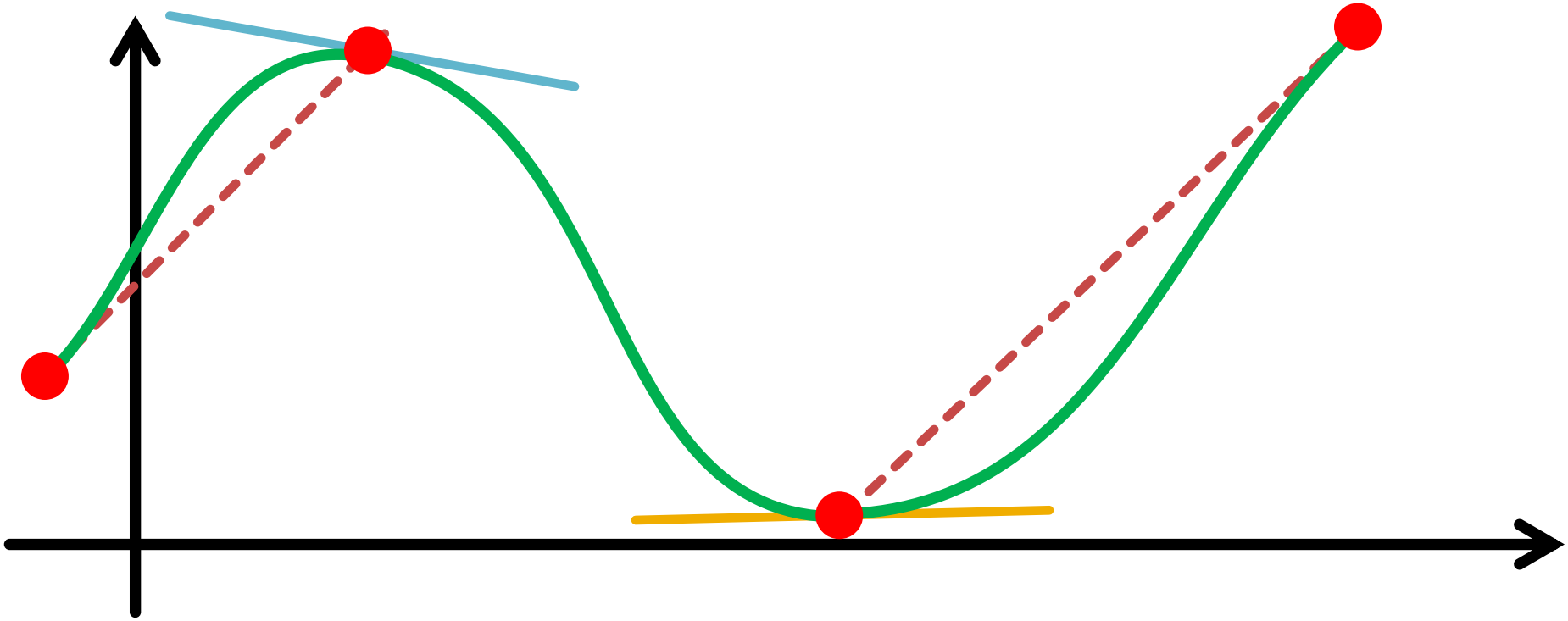
Catmull-Rom interpolation

Interpolating Data Points



Catmull-Rom interpolation

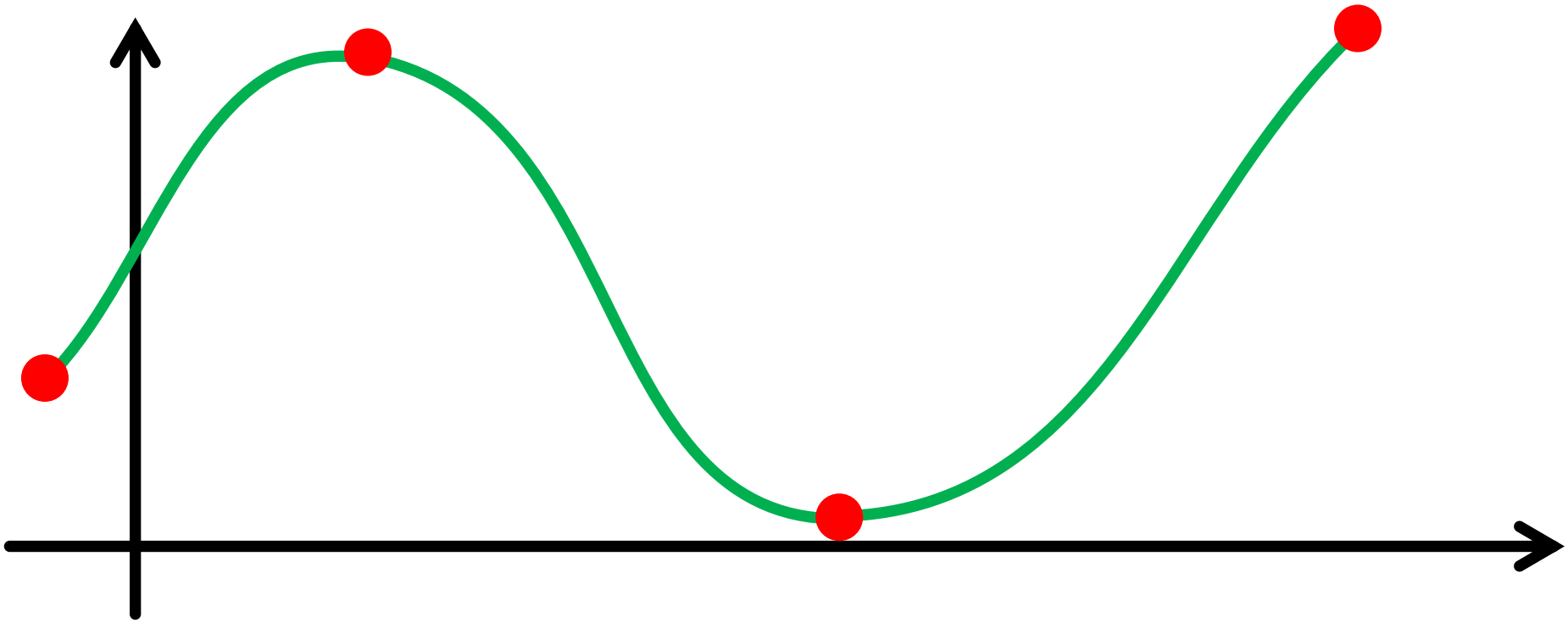
Interpolating Data Points



First and last need special treatment

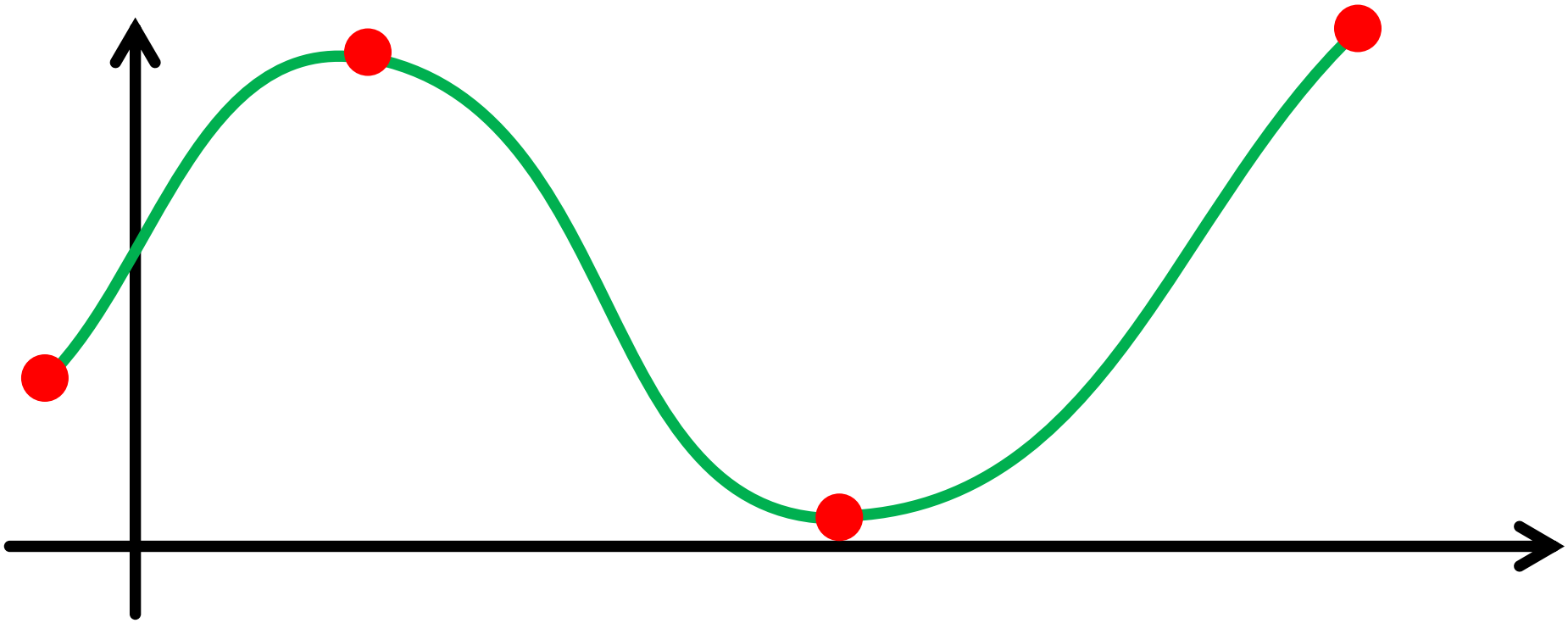
Catmull-Rom interpolation

Interpolating Data Points



Catmull-Rom interpolation

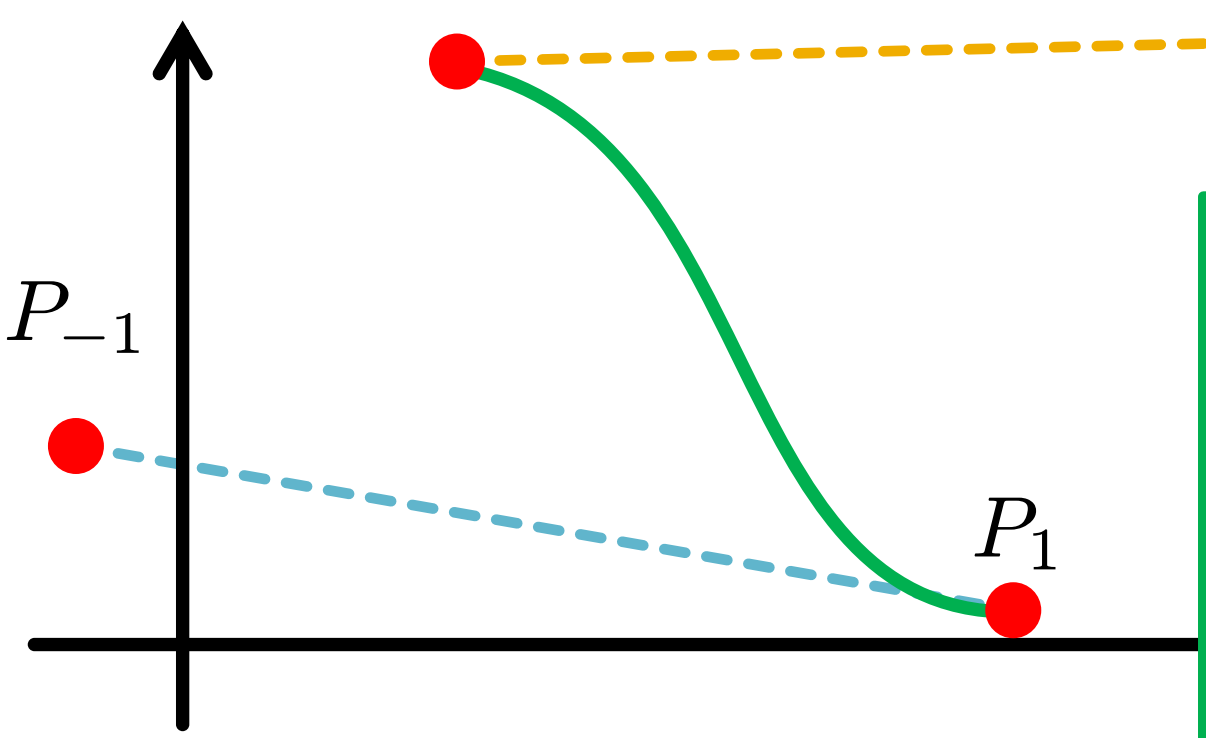
Interpolating Data Points



Three cubic segments with tangent continuity

Catmull-Rom interpolation

Hermite Formula



$$H_0 = P_0$$

$$H_1 = P_1$$

$$H_2 = \frac{1}{2}(P_1 - P_{-1})$$

$$H_3 = \frac{1}{2}(P_2 - P_0)$$

Catmull-Rom interpolation

Properties of Catmull-Rom

- **C^0 continuous**

  "Continuous" in the usual sense

- **C^1 continuous**

  Derivatives and values are continuous

Curves

$$\vec{\gamma}(t) = (\gamma_0(t), \gamma_1(t), \gamma_2(t))$$

Three cubic functions in one

Curves

$$\vec{\gamma}(t) = (\gamma_0(t), \gamma_1(t), \gamma_2(t))$$

Tangent to curve

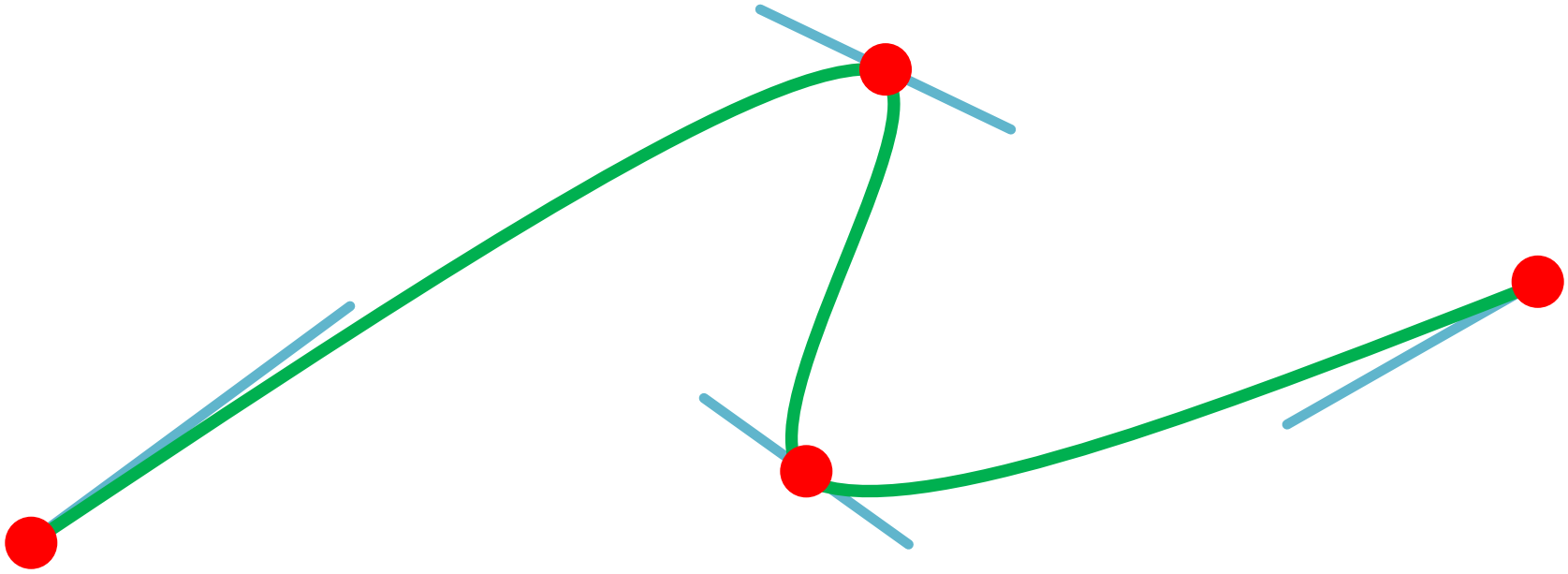
Slopes

