CS164: Differential Quantities on Surfaces

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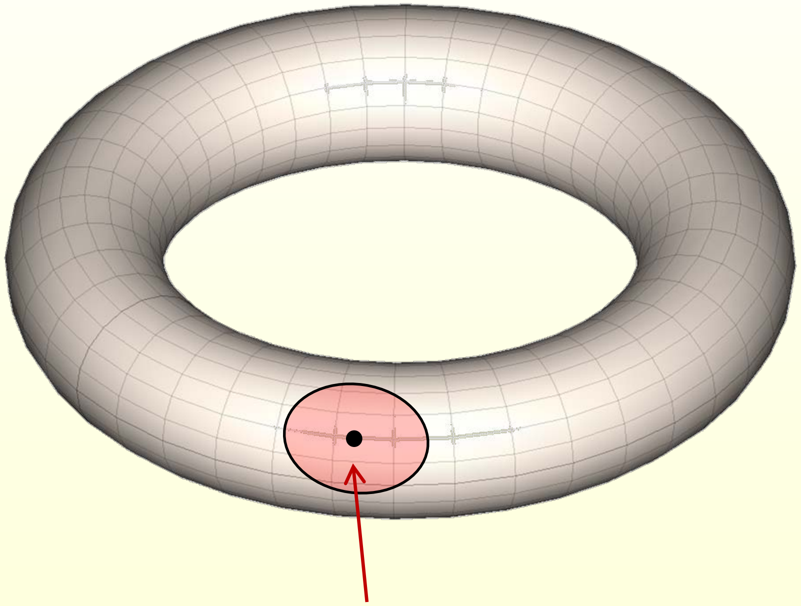
2-Manifolds and the Differential Geometry of Surfaces

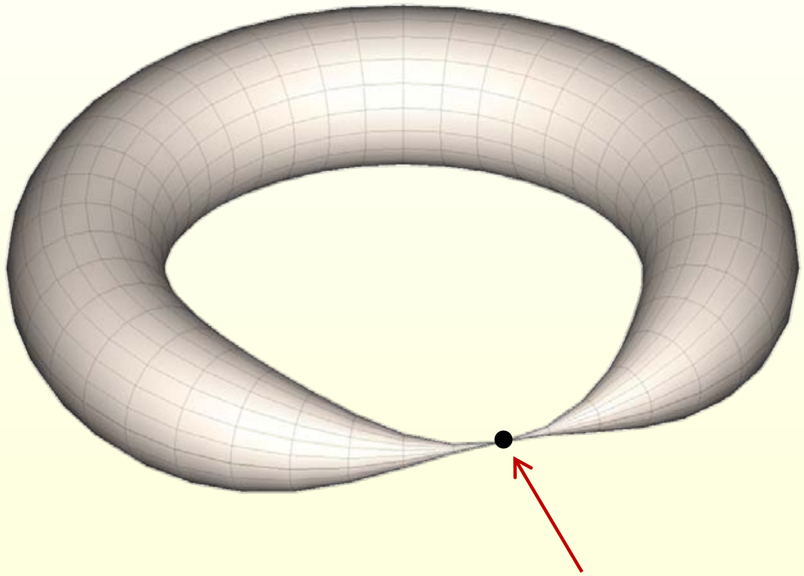
[Slides from Bronstein & Bronstein]

Manifolds

A topological space in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$ (topological disc) is called an *n*-dimensional (or *n*-) manifold



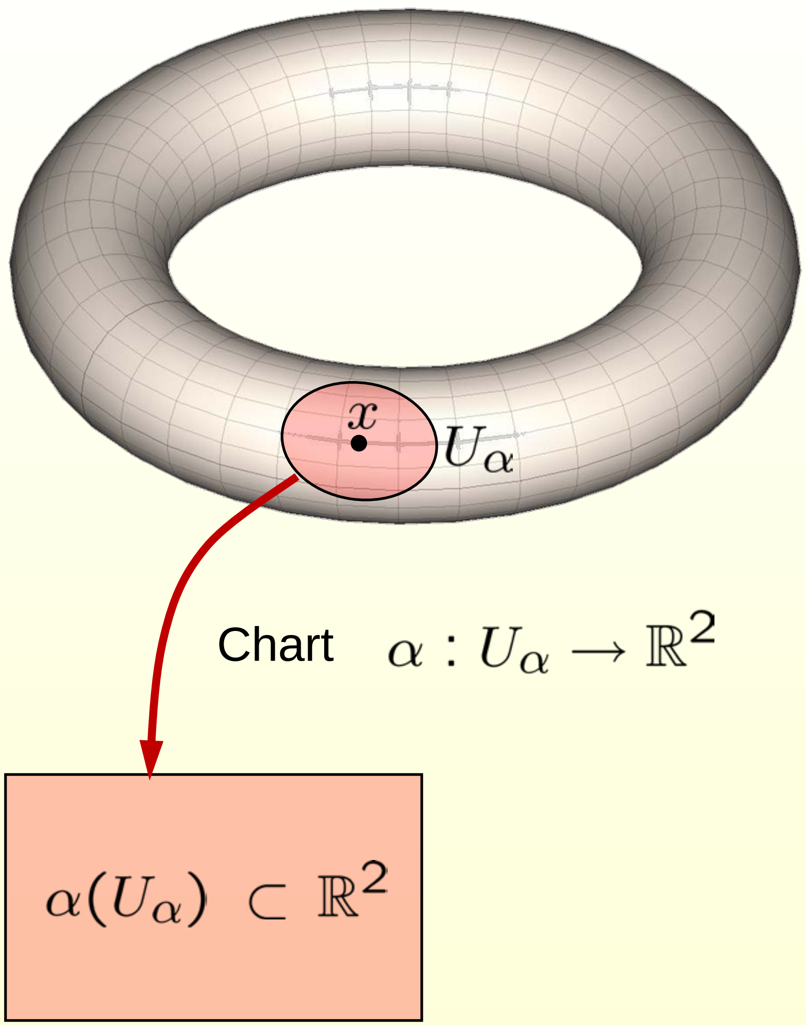
2-manifold



Not a manifold

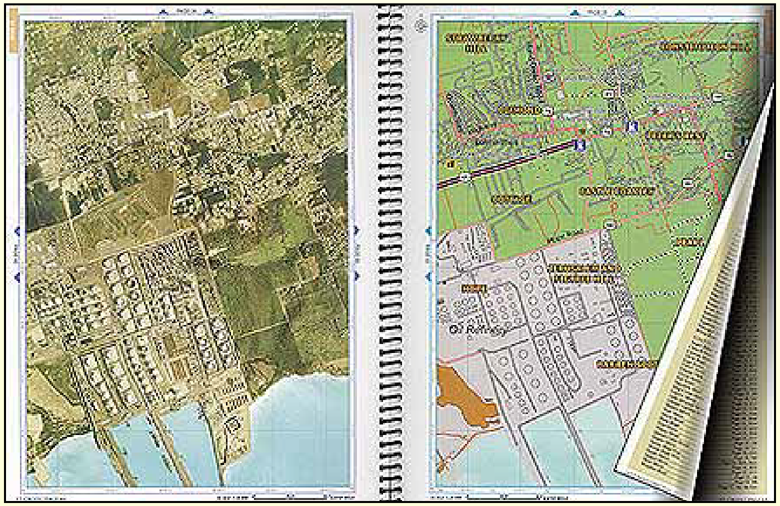
The earth is an example of a 2-manifold

Charts and Atlases

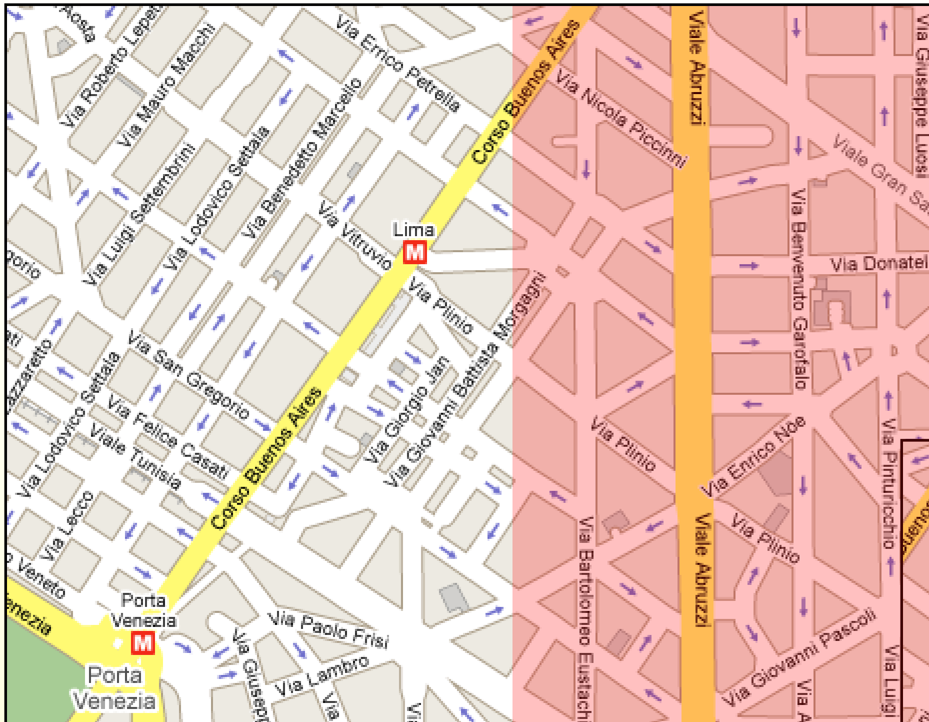


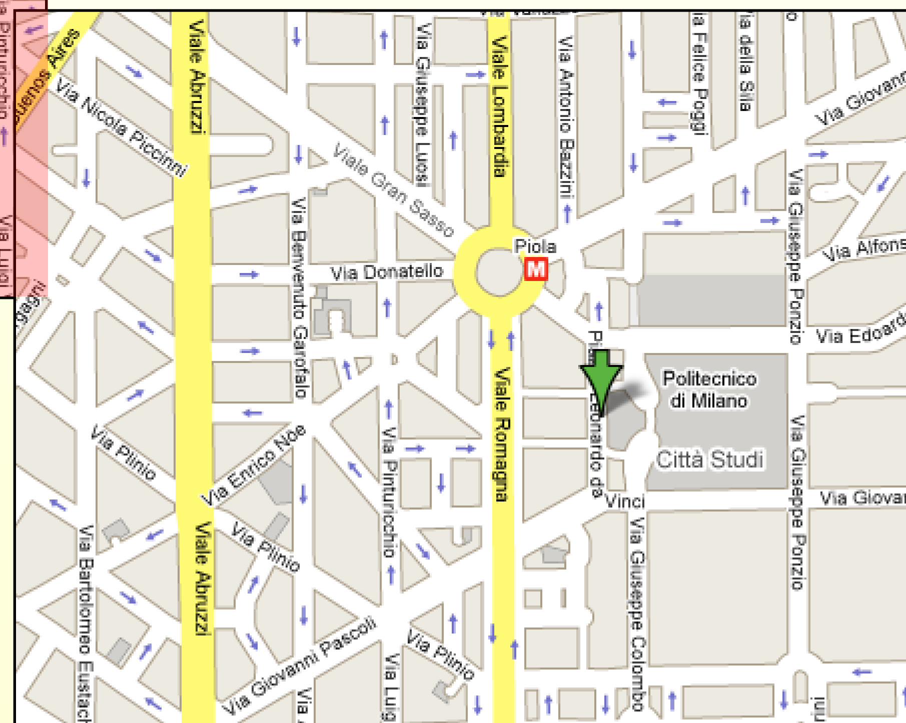
A homeomorphism $\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ from a neighborhood U_α of $x \in X$ to $\mathbb{R}^n$ is called a chart