$$\vec{\gamma}'(t) = (\gamma_0'(t), \gamma_1'(t), \gamma_2'(t))$$

Hermite formulas apply here

Curve Interface

$$\vec{\gamma}(t) = (\gamma_0(t), \gamma_1(t), \gamma_2(t))$$



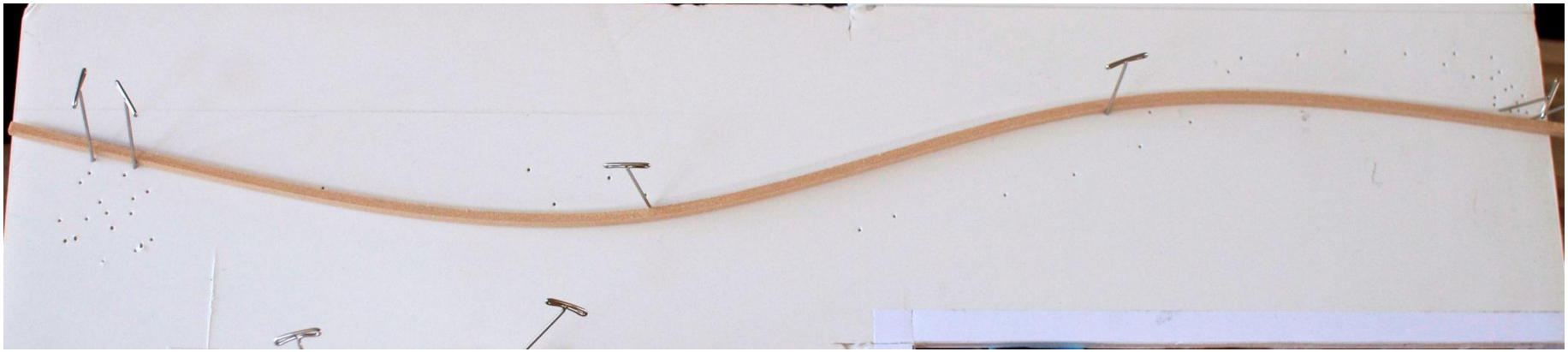
Hermite formulas apply here

Spline [splahyn]:

A sufficiently smooth piecewise polynomial with a high degree of smoothness where pieces meet.

http://en.wikipedia.org/wiki/Spline_%28mathematics%29

Origin of "Spline"



http://parametricwood2011.files.wordpress.com/2011/01/spline03.jpg

Woodworking

Drawing a Circle?

$$\vec{\gamma}(t) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right)$$

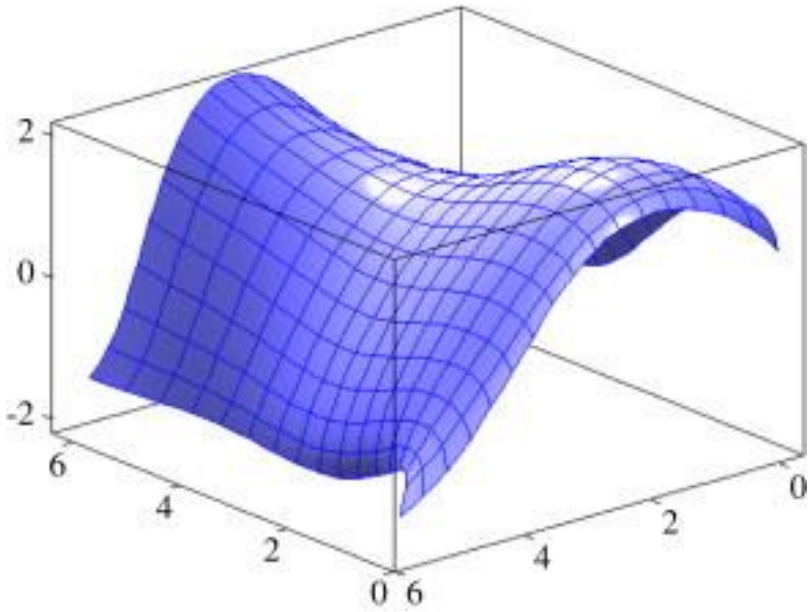
Circles aren't polynomial

Drawing a Circle?

$$\vec{\gamma}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \mapsto (1-t^2, 2t; 1+t^2)$$

Polynomial in projective space!

Tensor Product Patches



Easy generalization to surfaces where *x* and *y* slices are cubic curves.

ars.els-cdn.com/content/image/1-s2.0-S0377042709002064-gr4.jpg

Problem

We need an easy way to obtain points on our curve.

Blossoming a Polynomial

$$f(t) \text{ quadratic} \mapsto F(t_1, t_2)$$

$$f(t) \text{ cubic} \mapsto F(t_1, t_2, t_3)$$

$$\dots$$

$$f(t) \text{ degree } k \mapsto F(t_1, t_2, \cdots, t_k)$$

Degree *k* to *k* variables

Properties of a (Cubic) Blossom

- **Symmetric**

$$F(t_1, t_2, t_3) = F(t_1, t_3, t_2) = F(t_2, t_1, t_3) = \cdots$$

- **Affine**

$$F(\alpha u + (1-\alpha)v, t_2, t_3) =$$
$$\alpha F(u, t_2, t_3) + (1-\alpha)F(v, t_2, t_3)$$

- **Diagonal**

$$f(t) = F(t, t, t)$$

Blossoming Examples

$$f(t) = t^3 \mapsto F(t_1, t_2, t_3) = t_1 t_2 t_3$$

$$f(t) = t^2 \mapsto F(t_1, t_2, t_3) = \frac{1}{3}(t_1 t_2 + t_2 t_3 + t_1 t_3)$$

$$f(t) = t \mapsto F(t_1, t_2, t_3) = \frac{1}{3}(t_1 + t_2 + t_3)$$

$$f(t) = 1 \mapsto F(t_1, t_2, t_3) = 1$$

Blossoming Examples

$$f(t) = t^3 \mapsto F(t_1, t_2, t_3) = t_1 t_2 t_3$$

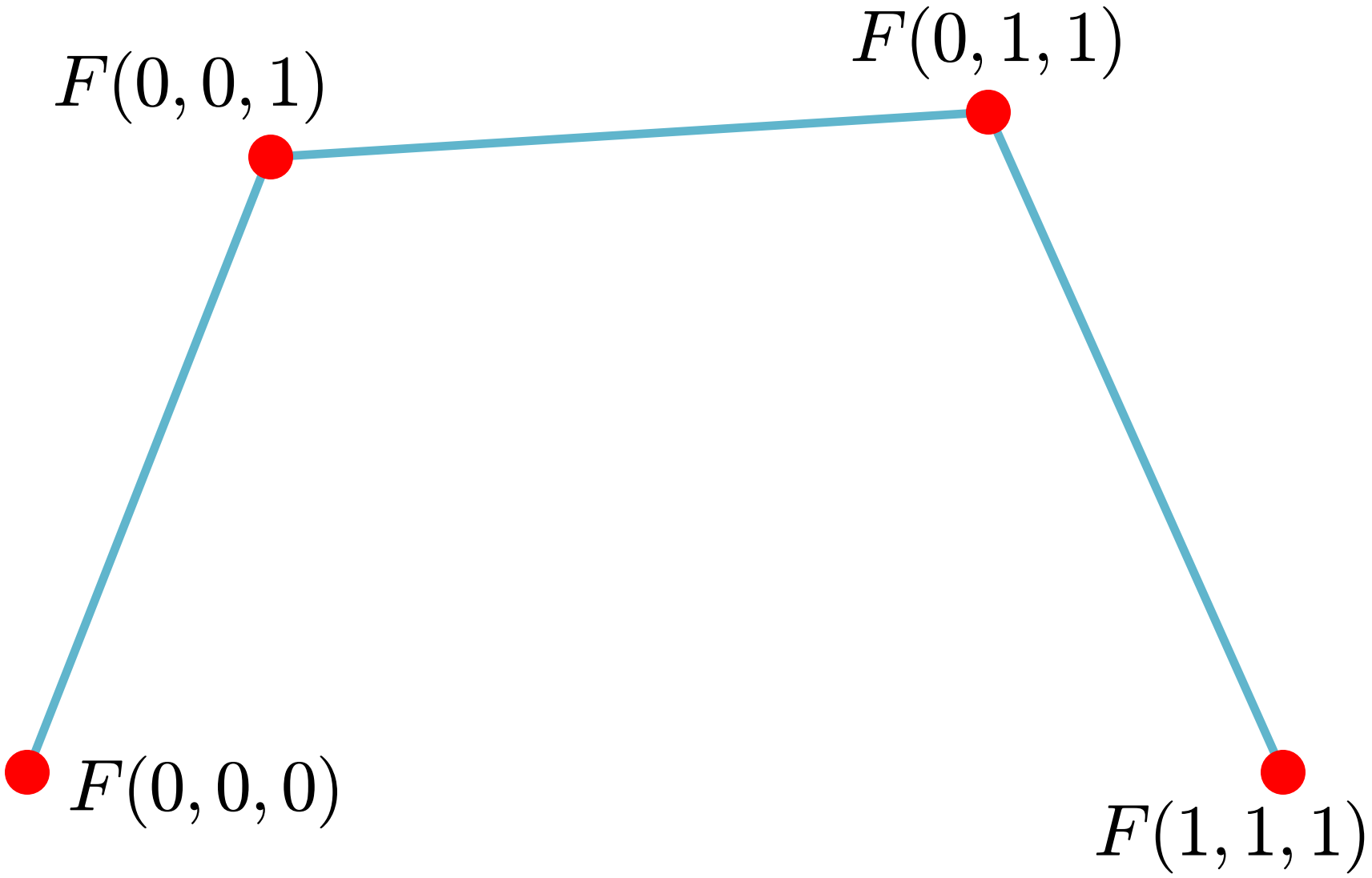
$$f(t) = t^2 \mapsto F(t_1, t_2, t_3) = \frac{1}{3}(t_1 t_2 + t_2 t_3 + t_1 t_3)$$

$$f(t) = t \mapsto F(t_1, t_2, t_3) = \frac{1}{3}(t_1 + t_2 + t_3)$$

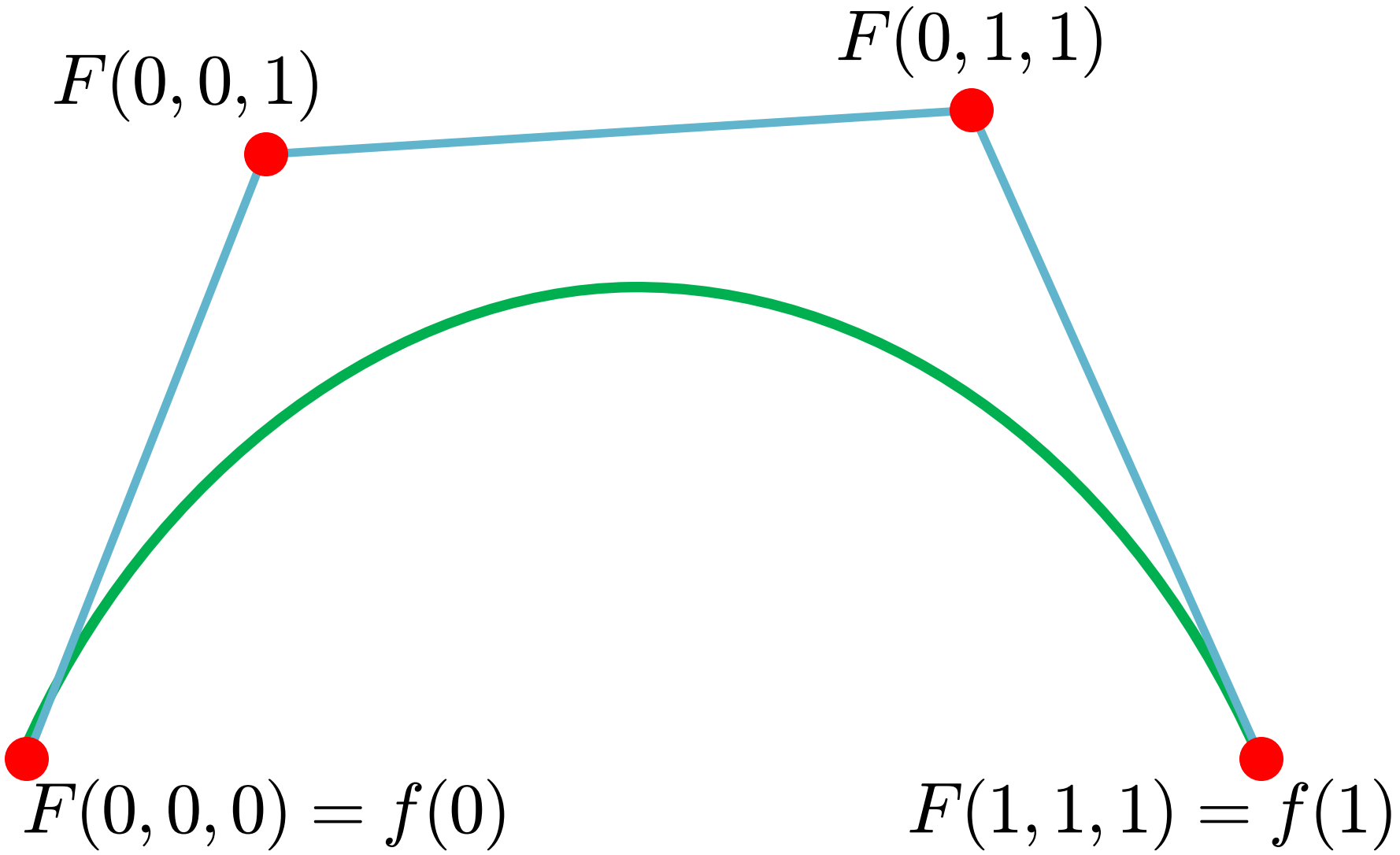
$$f(t) = 1 \mapsto F(t_1, t_2, t_3) = 1$$

More practice on next homework.