A collection of charts whose domains cover the manifold is called an atlas

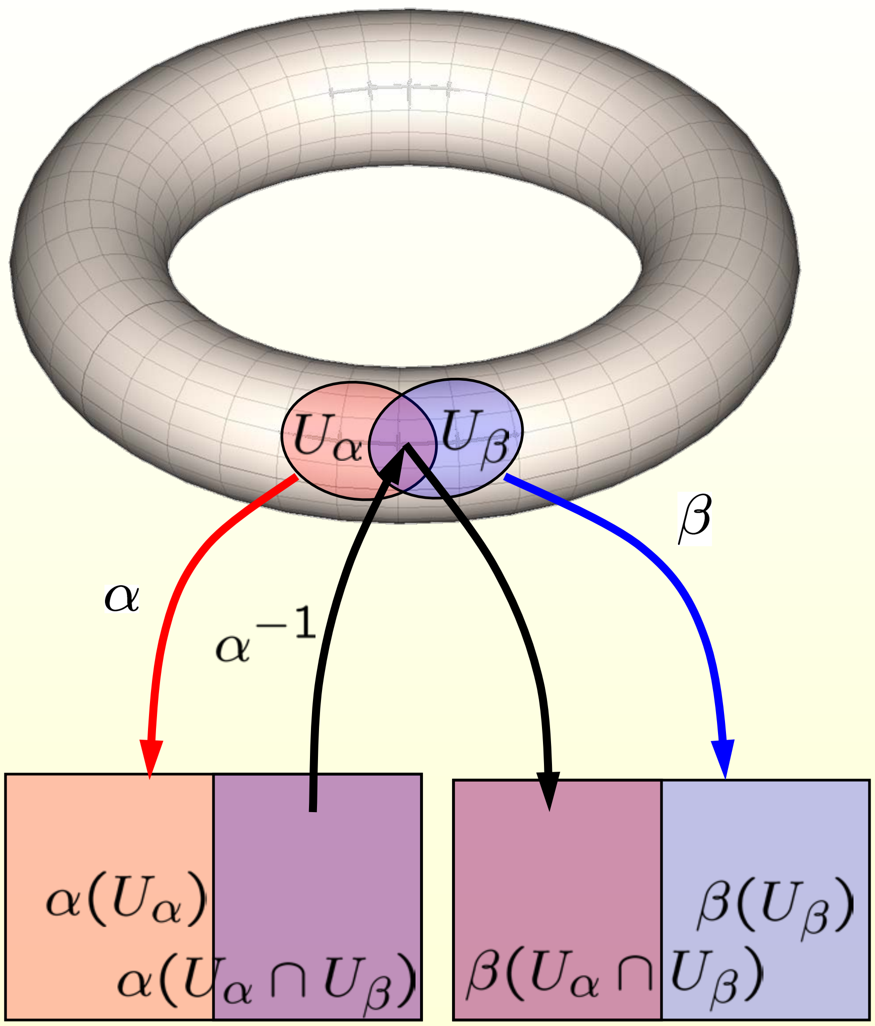


Charts and Atlases





Smooth Manifolds



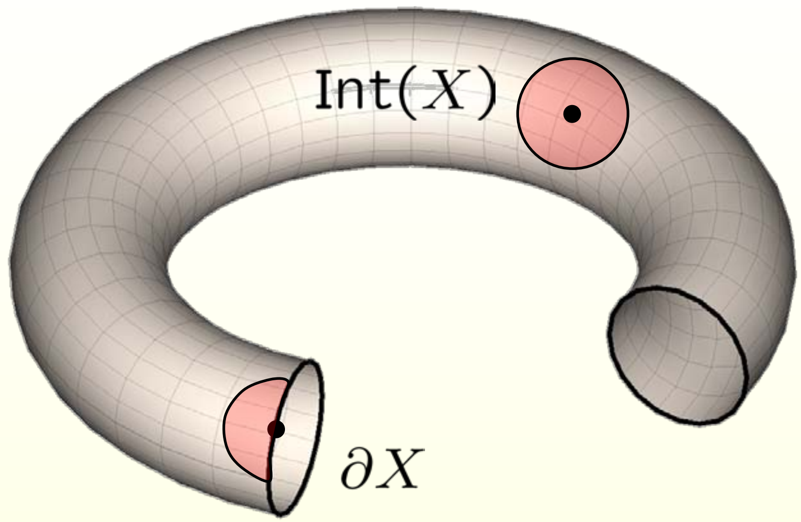
Given two charts $\alpha : U_\alpha \to \mathbb{R}^n$ and $\beta : U_\beta \to \mathbb{R}^n$ with overlapping domains, a change of coordinates is done by a transition function on $U_\alpha \cap U_\beta$

$$\beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^n$$

If all transition functions are $\mathcal{C}^r$, the manifold is said to be $\mathcal{C}^r$

A $\mathcal{C}^\infty$ manifold is called smooth

Manifolds with Boundary



A topological space in which every point has an open neighborhood homeomorphic to either

- topological disc $\mathbb{R}^n$; or
- topological half-disc $[0, \infty) \times \mathbb{R}^{n-1}$

is called a manifold with boundary

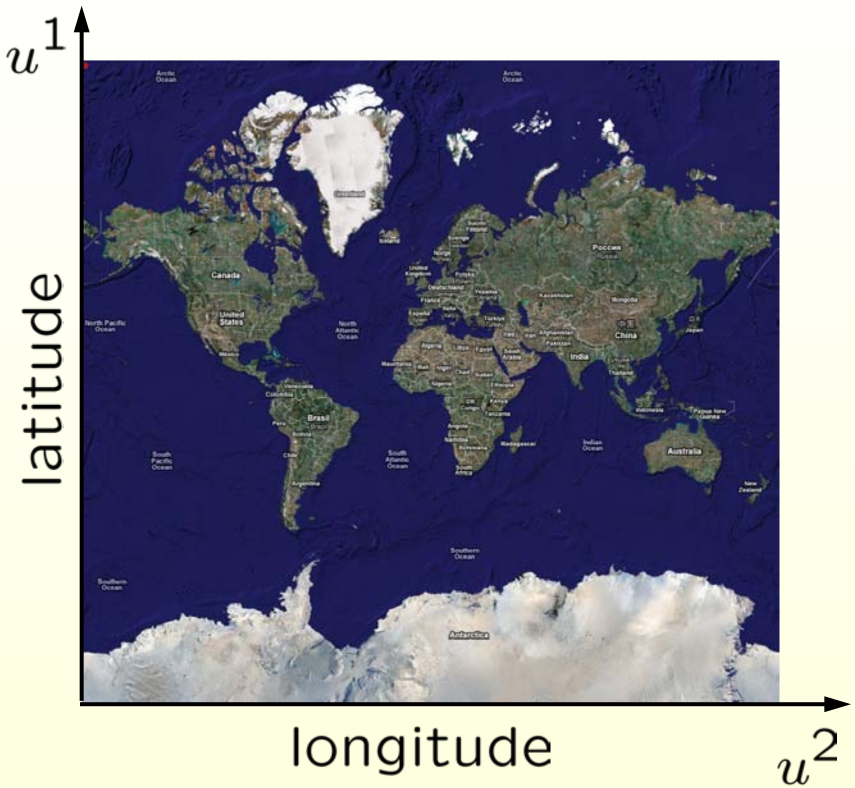
Points with disc-like neighborhood are called interior, denoted by $\mathrm{Int}(X)$

Points with half-disc-like neighborhood are called boundary, denoted by ∂X

Embedded Surfaces

- Boundaries of tangible physical objects are two-dimensional manifolds.
- They reside in (are embedded into, are subspaces of) the ambient three-dimensional Euclidean space.
- Such manifolds are called embedded surfaces (or simply surfaces).
- Can often be described by the map $x : U \subset \mathbb{R}^2 \rightarrow X \subset \mathbb{R}^3$
  - $U \subset \mathbb{R}^2$ is a parametrization domain.
  - the map $x(u) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$ is a global parametrization (embedding) of X.
- A smooth global parametrization does not always exist, nor is it easy to find.
- Sometimes it is more convenient to work with multiple charts.

Parametrization of the Earth



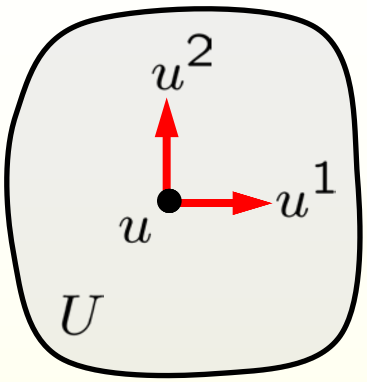
$$U = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-\pi, \pi]$$

$$\begin{aligned} x^1 &= r \cos u^2 \cos u^1 \\ x^2 &= r \sin u^2 \cos u^1 \\ x^3 &= r \sin u^1 \end{aligned}$$

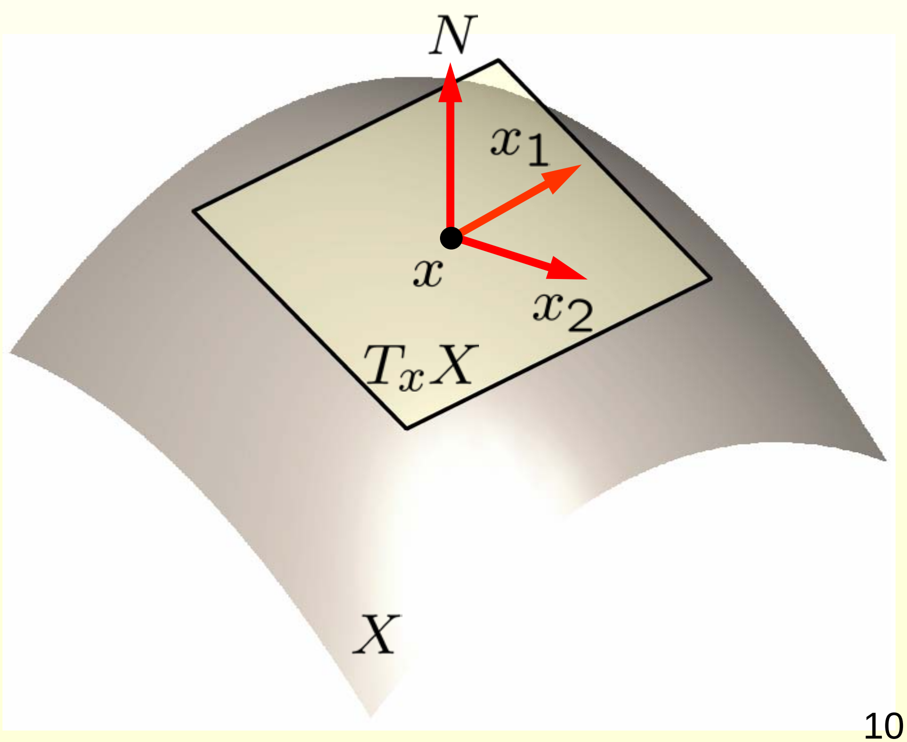
Tangent Plane and Normal

- At each point $u \in U$, we define a local system of coordinates

$$x_1 = \frac{\partial x}{\partial u^1} \qquad x_2 = \frac{\partial x}{\partial u^2}$$

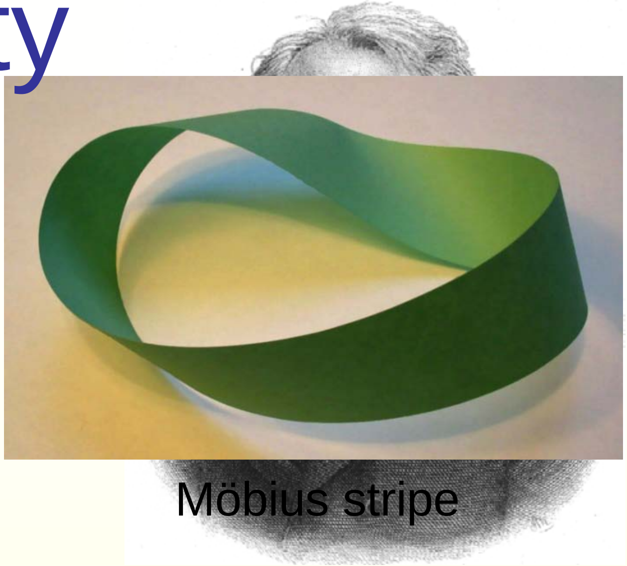


- A parametrization is regular if x_1 and x_2 are linearly independent.
- The plane $T_x X = \text{span}\{x_1, x_2\}$ is the tangent plane at $x = x(u)$.
- Provides a local Euclidean approximation of the surface.
- $N \perp T_x X$ is the normal to surface.



Orientability

- Normal is defined up to a sign.
- Partitions ambient space into inside and outside.
- A surface is orientable, if normal N depends smoothly on x.



Möbius stripe

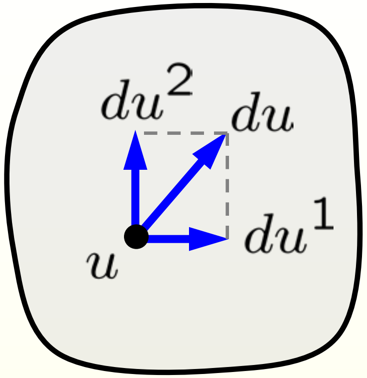
August Ferdinand Möbius

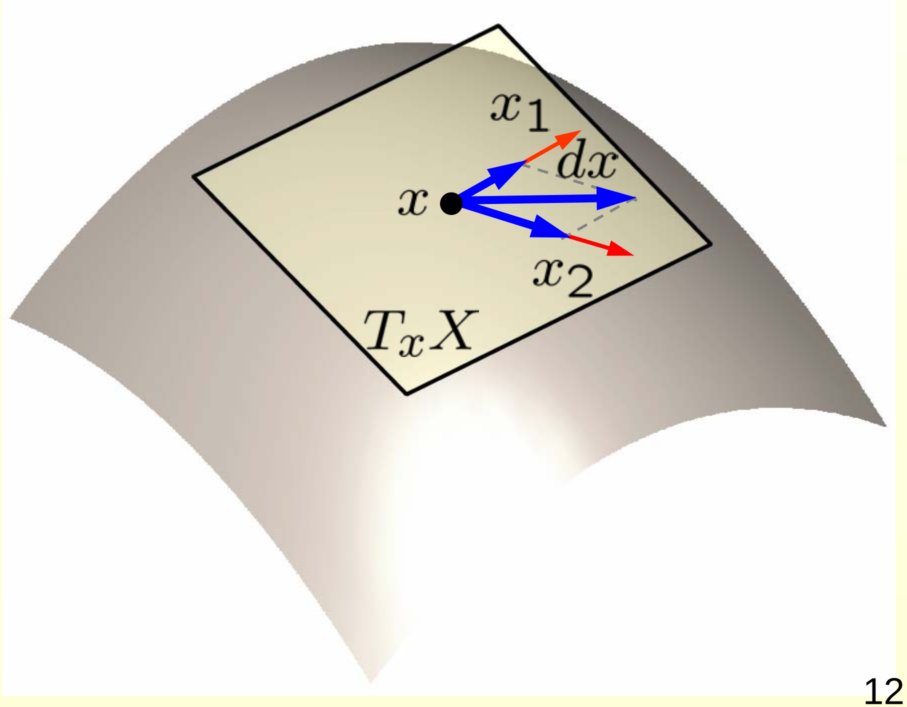
Felix Christian Klein
(1849-1925)

Klein bottle
(3D section)

First Fundamental Form

- Infinitesimal displacement on the chart du.
- Displaces x on the surface by
$$\begin{aligned} dx &= x(u+du) - x(u) \\ &= x_1 du^1 + x_2 du^2 \\ &= J du \end{aligned}$$
- J is the Jacobian matrix, whose columns are x_1 and x_2.