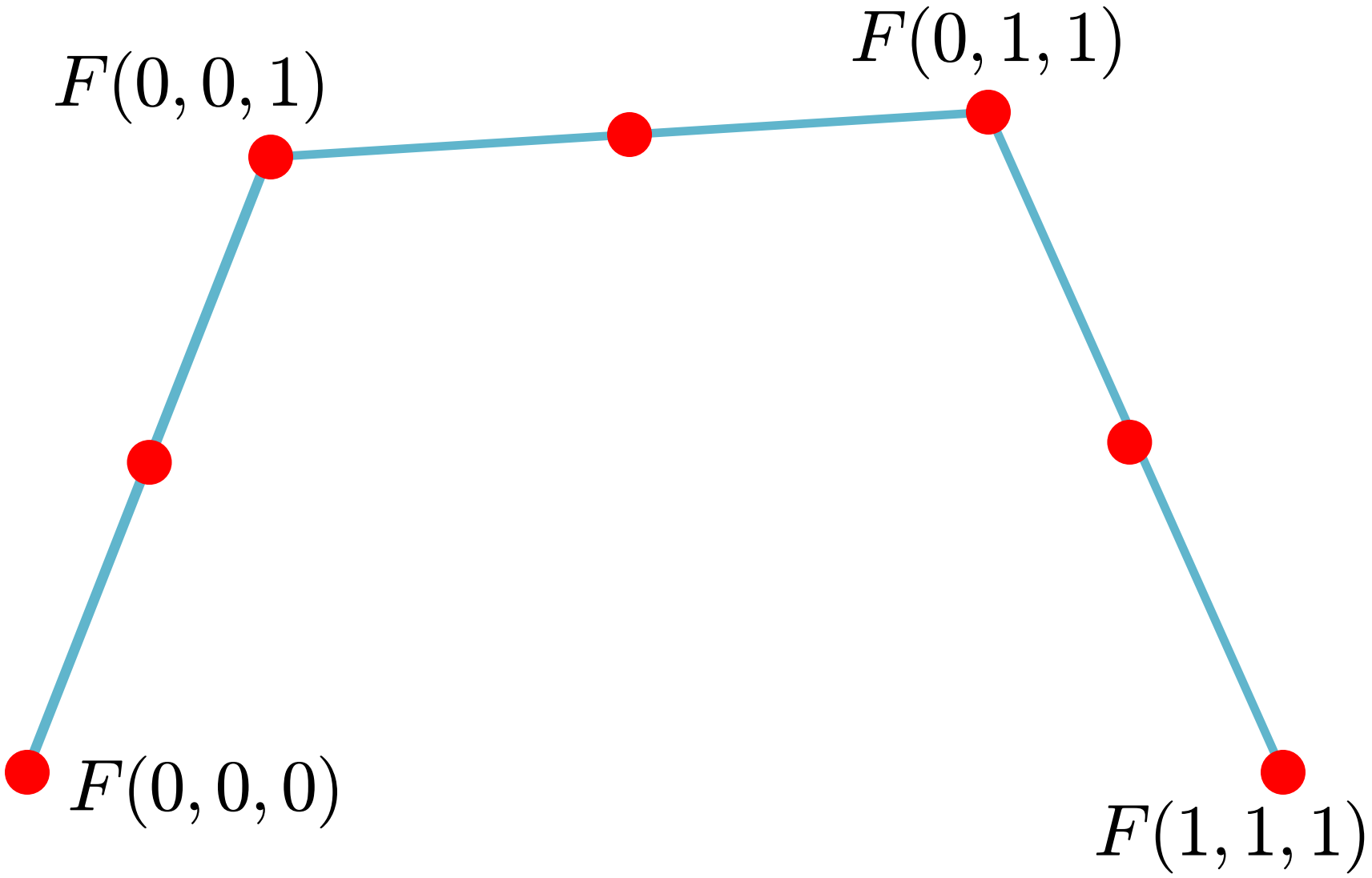
Cubic Control Polygon



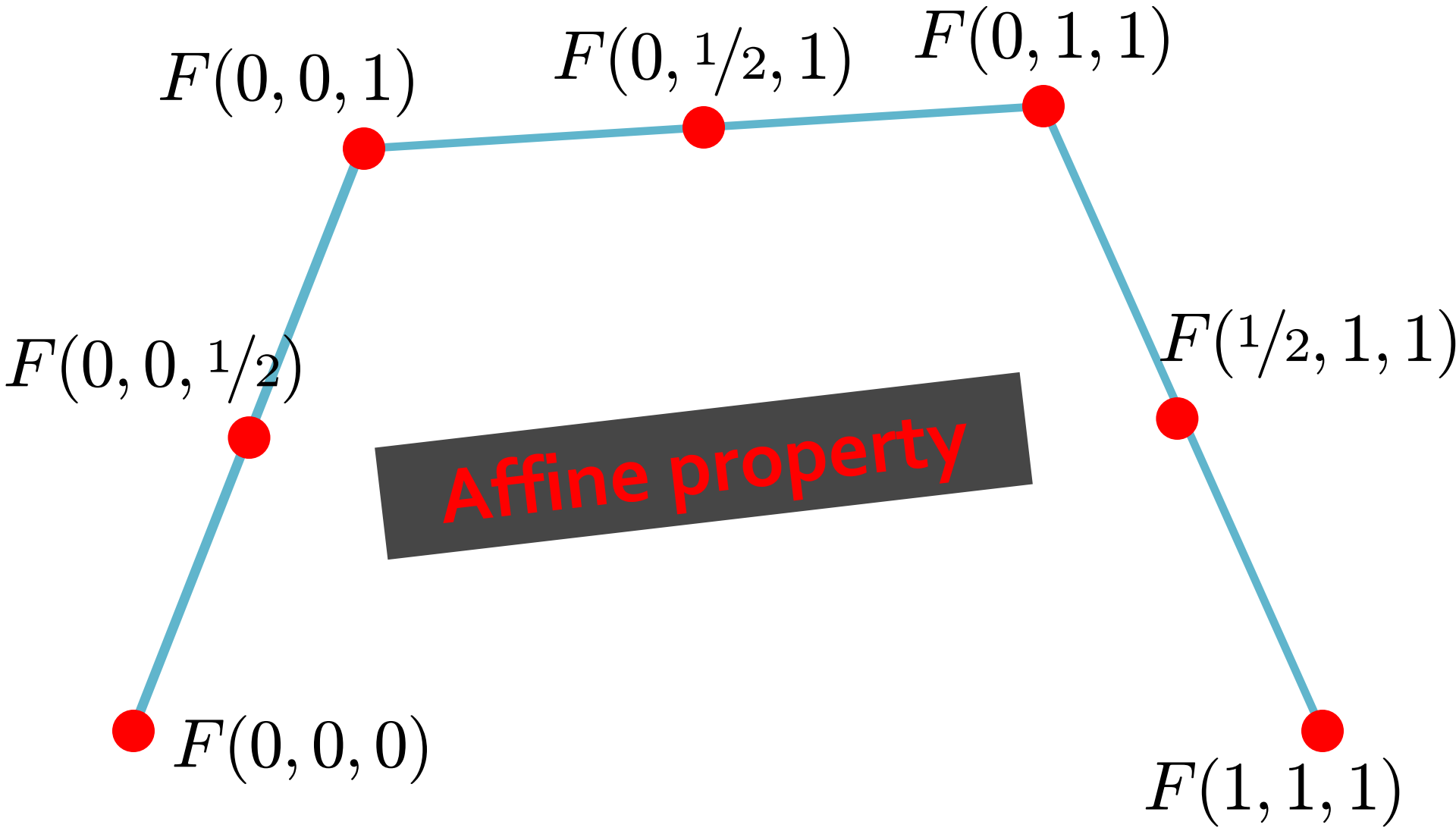
Cubic Control Polygon



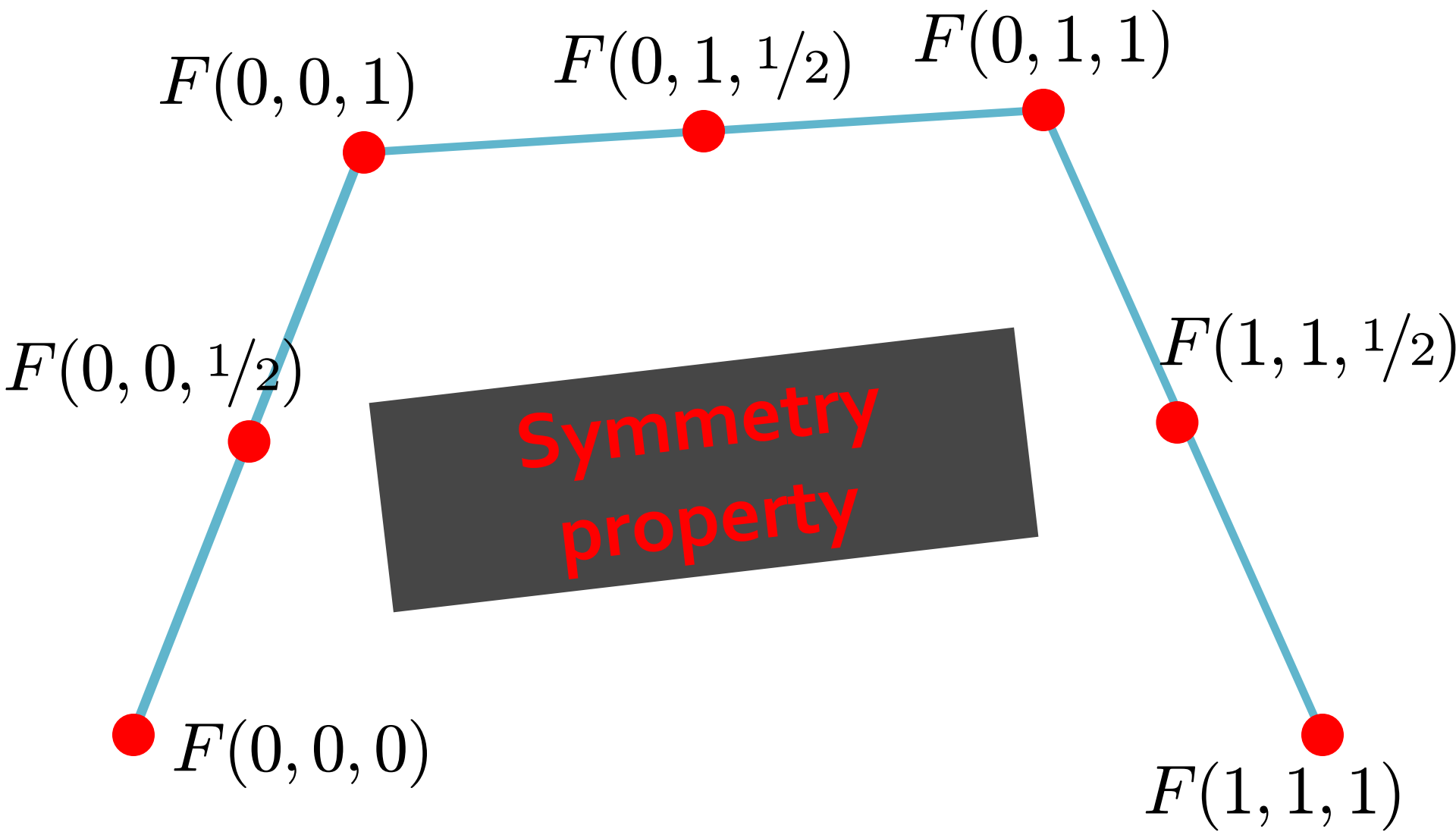
Cubic Control Polygon



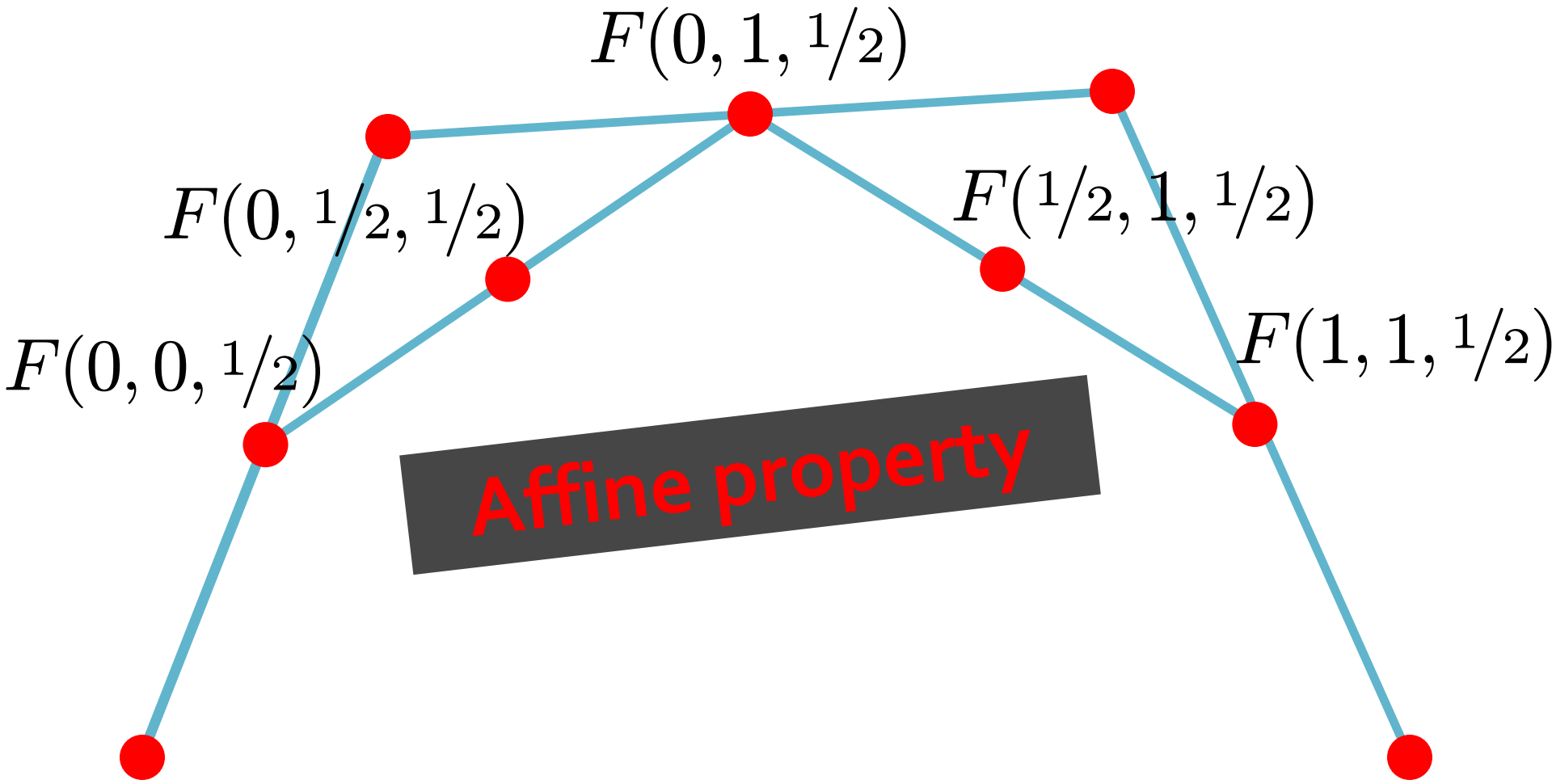
Cubic Control Polygon



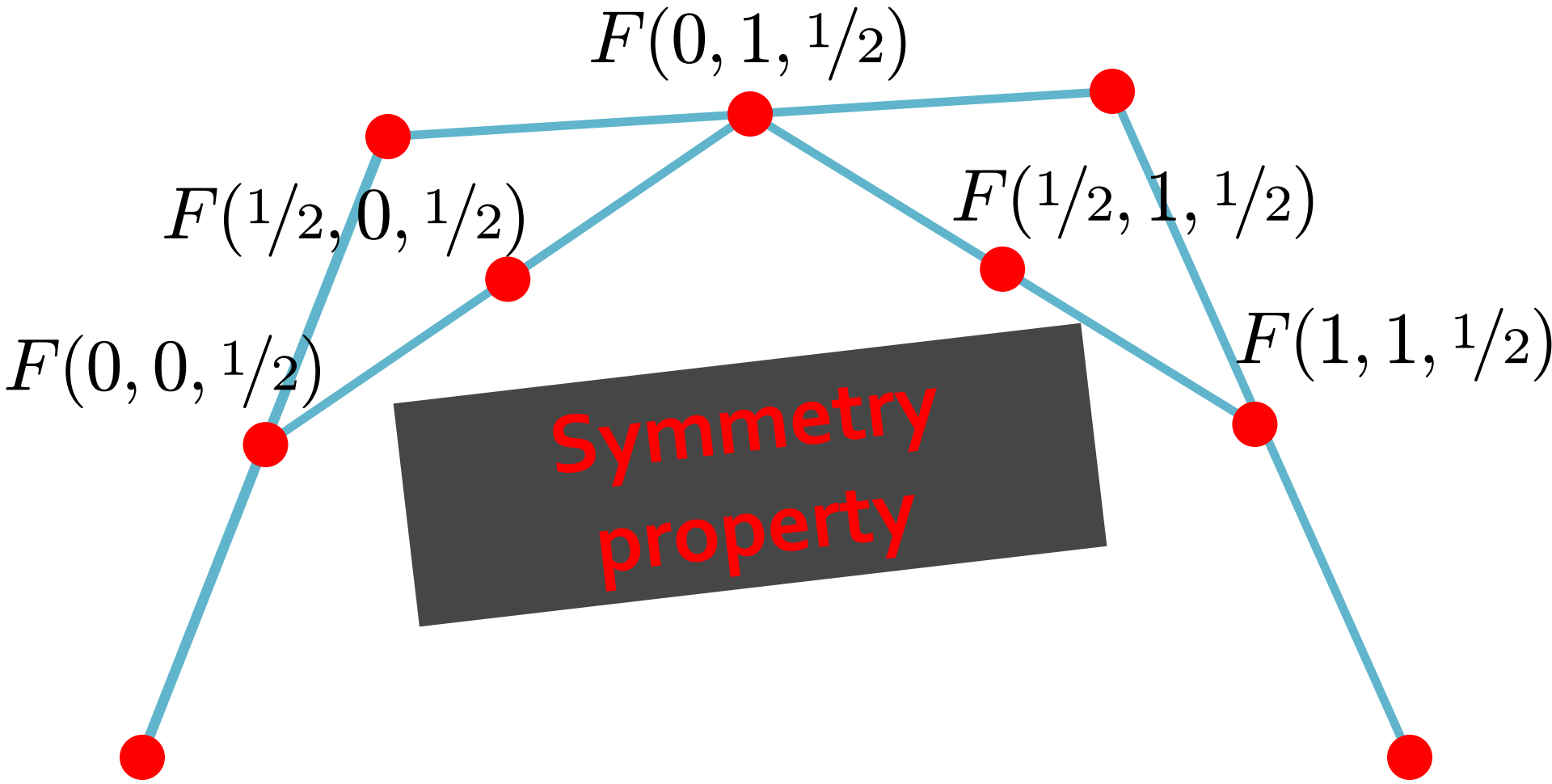
Cubic Control Polygon



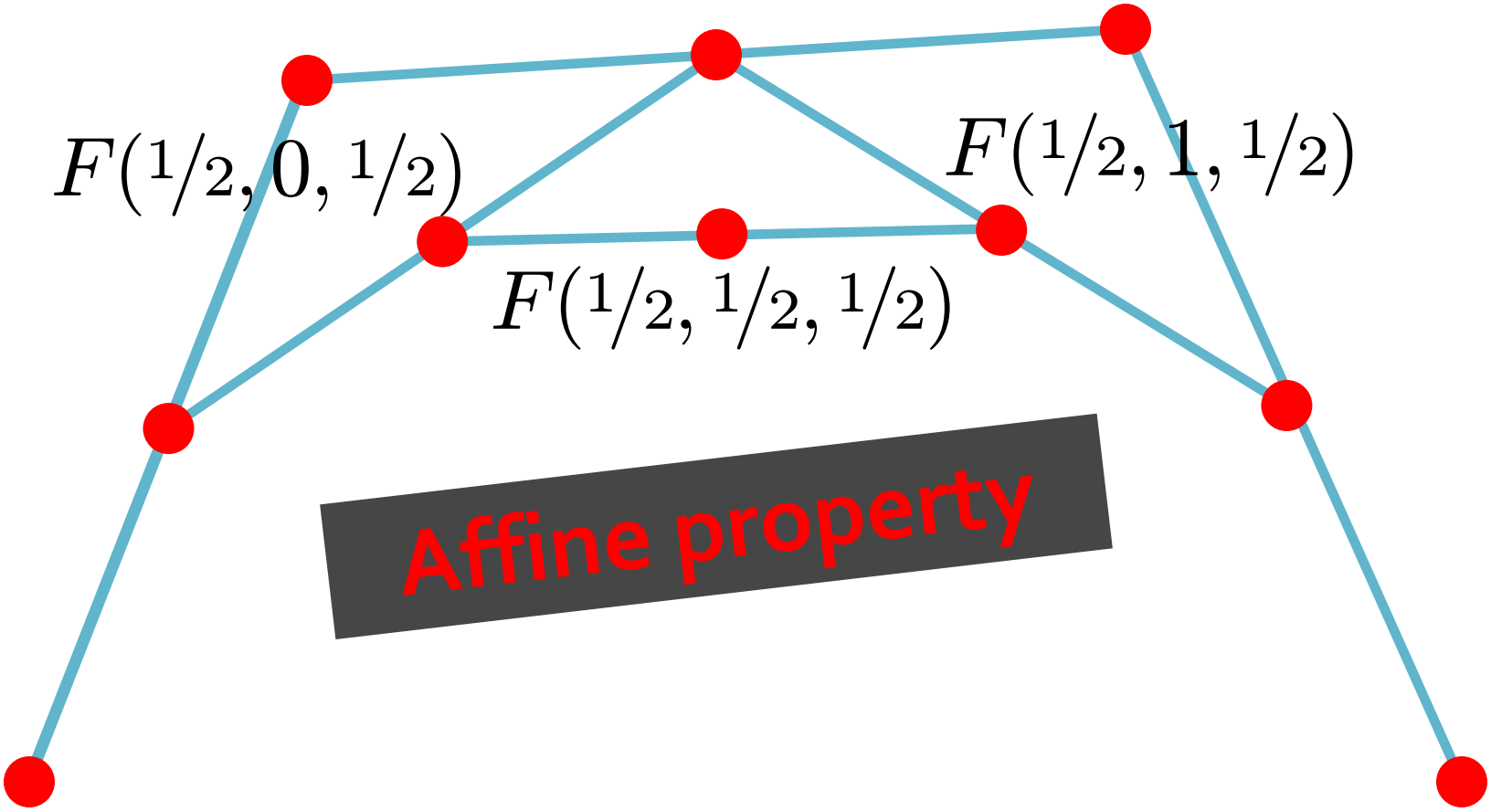
Cubic Control Polygon



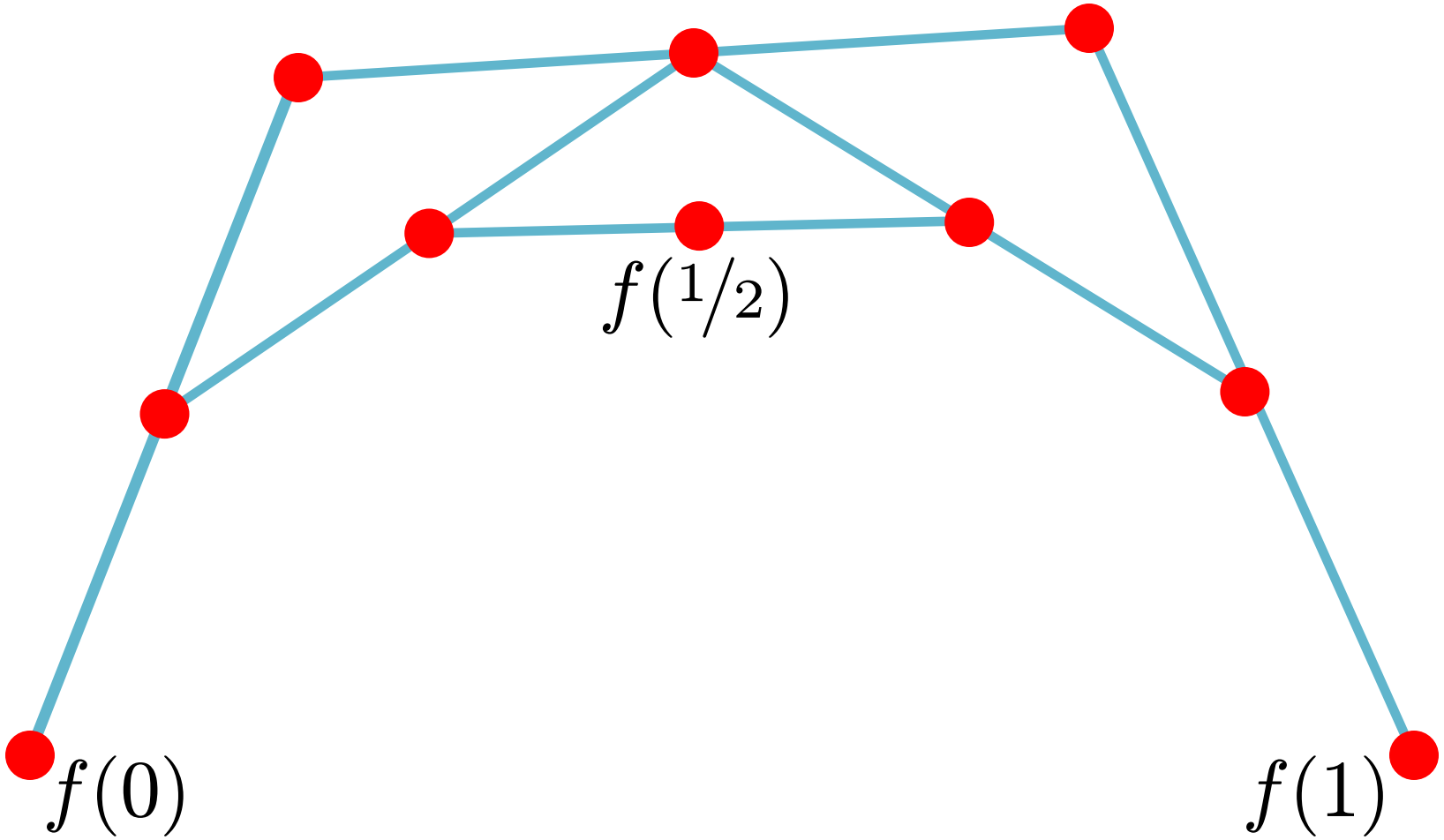
Cubic Control Polygon



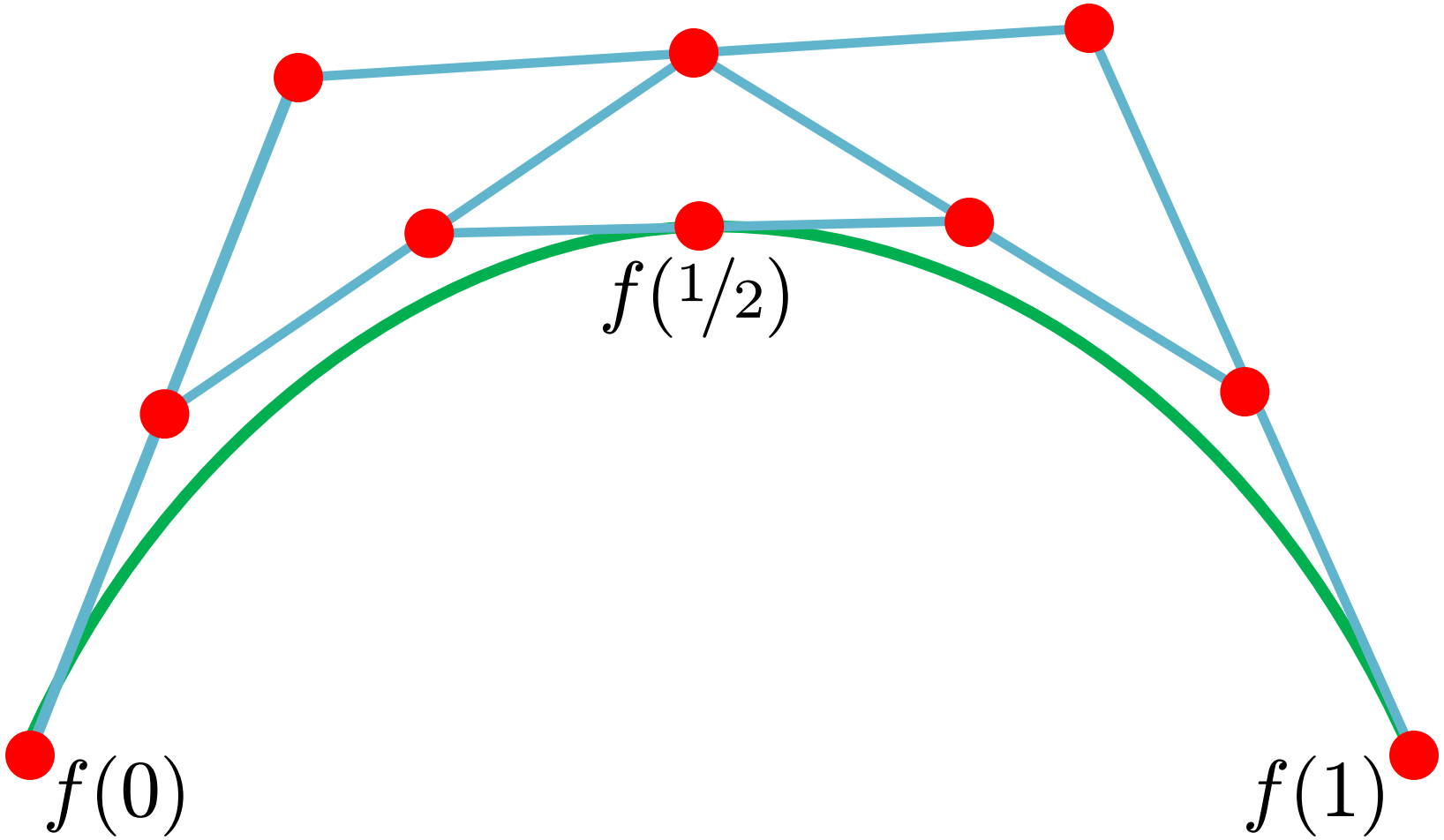
Cubic Control Polygon