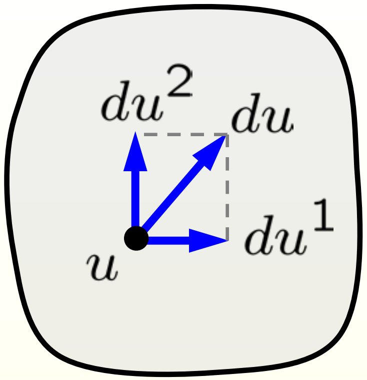




First Fundamental Form

- Length of the displacement

$$d\ell^2 = \|dx\|^2 = du^\top J^\top J du$$
$$= du^\top G du$$



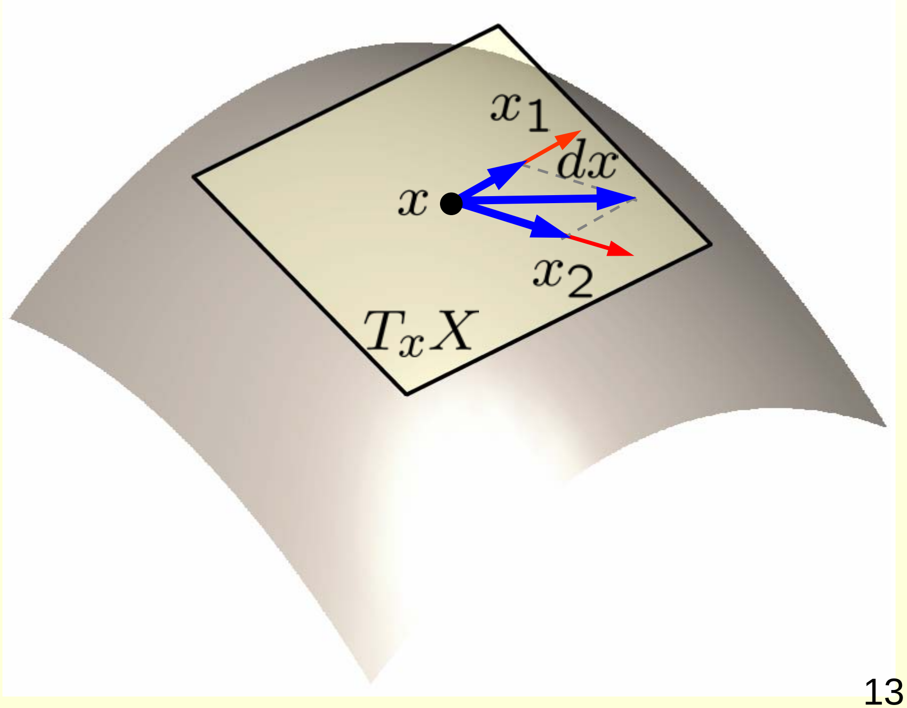
- G is a symmetric positive definite 2×2 matrix.
- Elements of G are inner products

$$g_{ij} = \langle x_i, x_j \rangle$$

- Quadratic form

$$d\ell^2 = du^\top G du$$

is the first fundamental form.



First Fundamental Form of the Earth

- Parametrization

$$x = (r \cos u^2 \cos u^1, r \sin u^2 \cos u^1, r \sin u^1)$$

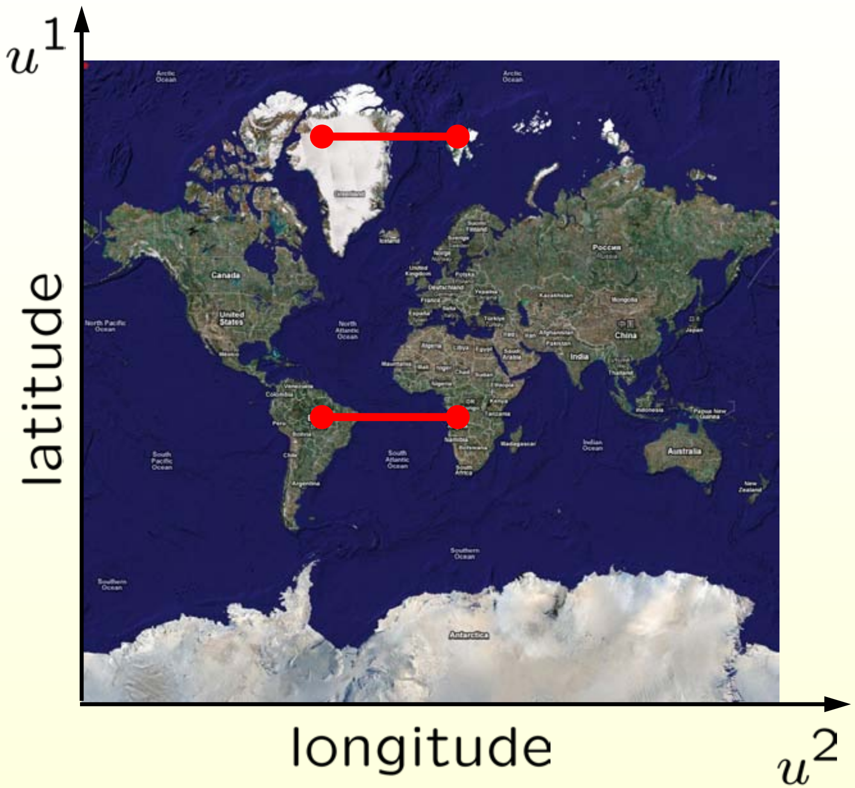
- Jacobian

$$\begin{aligned} x_1 &= (-r \cos u^2 \sin u^1, -r \sin u^2 \sin u^1, r \cos u^1) \\ x_2 &= (-r \sin u^2 \cos u^1, r \cos u^2 \cos u^1, 0) \end{aligned}$$

- First fundamental form

$$\begin{aligned} G &= \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle \\ \langle x_1, x_2 \rangle & \langle x_2, x_2 \rangle \end{pmatrix} \\ &= r \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix} \end{aligned}$$

First Fundamental Form of the Earth



$$G = r \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix}$$

First Fundamental Form

- Smooth curve on the chart:

$$\gamma : [a,b] \to U$$

- Its image on the surface:

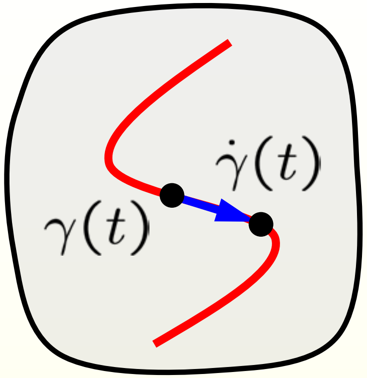
$$\Gamma = x \circ \gamma$$

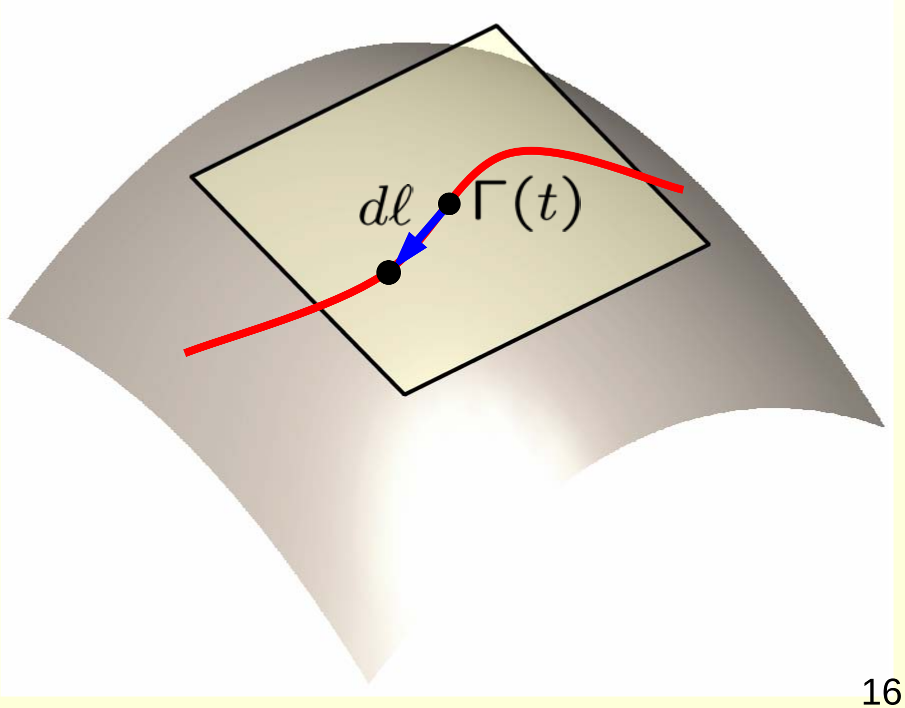
- Displacement on the curve: dt
- Displacement in the chart:

$$\begin{aligned} d\gamma &= \gamma(t+dt) - \gamma(t) \\ &= \dot{\gamma}(t)dt \end{aligned}$$

- Length of displacement on the surface:

$$d\ell = \sqrt{\dot{\gamma}(t)^{\mathsf{T}} G(\gamma(t)) \dot{\gamma}(t)}\, dt$$





Length, Intrinsic Geometry

- Length of the curve

$$L(\Gamma) = \int_\Gamma d\ell = \int_a^b \sqrt{\dot{\gamma}(t)^\top G(\gamma(t))\dot{\gamma}(t)}\, dt$$

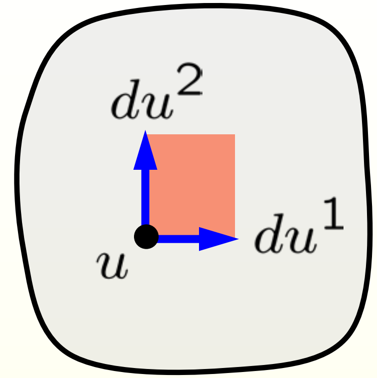
- First fundamental form induces a length metric (intrinsic metric)

$$d_X(x_1, x_2) = \min_{\substack{\Gamma \\ \Gamma(0)=x_1, \Gamma(1)=x_2}} L(\Gamma)$$

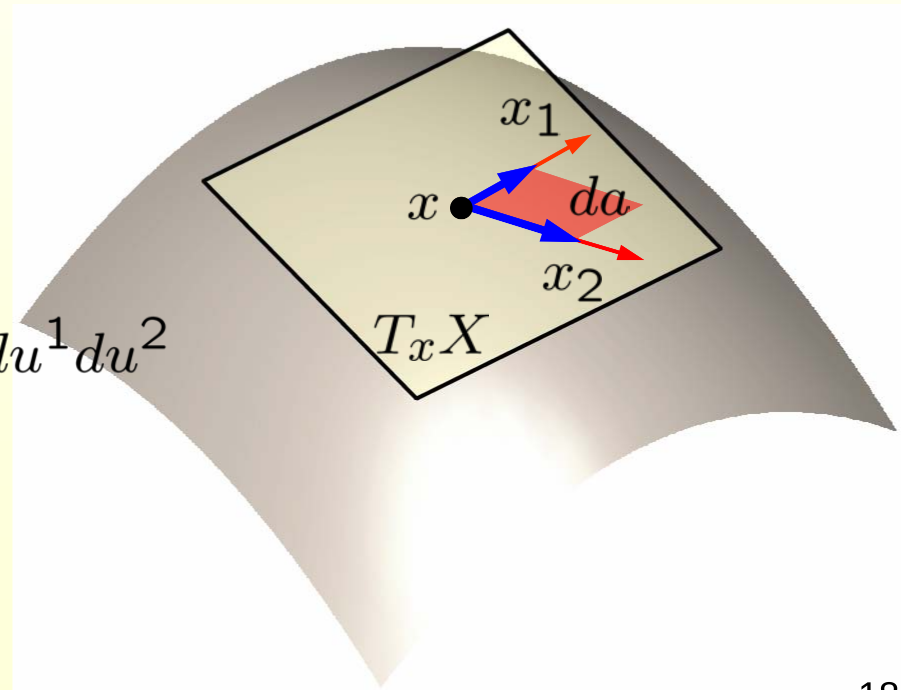
- Intrinsic geometry of the shape is completely described by the first fundamental form.
- First fundamental form is invariant to isometries.

Area

- Differential area element on the chart: rectangle $du^1 \times du^2$
- Copied by x to a parallelogram $du^1 x_1 \times du^2 x_2$ in tangent space.
- Differential area element on the surface:



$$
\begin{aligned}
da &= \|du^1 x_1 \times du^2 x_2\| \\
&= \|x_1 \times x_2\| du^1 du^2 \\
&= \sqrt{\|x_1\|^2 \|x_2\|^2 - \langle x_1, x_2 \rangle^2}\, du^1 du^2 \\
&= \sqrt{g_{11} g_{22} - g_{12}^2}\, du^1 du^2 \\
&= \sqrt{\det G}\, du^1 du^2
\end{aligned}
$$



Area

- Area or a region $\Omega \subseteq X$ charted as $\Omega = x(\omega \subseteq U)$

$$\mu(\Omega) = \int_{\Omega} da = \int_{\omega} \sqrt{\det G} du^1 du^2$$

- Relative area

$$\nu(\Omega) = \frac{\mu(\Omega)}{\mu(X)}$$

- Probability of a point on X picked at random (with uniform distribution) to fall into Ω .