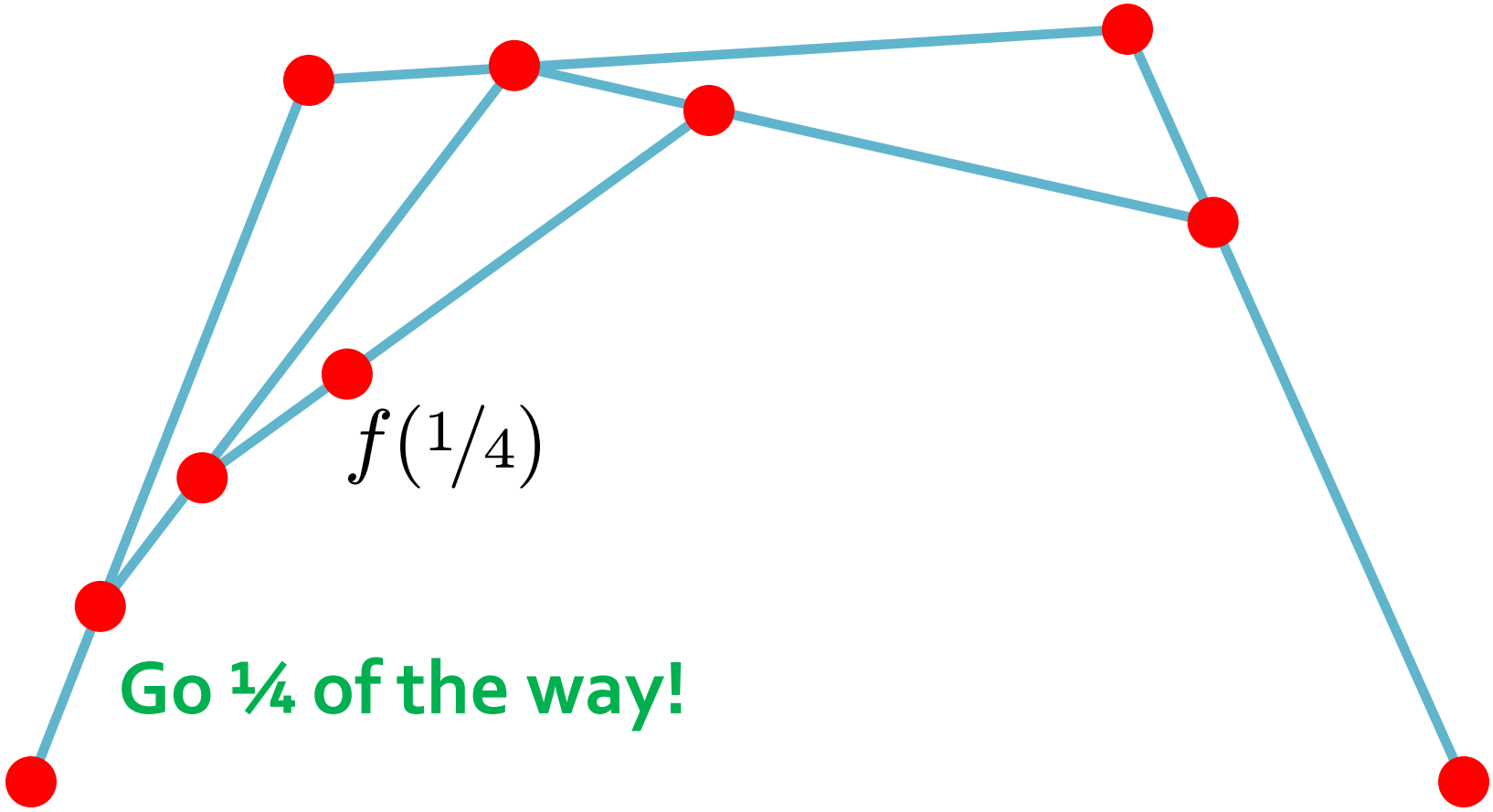
Cubic Control Polygon



Cubic Control Polygon



Cubic Control Polygon

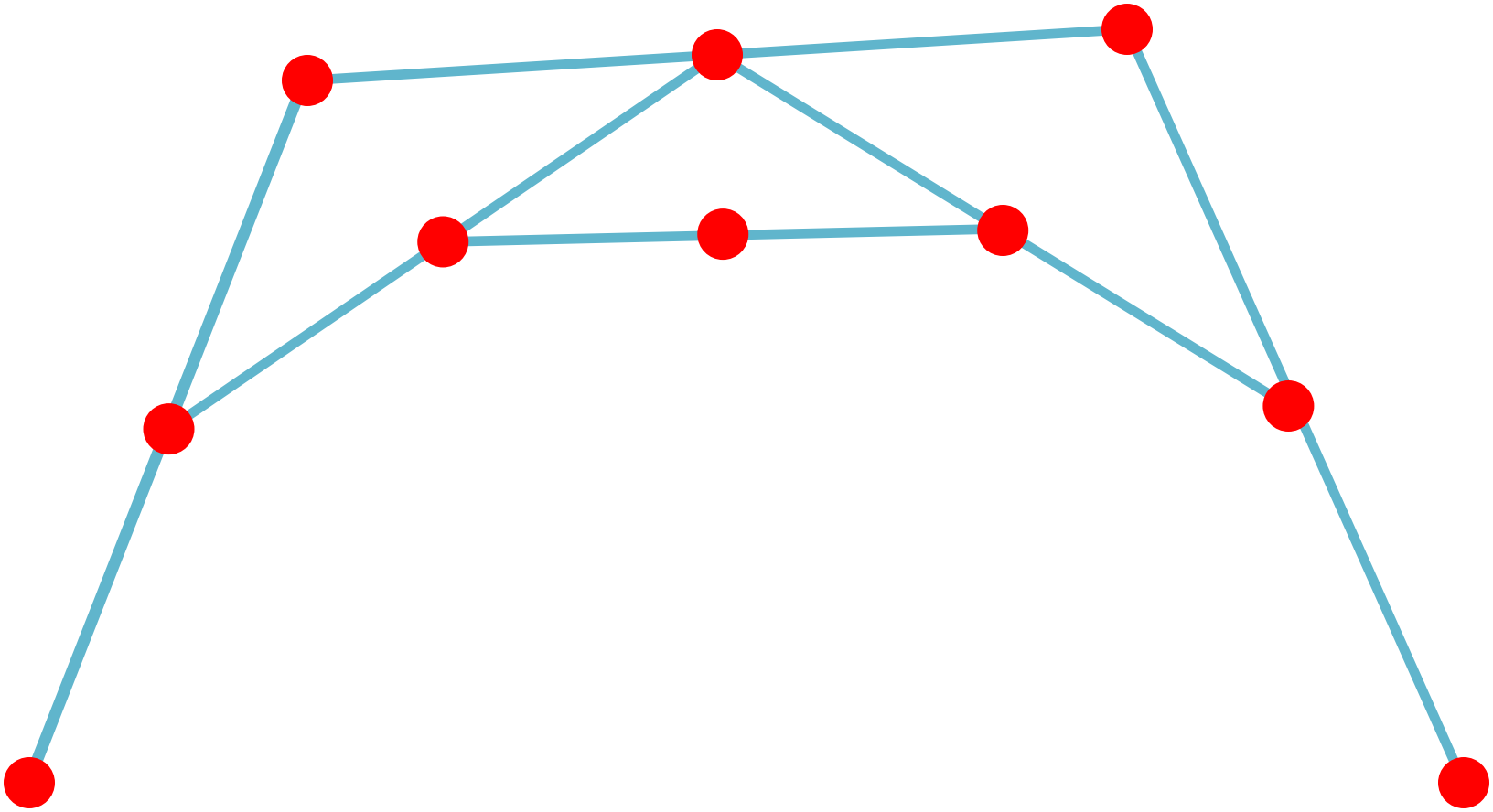


What Happened?

We can use the four control points to find any point $f(t)$ on the cubic curve.

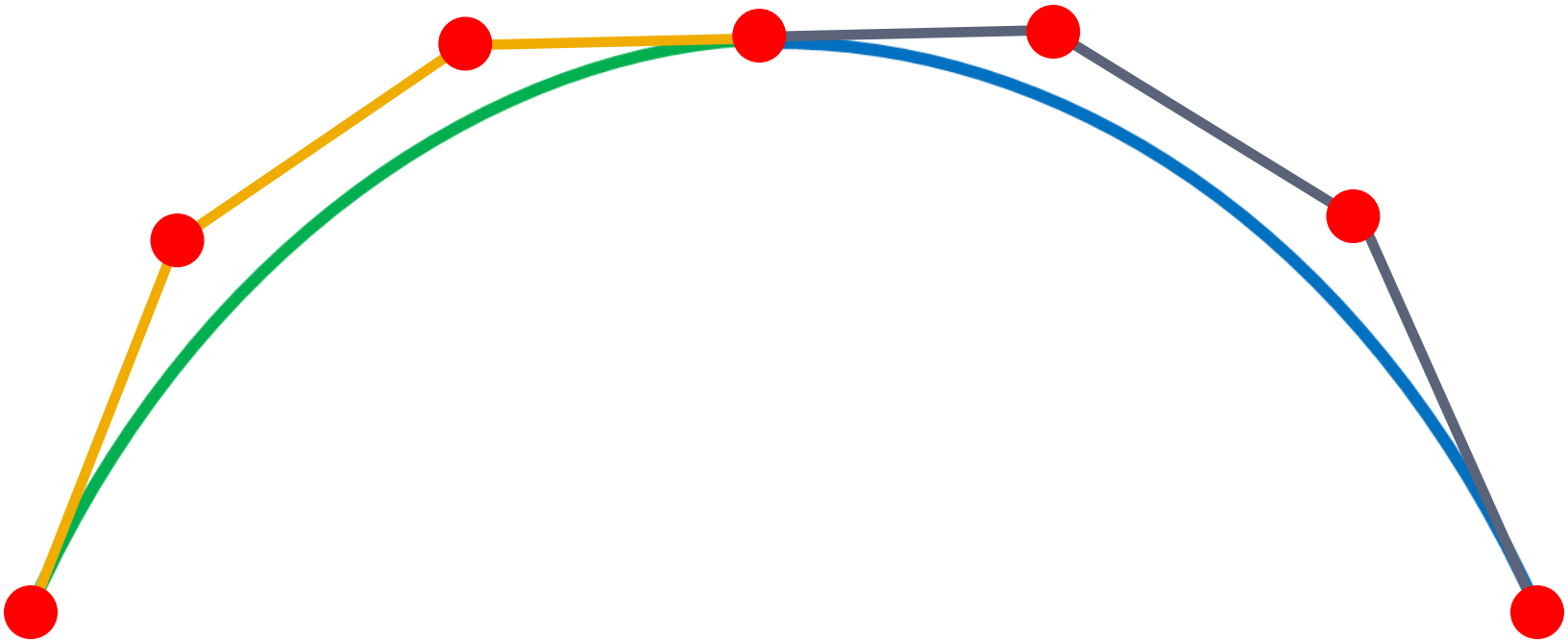