Formally $\mu(\Omega), \nu(\Omega)$

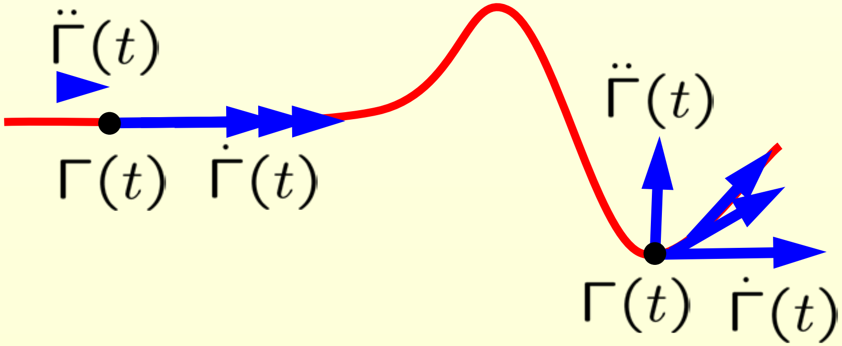
- are measures on X.

Curvature in the Plane

- Let $\Gamma : [0, L] \rightarrow \mathbb{R}^2$ be a smooth curve parameterized by arclength

$$\int_a^b \|\dot{\Gamma}(t)\| dt = |a - b|$$

- Γ trajectory of a race car driving at constant velocity.
- $\dot{\Gamma}$ velocity vector (rate of change of position), tangent to path.
- $\ddot{\Gamma}$ acceleration (curvature) vector, perpendicular to path.
- $\kappa = \|\ddot{\Gamma}\|_2$ curvature, measuring rate of rotation of velocity vector.

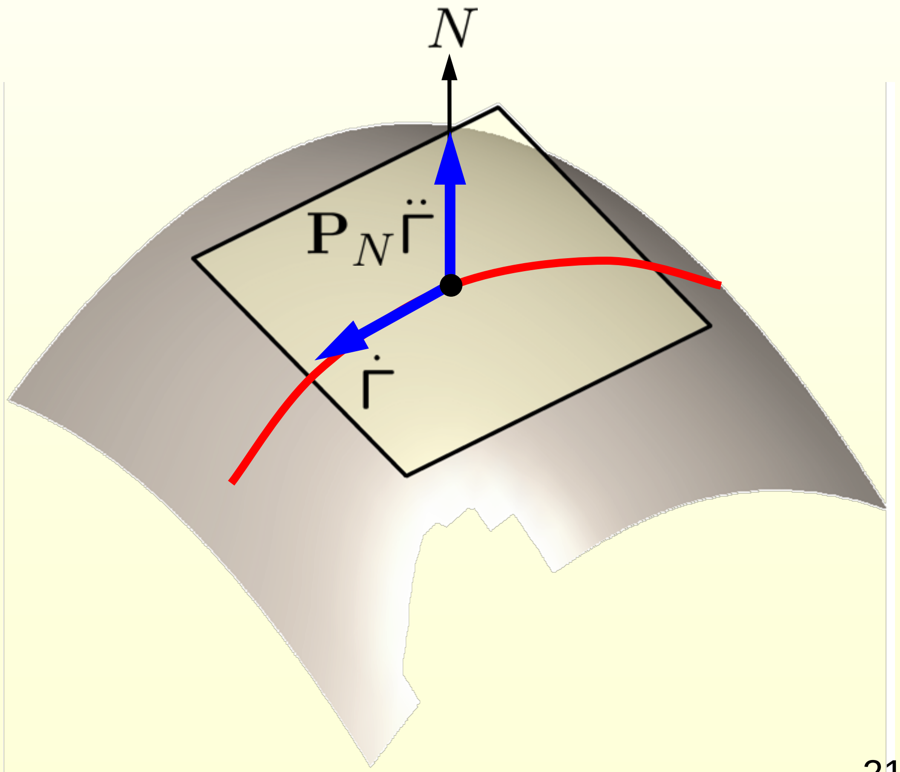


Curvature on the Surface

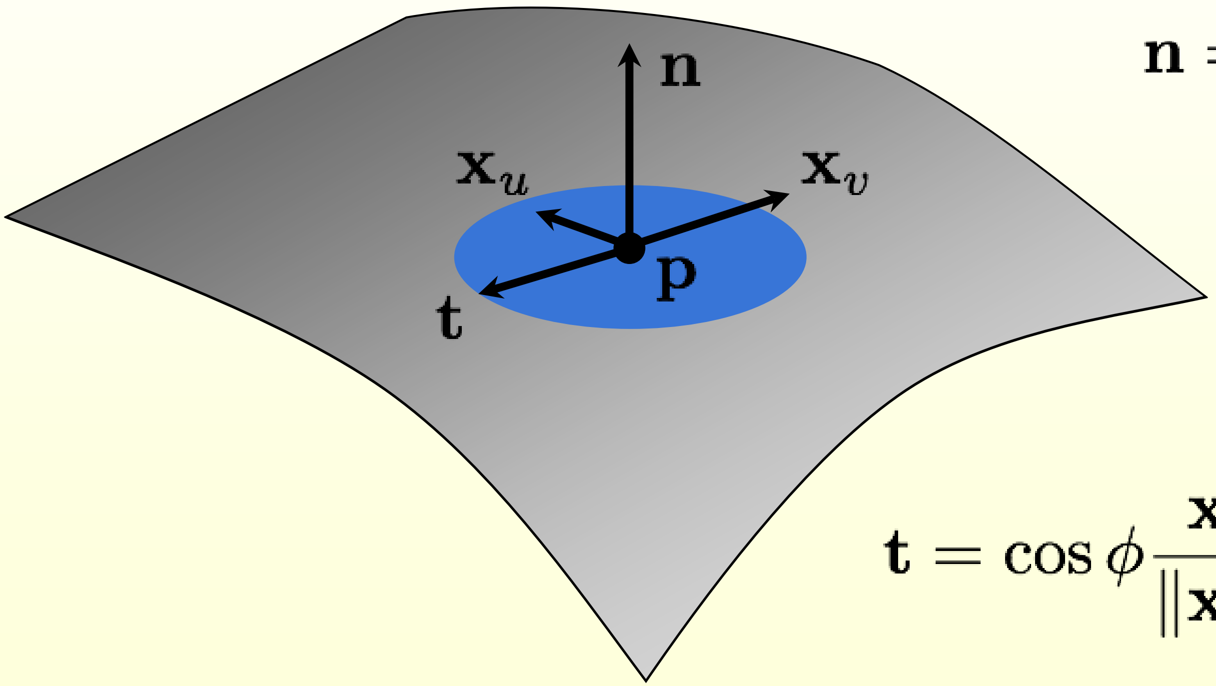
- Now the car drives on terrain X.
- Trajectory described by $\Gamma : [0, L] \to X$
- Curvature vector $\ddot{\Gamma}$ decomposes into
  - $\mathbf{P}_{T_\Gamma X} \ddot{\Gamma}$ geodesic curvature vector.
  - $\mathbf{P}_N \ddot{\Gamma}$ normal curvature vector.
- Normal curvature $\kappa_n = \langle N, \ddot{\Gamma} \rangle$
- Curves passing in different directions

  have different values of κ_n.

  Said differently:
- A point $x \in X$ has multiple curvatures!



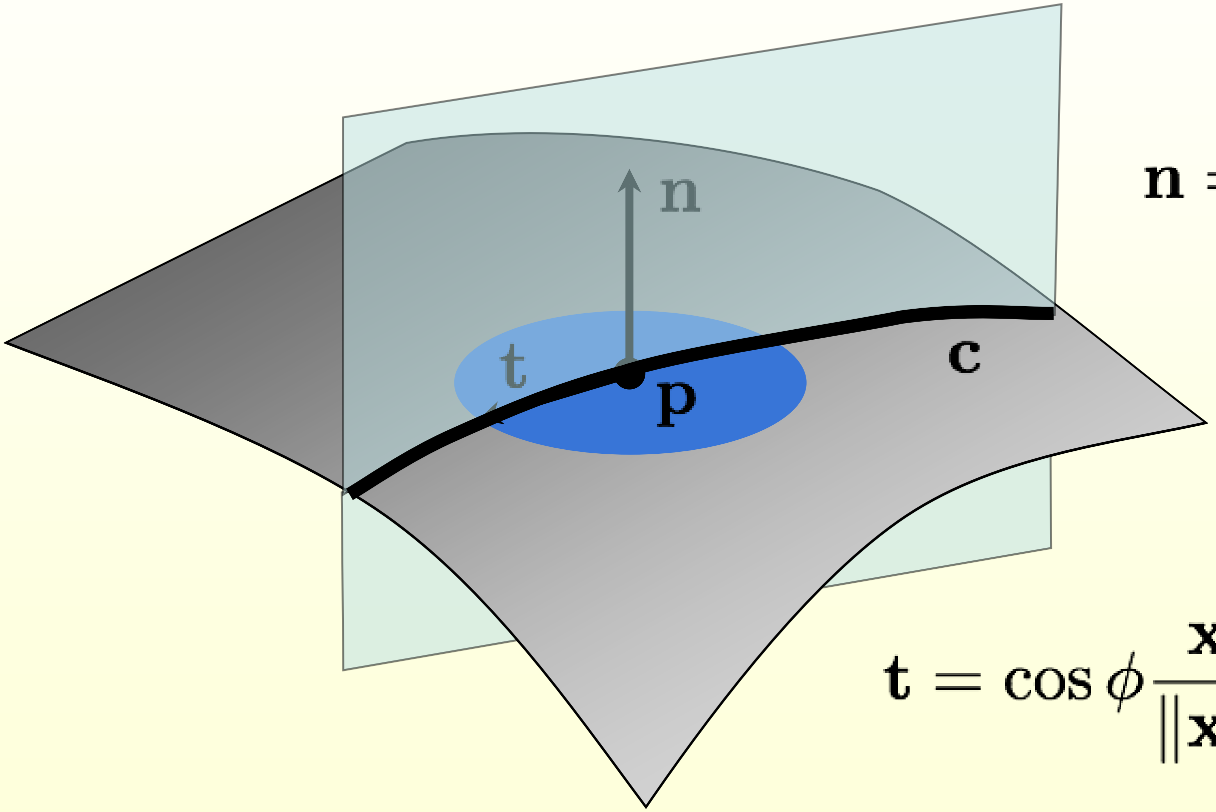
Normal Curvature



$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature



$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Principal Curvatures

- For each direction $v \in T_x X$, a curve Γ passing through $\Gamma(0) = x$ in the direction $\dot{\Gamma}(0) = v$ may have a different normal curvature κ_n.
- Principal curvatures

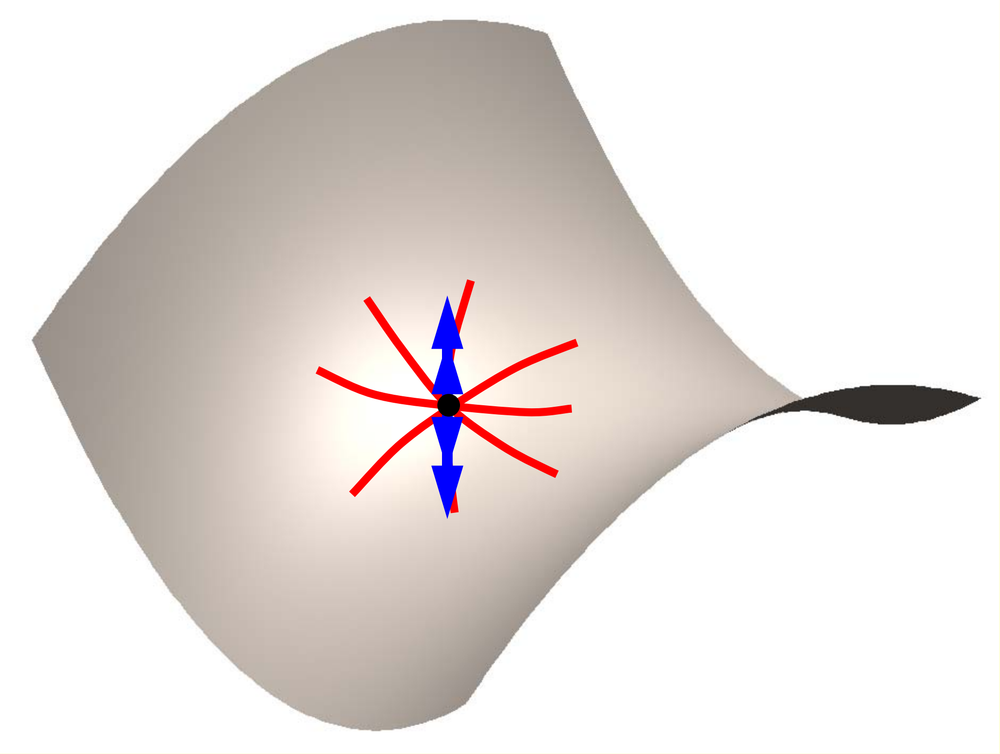
$$\kappa_1 = \min_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle$$

$$\kappa_2 = \max_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle$$

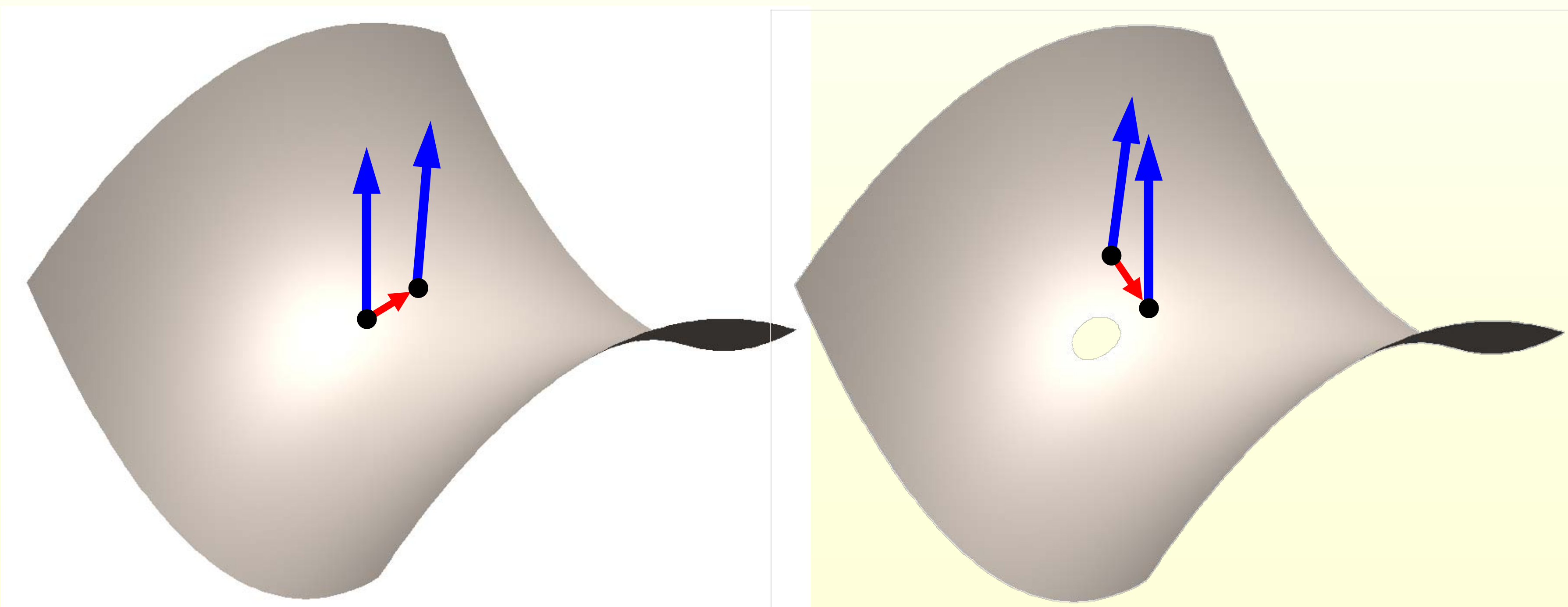
- Principal directions

$$T_1 = \arg \min_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle$$

$$T_2 = \arg \max_{v \in T_x X} \langle N, \ddot{\Gamma} \rangle$$



- Sign of normal curvature = direction of rotation of normal to surface.
  - $\kappa_n > 0$ a step in direction $\dot{\Gamma}$ rotates N in same direction.
  - $\kappa_n < 0$ a step in direction $\dot{\Gamma}$ rotates N in opposite direction.



Curvature: A Different View

- A plane has a constant normal vector, e.g. $N = (0, 0, 1)$
- We want to quantify how a curved surface is different from a plane.
- Rate of change of N i.e., how fast the normal rotates.
- Directional derivative of N at point $x \in X$ in the direction $v \in T_x X$

$$D_v N = \lim_{t \to 0} \frac{1}{t}(N(\Gamma(t)) - N(x)) = \frac{d}{dt} N(\Gamma(t))\Big|_{t=0}$$

$\Gamma : (-\epsilon, +\epsilon) \to X$ is an arbitrary smooth curve with $\Gamma(0) = x$
and $\dot{\Gamma}(0) = v$.

Curvature

- $D_v N$ is a vector in $\mathbb{R}^3$ measuring the change in N as we make differential steps in the direction v.
- Differentiate $1 = \langle N, N \rangle$ w.r.t. t

$$0 = \frac{d}{dt}\langle N, N \rangle = 2\langle D_v N, N \rangle$$

- Hence $D_v N \perp N$ or $D_v N \in T_x X$.
- Shape operator (a.k.a. Weingarten map): is the map $S : T_x X \to T_x X$ defined by

$$S(v) = -D_v N$$

Julius Weingarten
(1836-1910)

Shape Operator

- Can be expressed in parametrization coordinates as $S(v) = Sv$

  S is a 2×2 matrix satisfying

$$\begin{pmatrix} S(x_1) \\ S(x_2) \end{pmatrix} = S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Multiply by (x_1, x_2)

$$\begin{pmatrix} S(x_1) \\ S(x_2) \end{pmatrix} (x_1, x_2) = S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (x_1, x_2)$$

where

$$B = SG$$

$$B = \begin{pmatrix} \langle S(x_1), x_1 \rangle & \langle S(x_1), x_2 \rangle \\ \langle S(x_2), x_1 \rangle & \langle S(x_2), x_2 \rangle \end{pmatrix} = - \begin{pmatrix} \langle \partial_{u^1} N, x_1 \rangle & \langle \partial_{u^1} N, x_2 \rangle \\ \langle \partial_{u^2} N, x_1 \rangle & \langle \partial_{u^2} N, x_2 \rangle \end{pmatrix}$$

Shape Operator

- The shape operator is a 2 by 2 matrix defined at each point of *S*. At each point x it is a linear map from T_x to T_x.
- It defines the locally best quadratic approximation to the surface *S* at the point *x*, parametrized over the tangent plane T_x .
- Its eigenvalues are the principal curvatures, and its eigenvectors the principal curvature directions at *x*.

Second Fundamental Form

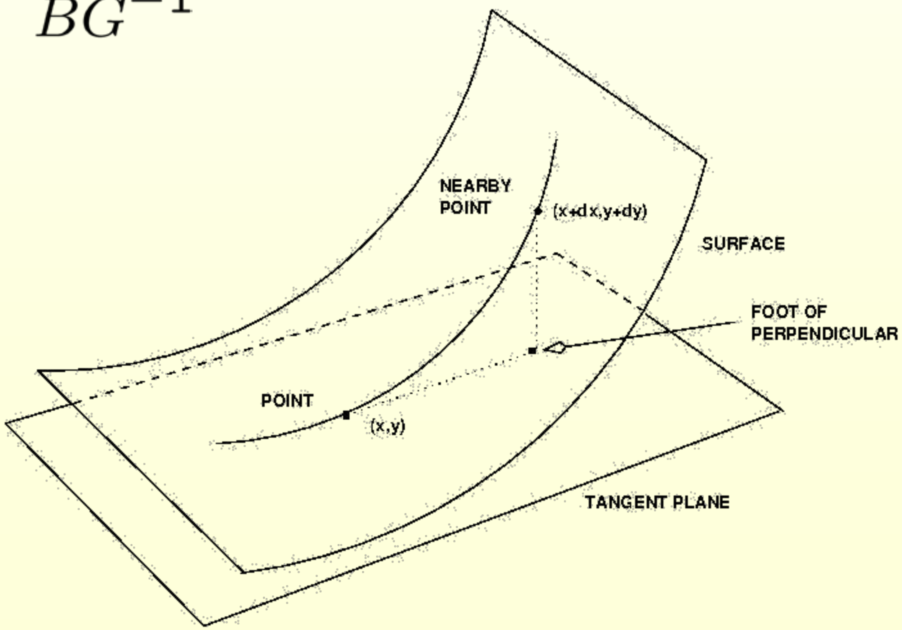
- The matrix B gives rise to the quadratic form

$$B(v, w) \;=\; \langle S(v), w \rangle \;=\; w^{\top} B v$$

  called the second fundamental form.

- Related to shape operator and first fundamental form by identity

$$S \;=\; B G^{-1}$$



Principal Curvatures Encore

- Let $\Gamma : [0, L] \to X$ be a curve on the surface.
- Since $\dot{\Gamma} \in T_x X$, $\langle \dot{\Gamma}, N \rangle = 0$.
- Differentiate w.r.t. to t

$$0 = \frac{d}{dt}\langle \dot{\Gamma}, N \rangle = \langle \ddot{\Gamma}, N \rangle + \langle \dot{\Gamma}, \frac{d}{dt} N \rangle$$

$$\kappa_n = \langle \ddot{\Gamma}, N \rangle = \langle \dot{\Gamma}, -D_{\dot{\Gamma}} N \rangle = B(\dot{\Gamma}, \dot{\Gamma}) = \dot{\Gamma}^\top B \dot{\Gamma}$$

- $\kappa_1 \leq \dot{\Gamma}^\top B \dot{\Gamma} \leq \kappa_2$
- κ_1 is the smallest eigenvalue of *S*.
- κ_2 is the largest eigenvalue of *S*.
- T_1, T_2 are the corresponding eigenvectors.

Second Fundamental Form of the Earth

- Parametrization $x = (r \cos u^2 \cos u^1, r \sin u^2 \cos u^1, r \sin u^1)$
- Normal

$$
\begin{aligned}
N &= (\cos u^2 \cos u^1, \sin u^2 \cos u^1, \sin u^1) \\
\partial_{u^1} N &= (-\cos u^2 \sin u^1, -\sin u^2 \sin u^1, \cos u^1) \\
\partial_{u^2} N &= (-\sin u^2 \cos u^1, \cos u^2 \cos u^1, 0)
\end{aligned}
$$

- Second fundamental form

$$
B = -\begin{pmatrix} \langle \partial_{u^1} N, x_1 \rangle & \langle \partial_{u^1} N, x_2 \rangle \\ \langle \partial_{u^2} N, x_1 \rangle & \langle \partial_{u^2} N, x_2 \rangle \end{pmatrix} = -\frac{1}{r} G = \begin{pmatrix} -1 & 0 \\ 0 & -\cos^2 u^1 \end{pmatrix}
$$

Shape Operator of the Earth

- First fundamental form

$$G = r\begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix}$$

- Second fundamental form

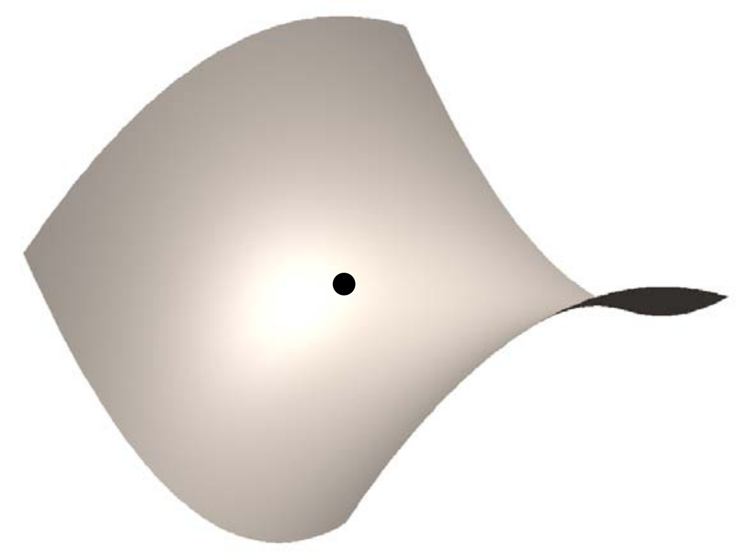
$$B = \begin{pmatrix} -1 & 0 \\ 0 & -\cos^2 u^1 \end{pmatrix}$$

- Shape operator $S = BG^{-1} = -\frac{1}{r}I$
- Constant at every point.

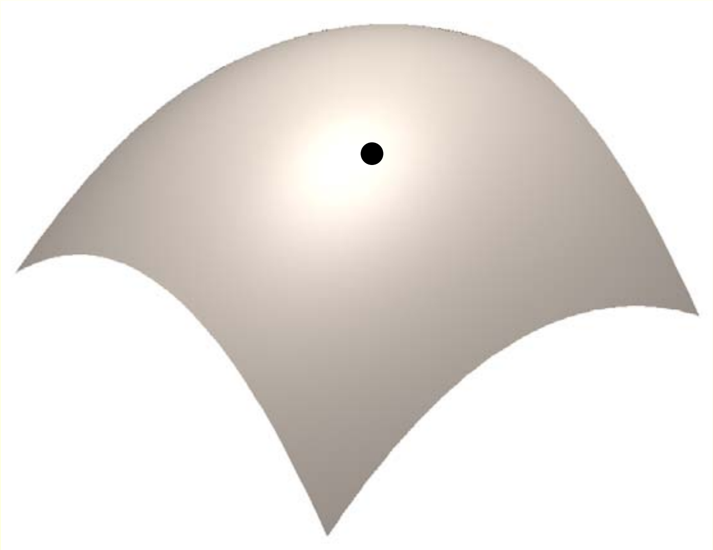
- Is there connection between algebraic invariants of the shape operator S (trace, determinant) with geometric invariants of the shape?

Mean and Gaussian Curvatures

- Mean curvature $H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\text{trace}\, S$
- Gaussian curvature $K = \kappa_1 \kappa_2 = \det\, S$ isometric invariant!