De Casteljau's Algorithm

Extra Observation



Structure when evaluating $t=½$

Extra Observation

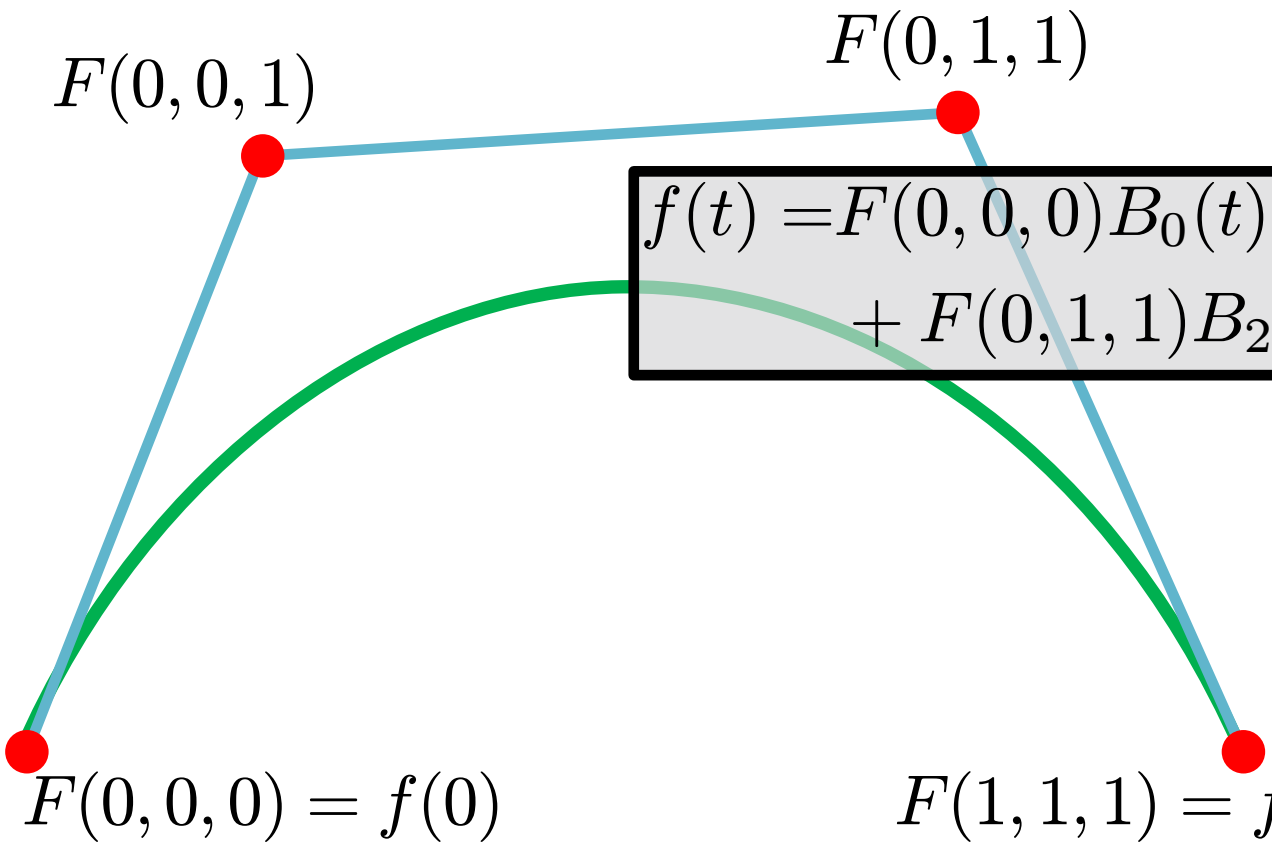


Two new control polygons!

Curve Subdivision

Recursively apply De Casteljau's algorithm to split control polygons into parts.