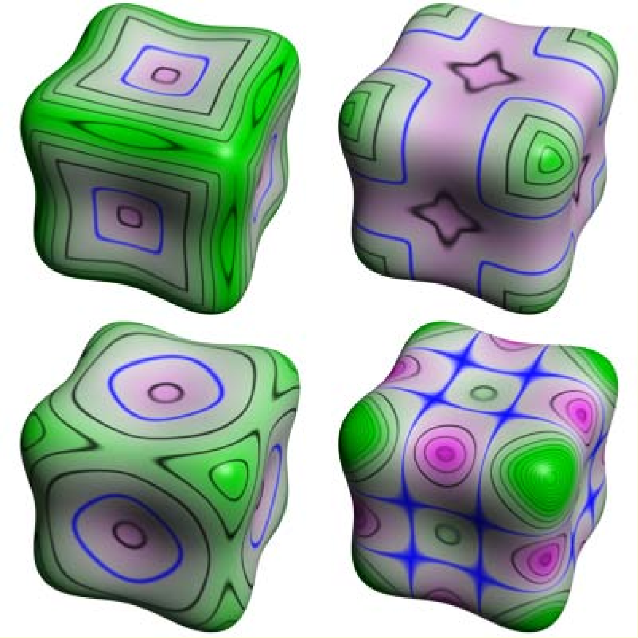
hyperbolic point $K < 0$



elliptic point $K > 0$

Mean and Gaussian Curvatures

$$\kappa_1 = \max_{\phi} \kappa_n(\phi)$$



$$\kappa_2 = \min_{\phi} \kappa_n(\phi)$$

$$H = \frac{1}{2}(\kappa_1 + \kappa_2)$$

$$K = \kappa_1 \cdot \kappa_2$$

Extrinsic v.s. Intrinsic Geometry

- First fundamental form describes completely the intrinsic geometry.
- Second fundamental form describes completely the extrinsic geometry – the “layout” of the shape in ambient space.
- First fundamental form is invariant to isometry.
- Second fundamental form is invariant to rigid motion (congruence).
- If X and $f(X)$ are congruent (i.e., $f \in \mathrm{Iso}(\mathbb{R}^3)$), then they have identical intrinsic and extrinsic geometries.
- Fundamental theorem: a map preserving the first and the second fundamental forms is a congruence.

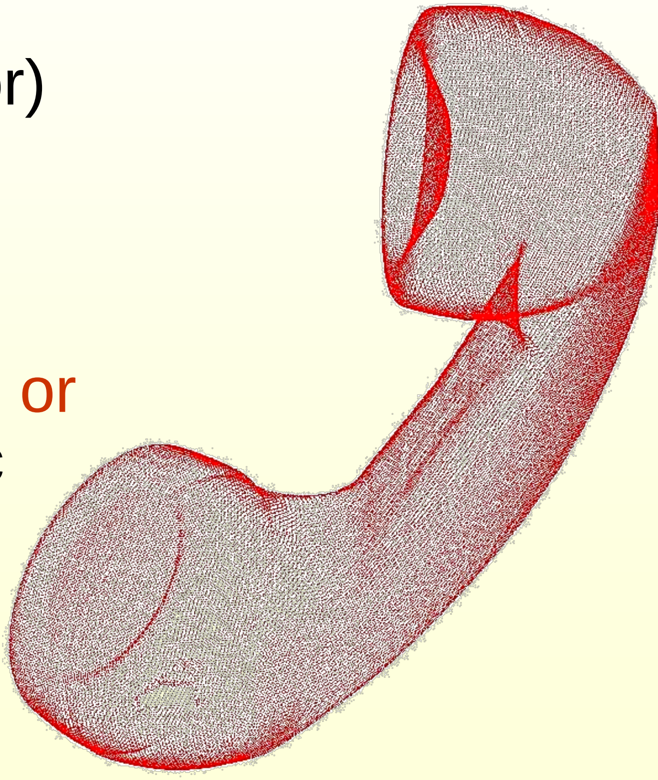
  Said differently: an isometry preserving the second fundamental form is a restriction of Euclidean isometry.

Differential Quantities on a Point Cloud

Goal: PCD as First-Class Geometric Models

- PCD = “point cloud data”
  - unorganized collection of points sampled from the surface (or interior) of an object, with noise added
  - typical output of a 3-D scanning process
- no connectivity information or manifold or mesh structure hard to use geometric methods
- no regular sampling

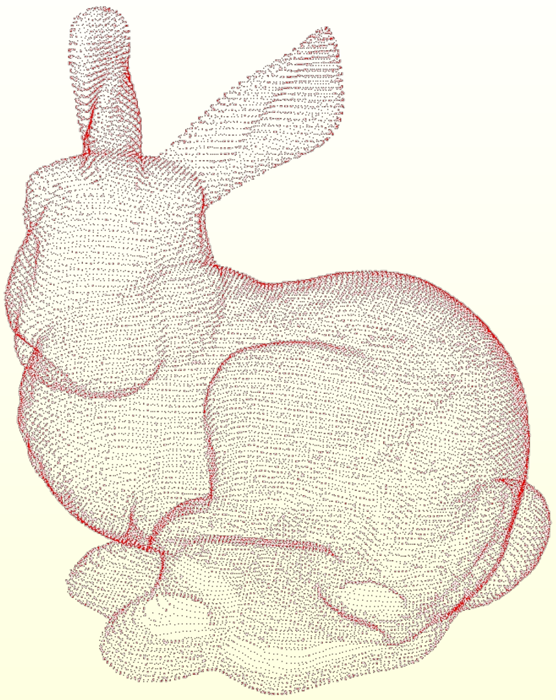
  hard to use signal processing tools



The Normal Estimation Problem

- Given

  Noisy PCD sampled from a curve/surface

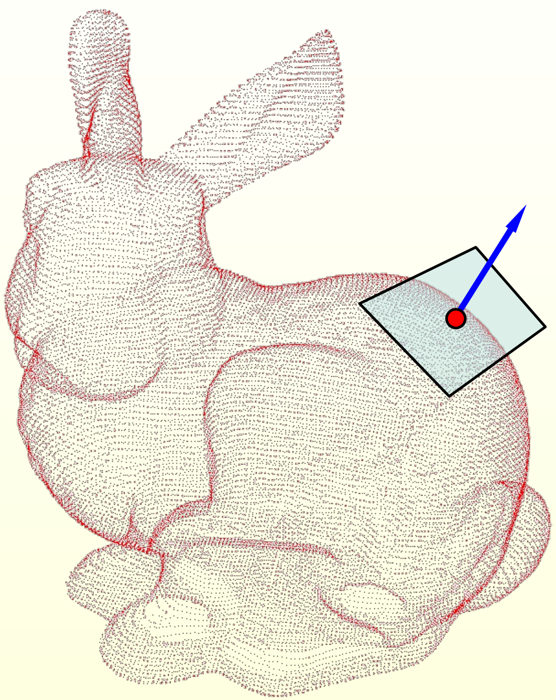


The Normal Estimation Problem

- Given
  Noisy PCD sampled from a curve/surface

- Goal
  Compute surface normals at each point *p*

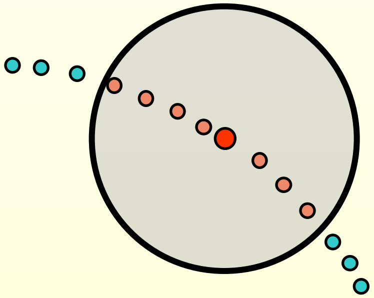
  Error bound the normal estimates

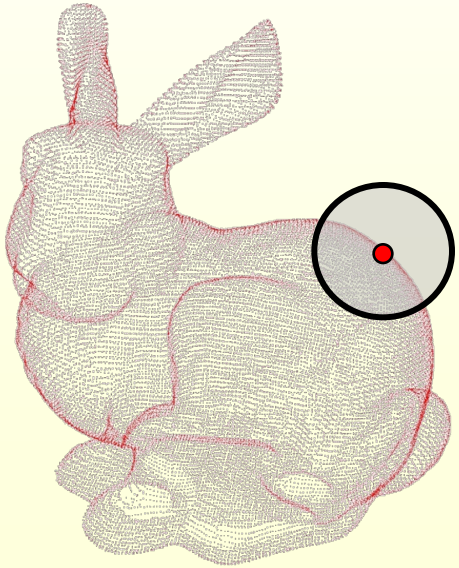


Without first computing a mesh

A Standard Solution

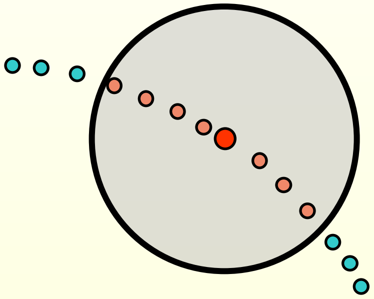
Use *least square fit* to a neighborhood of radius *r* around point *p*

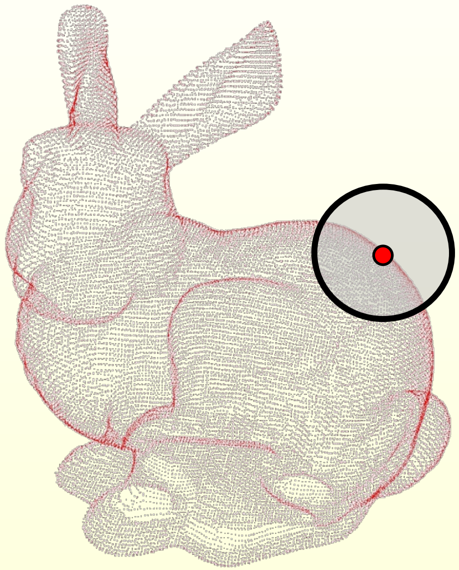




A Standard Solution

Use *least square fit* to a neighborhood of radius *r* around point *p*

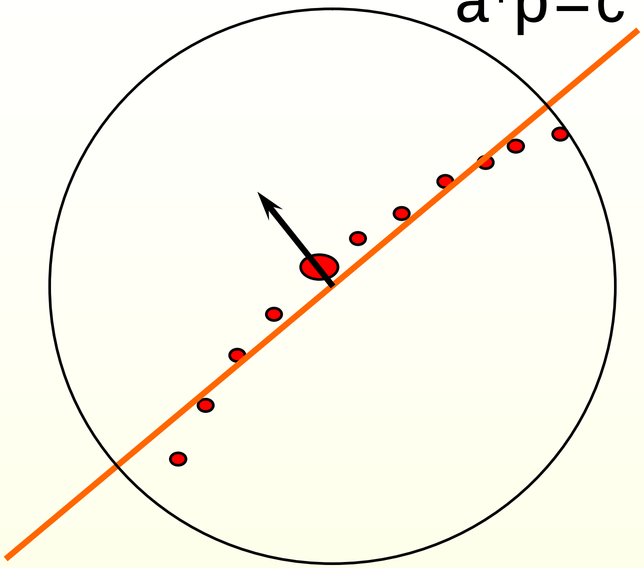




PROBLEM !!

what neighborhood size to choose?

Least Square Fit



- Assume

  best fit hyperplane: $a^T p = c$

- Minimize

$$\frac{1}{k}\sum_{i=1}^{k}(a^T p_i - c)^2$$

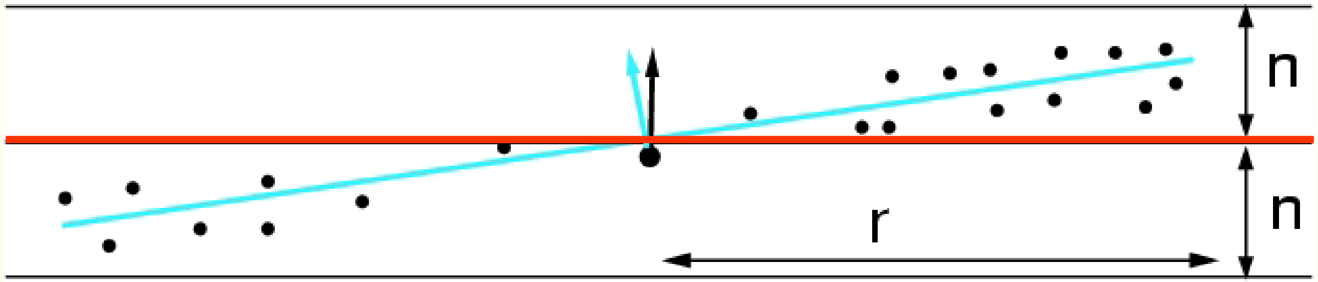
- Reduces to the eigen-analysis of the **covariance matrix** of the p_i

- Smallest eigenvector of M is the estimate of the normal

$$M = \frac{1}{k}\sum_{i=1}^{k}\left(p_i - \overline{p}\right)\left(p_i - \overline{p}\right)^T$$

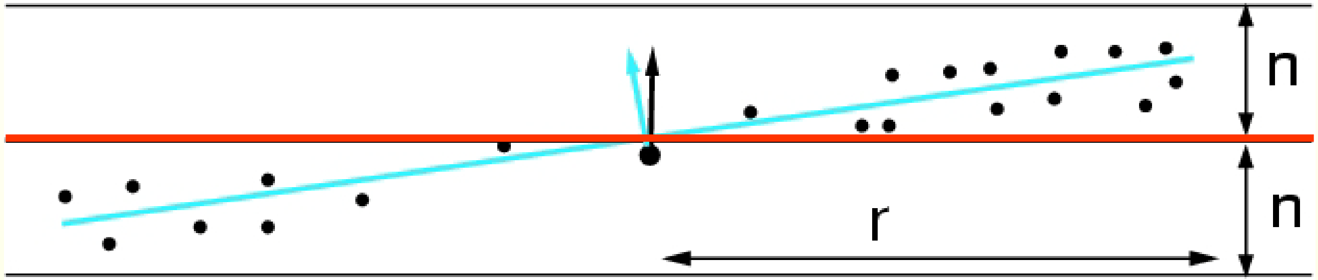
Deceptive Case

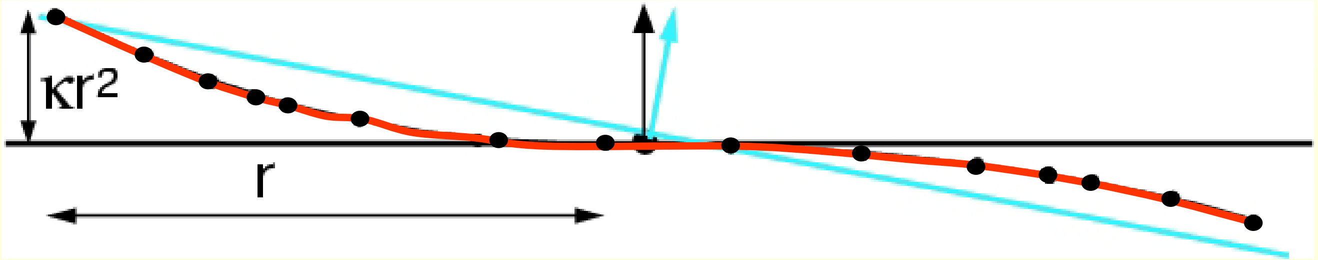
Collusive noise



Deceptive Cases

Collusive noise





Curvature effect

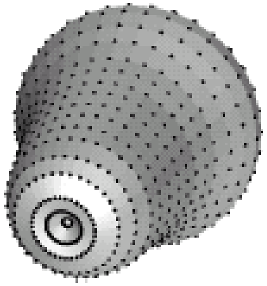
Important Issue

- Study the effects on normal computation of
  - *curvature*
  - *sampling density*
  - *noise*
  - *neighborhood size*
- Use this insight to choose an optimal neighborhood size
- Compute bound on the estimation error
- Direction of normal?