Bernstein Polynomials



$$f(t) = F(0,0,0)B_0(t) + F(0,0,1)B_1(t) + F(0,1,1)B_2(t) + F(1,1,1)B_3(t)$$

$$B_0(t) = (1-t)^3$$

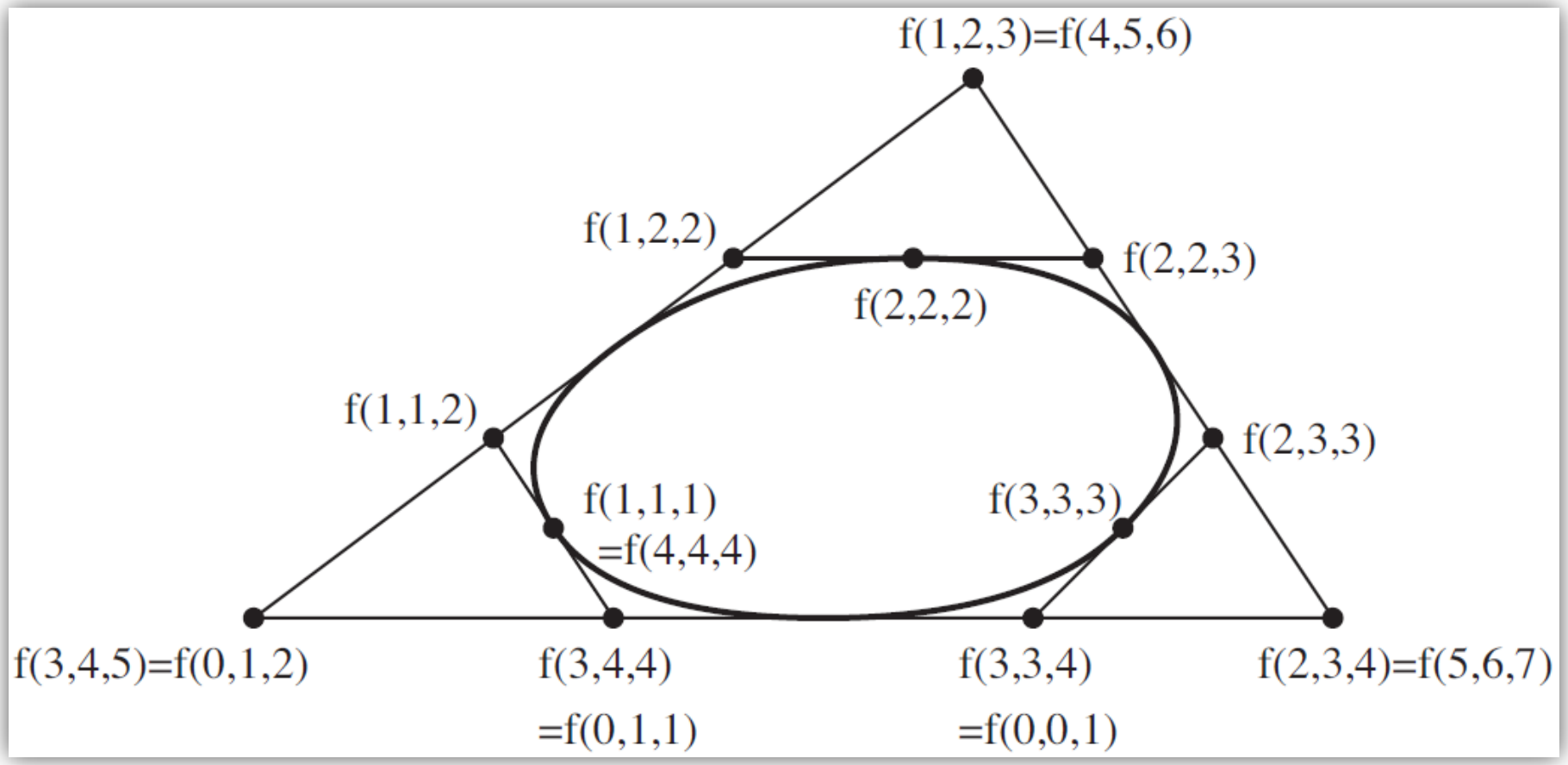
$$B_1(t) = 3t(1-t)^3$$

$$B_2(t) = 3t^2(1-t)$$

$$B_3(t) = t^3$$

Alternative cubic basis

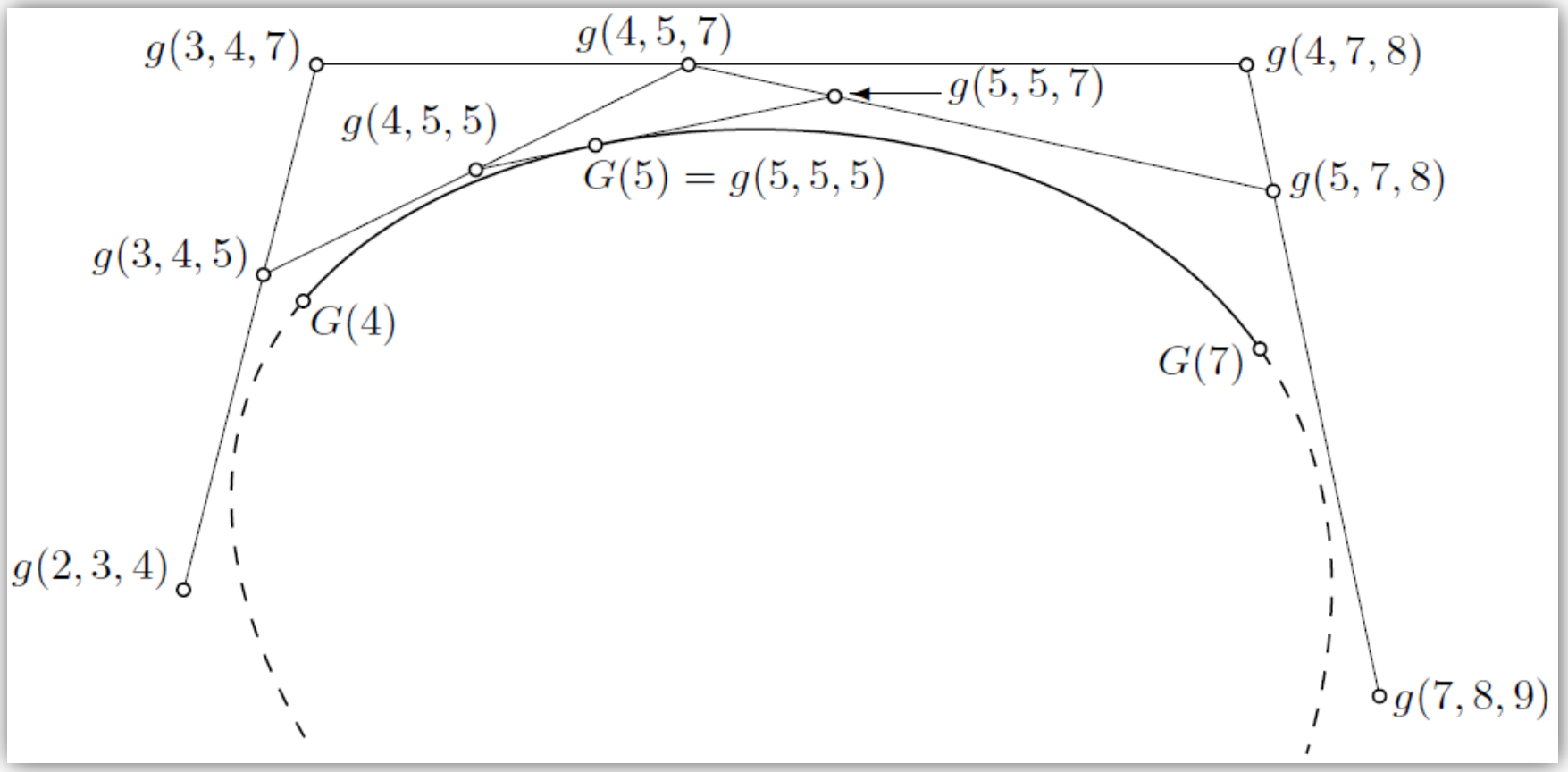
Putting Curves Together



http://graphics.stanford.edu/courses/cs348a-12-winter/Handouts/handout24.pdf

Make sure tangents match up

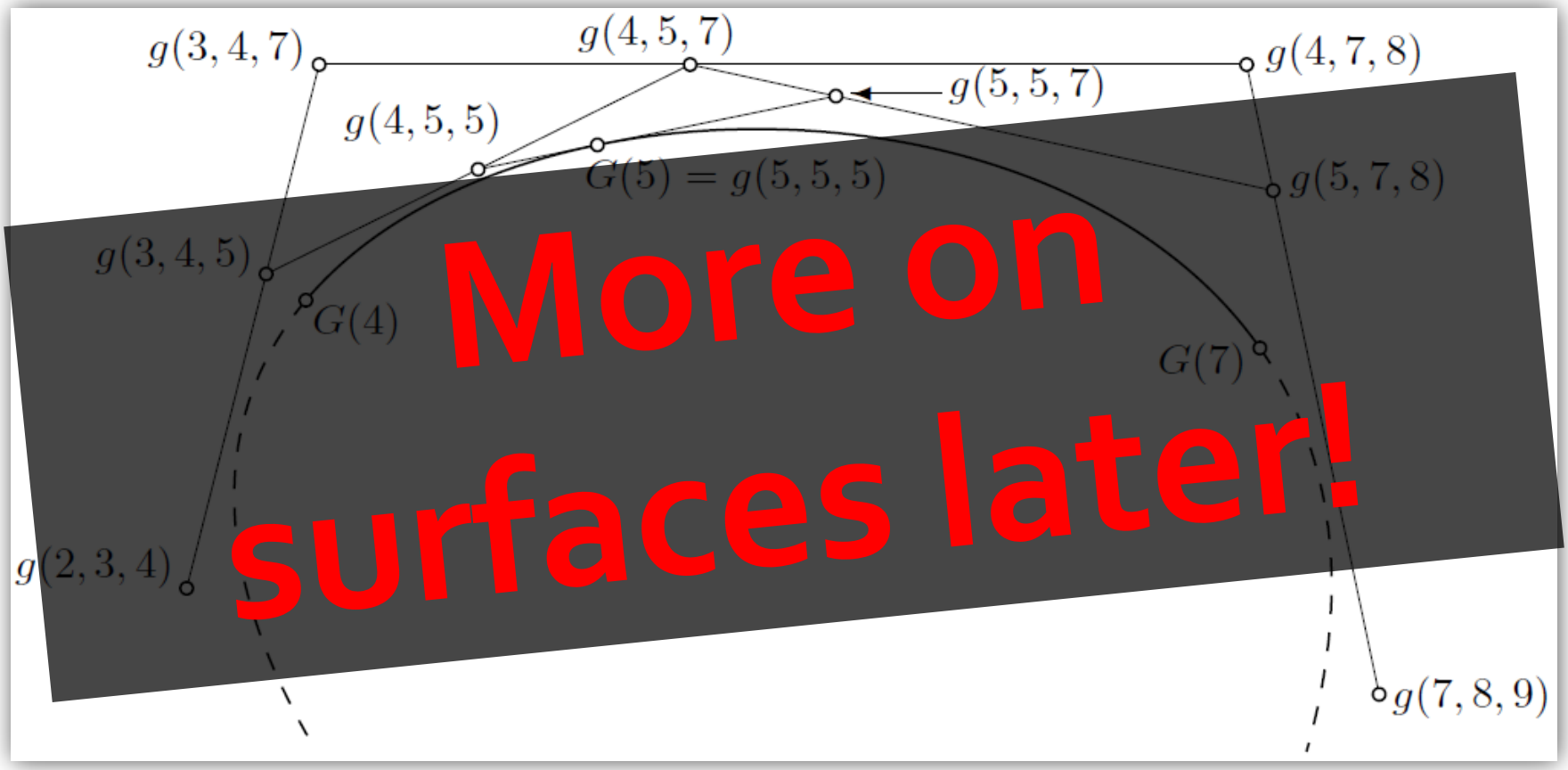
Putting Curves Together



http://graphics.stanford.edu/courses/cs348a-12-winter/Handouts/handout19.pdf

de Boor control points

Putting Curves Together



http://graphics.stanford.edu/courses/cs348a-12-winter/Handouts/handout19.pdf

de Boor control points

http://eikons.nl/wp-content/uploads/2012/02/car_studiolighting3.jpeg

Geometric Modeling

CS 148, Summer 2012
Introduction to Computer Graphics and Imaging

Justin Solomon