Assumptions

- Noise
  - Independent of measurement
  - Zero mean
  - Variance is known (noise need not be bounded)
- Data
  - Sampling criterion satisfied
  - *Evenly distributed* data
    - To prevent biased estimates
  - Curvature is bounded

Sampling Criteria (2D)



Sampling density

- lower bound (like Nyquist rate)
- upper bound (to prevent biased fits)

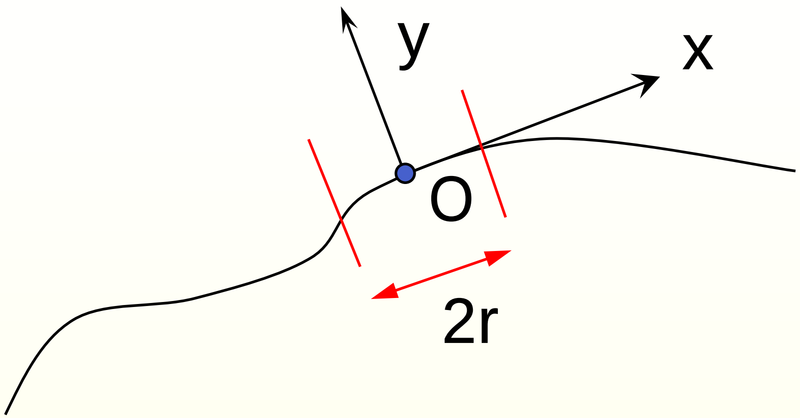
Evenly distributed

Number of points in a disc of radius r bounded *above* and *below* by $\Theta(1)r\rho$

ρ is the sampling density

(ε,δ) sampling condition [Dey et. al.] implies evenly distributed.

Modeling (2D)



- At a point O
  - Points of PCD inside a disc of radius *r* comes from a segment of the curve
  - $y = g(x)$ define the curve for all $x \in [-r,r]$
  - Bounded curvature: $|g''(x)| < \kappa$ for all x
  - Additive Noise(n) in y-direction $(x, g(x)+n)$
  - κr, σ_n / r assumed to be small

Proof Idea

- Eigen-analysis of covariance matrix

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

Proof Idea

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

- covariance matrix
- let,

$$\alpha = (|m_{12}| + m_{22}) / m_{11}$$

Proof Idea

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

- covariance matrix
- let, $\alpha = (|m_{12}| + m_{22})/m_{11}$
- error angle bounded by,

$$\tan^{-1}\left(\frac{\alpha(1+\alpha)}{(1-\alpha)^2}\right) \approx \alpha$$

- to bound estimation error, need to bound α

Final Result in 2D

$$\alpha \leq \Theta(1)\kappa r + \Theta(1)\frac{\sigma_n}{\sqrt{\varepsilon\rho r^3}} + \Theta(1)\frac{\sigma_n^2}{r^2}$$

- $\kappa = 0$,

take as large a neighborhood as possible

Final Result in 2D

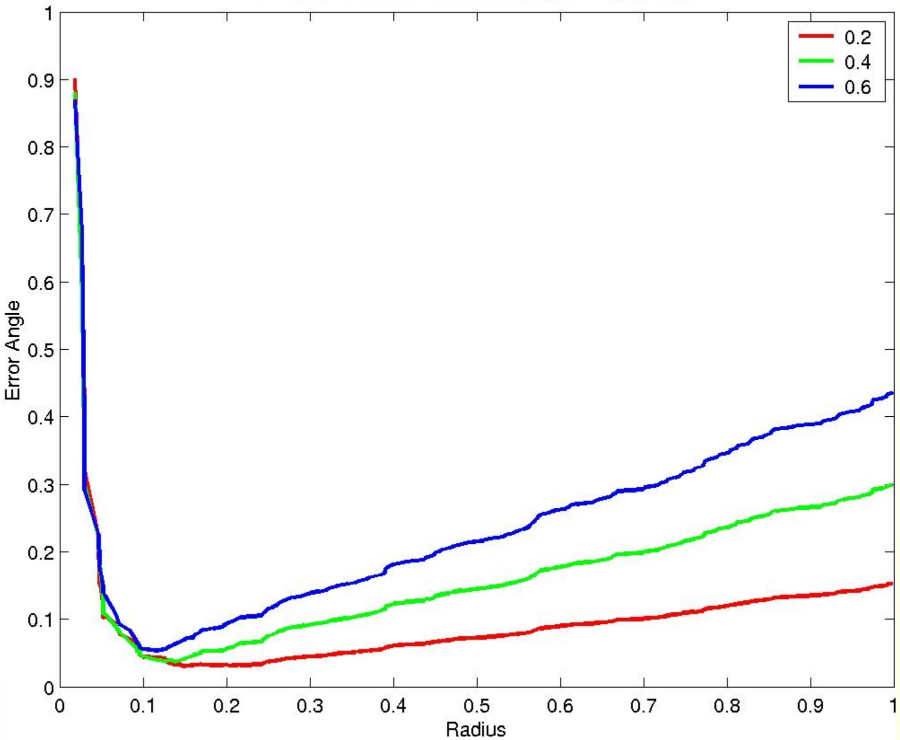
$$\alpha \leq \boxed{\Theta(1)\kappa r} + \Theta(1)\frac{\sigma_n}{\sqrt{\varepsilon\rho r^3}} + \Theta(1)\frac{\sigma_n^2}{r^2}$$

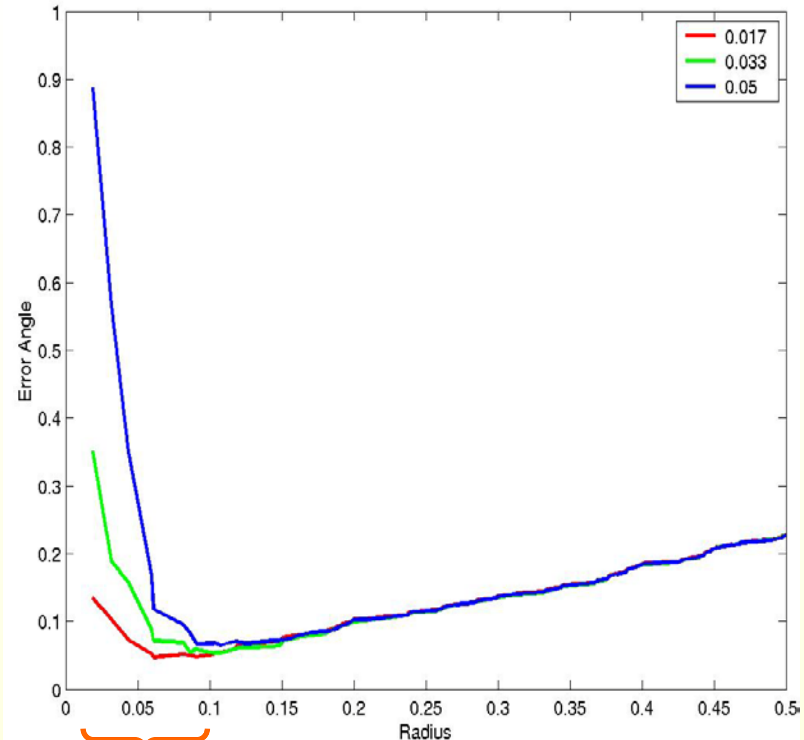
- κ = *0*, take as large a neighborhood as possible
- σ_n = *0*

  take as small a neighborhood as possible

Experiments in 2D

κ





$$error \le \Theta(1)\kappa r \; + \; \Theta(1)\frac{\sigma_n}{\sqrt{\varepsilon \rho r^3}} + \Theta(1)\frac{\sigma_n^2}{r^2}$$

Result for 3D

A similar but involved analysis results in,

$$error \leq \Theta(1)\kappa r + \Theta(1)\sigma_n /(r^2 \sqrt{\varepsilon\rho}) + \Theta(1)\sigma_n^2 / r^2$$

A good choice of *r* is,

$$r = \left(\frac{1}{\kappa} \left(c_1 \frac{\sigma_n}{\sqrt{\varepsilon\rho}} + c_2 \sigma_n^2 \right) \right)^{1/3}$$

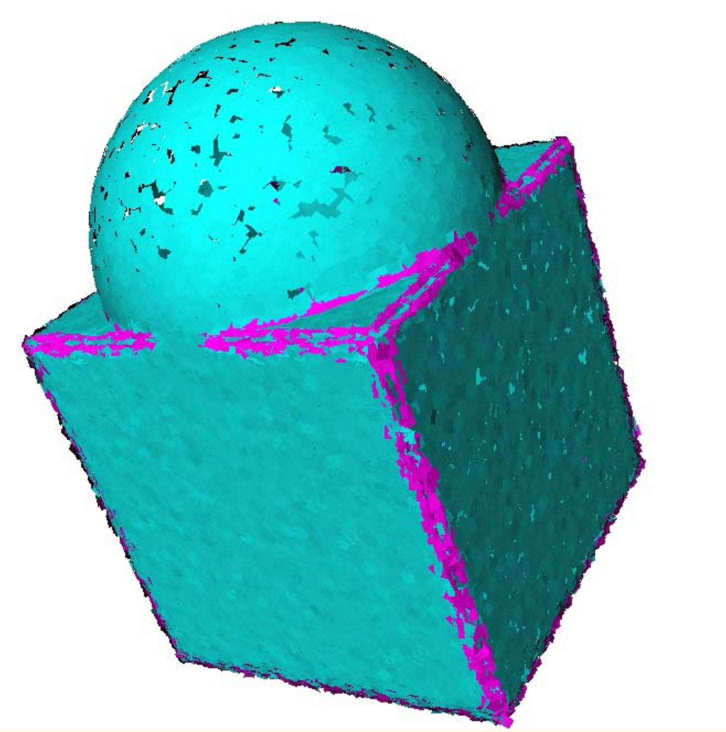
How can we use this result?

$$r = \left(\frac{1}{\kappa} \left(c_1 \frac{\sigma_n}{\sqrt{\varepsilon\rho}} + c_2 \sigma_n^2 \right) \right)^{1/3}$$

- Need to
  - know $\epsilon, \ \sigma_n$
  - estimate suitable values for $c_1, \ c_2$
  - estimate $\rho, \ \kappa$ locally

Estimating c_1, c_2

Exact normals known at *almost* all points



$$r = \left(\frac{1}{\kappa} \left(c_1 \frac{\sigma_n}{\sqrt{\varepsilon\rho}} + c_2 \sigma_n^2 \right) \right)^{1/3}$$

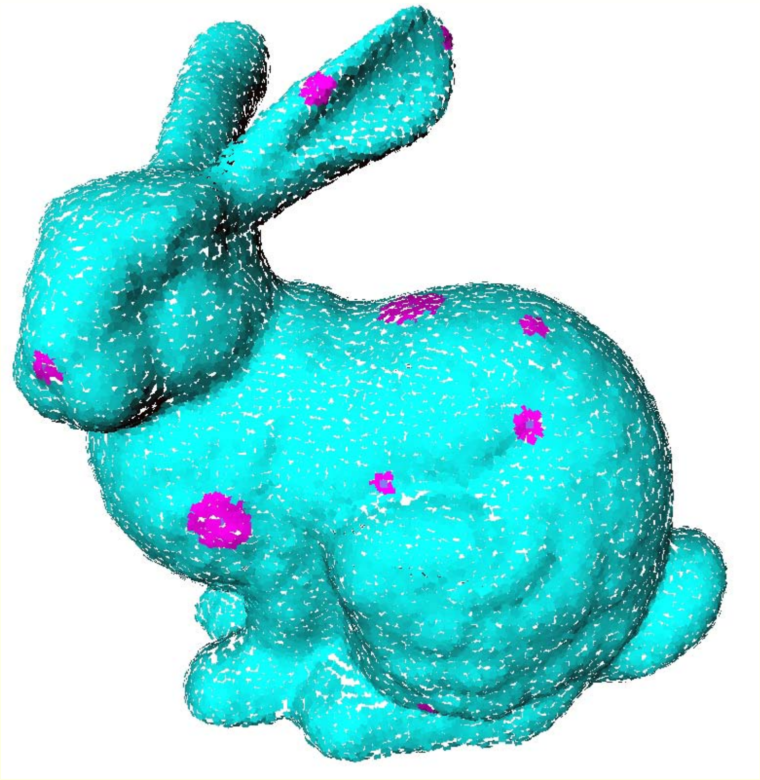
$c_1=1$, $c_2=4$

- same constants used for following results

Algorithm

- For each point, start with $k = 15$
- Iterate and refine (maximum of 10 steps)
  - Compute r, ρ, κ [Gumhold et al.] locally
  - Use them to compute r_{new}
  - $k_{new} = \pi r_{new}^2 \rho_{old}$
  - Stop if
    - $k > threshold$
    - k saturates

Effect of Curvature on Neighborhood Size



1x noise

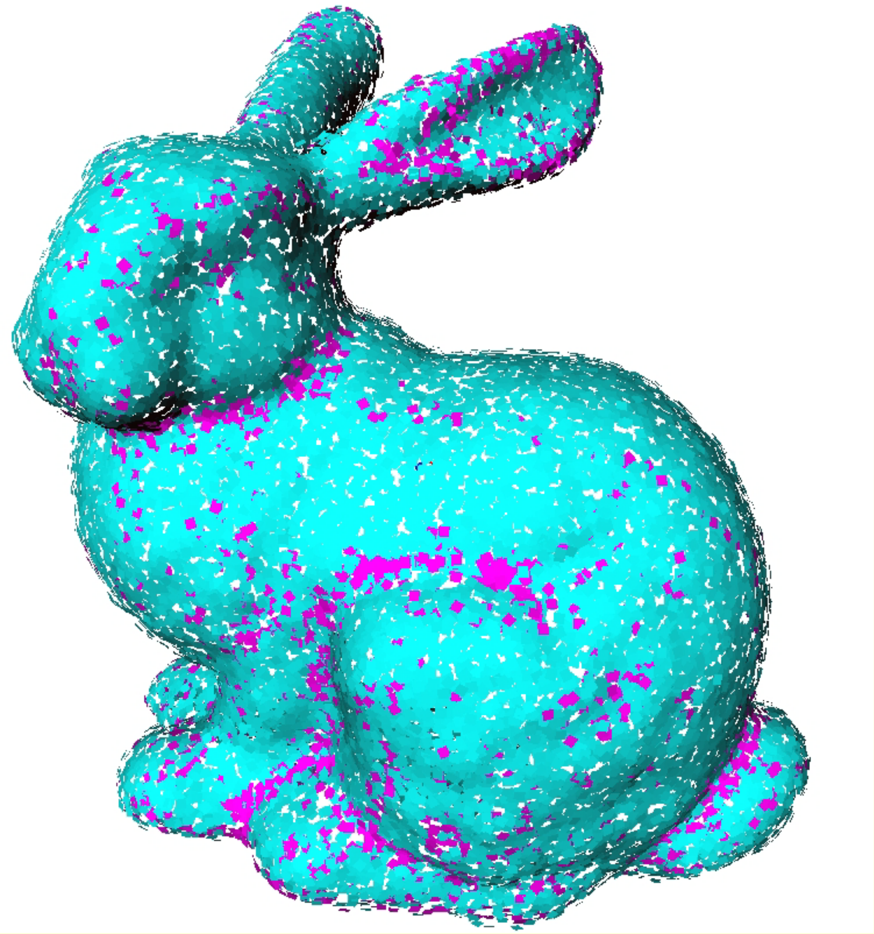
Effect of Noise on Neighborhood Size

1x noise

2x noise

Estimation Error > 5

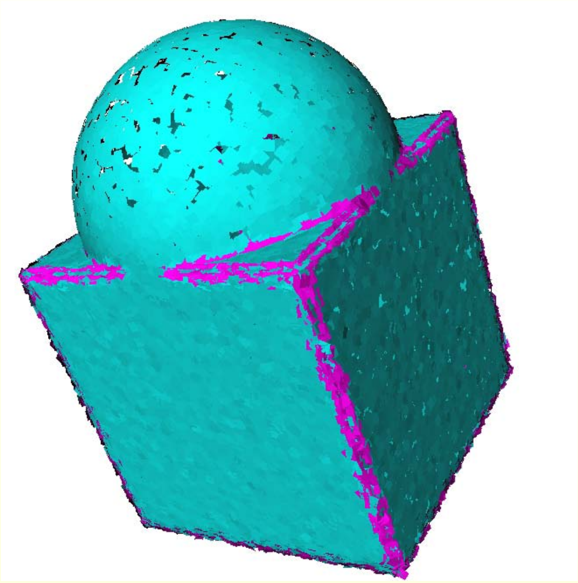
1x noise



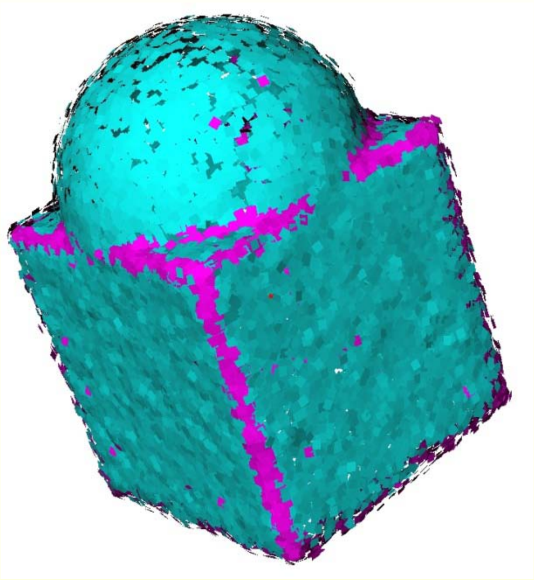
2x noise

Increasing Noise

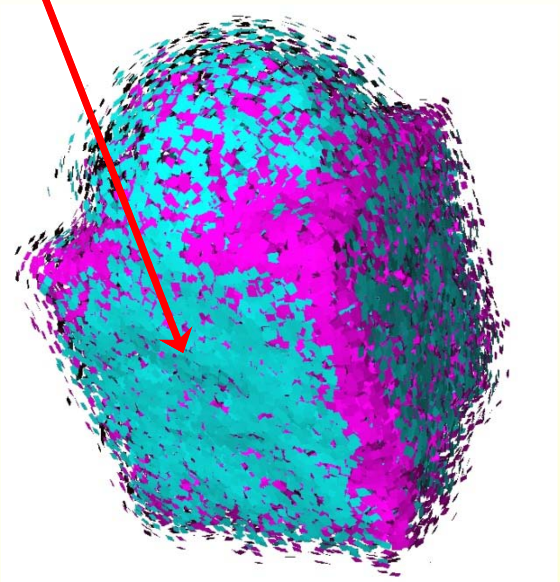
Can still get good estimates in flat areas



1x noise



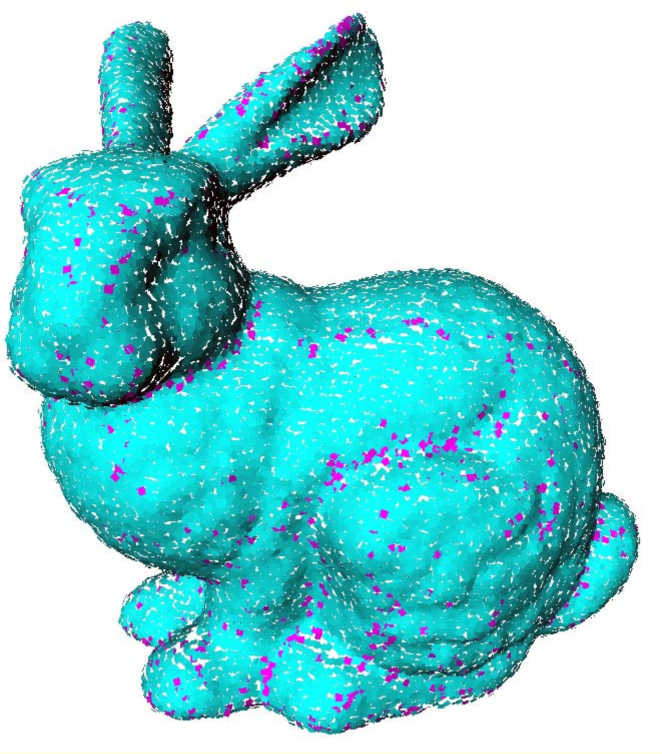
2x noise



4x noise

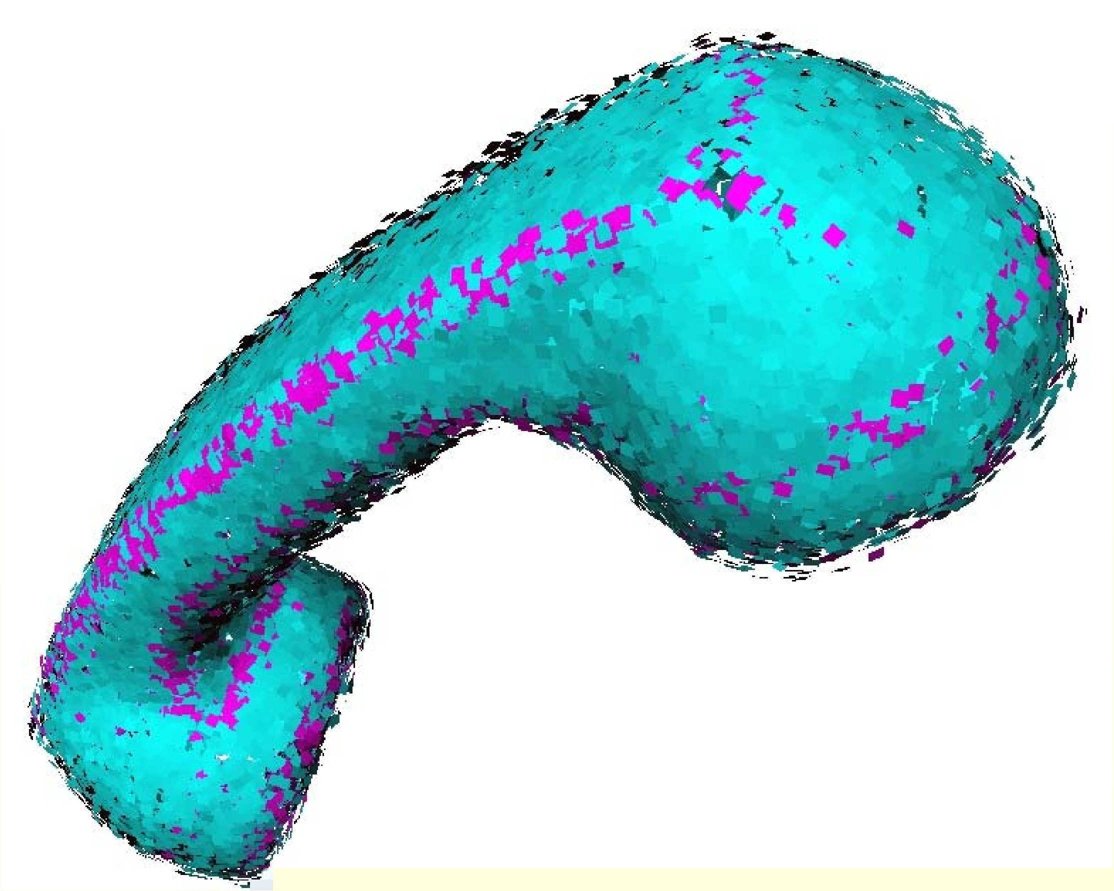
Different Noise Distribution (same variance)

uniform



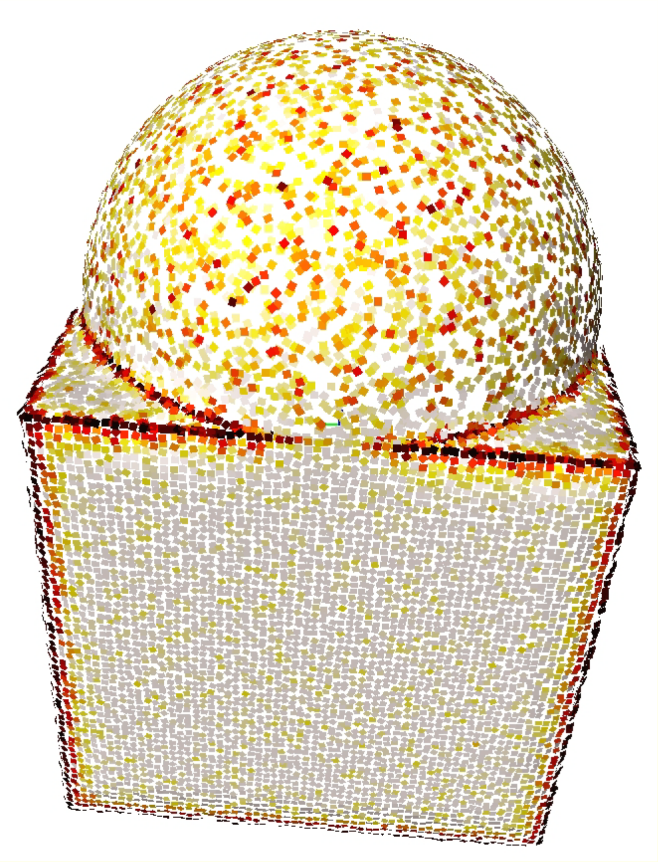
gaussian

Result: Phone



1x noise

Varying Neighborhood Size



Neighborhood size at all points being shown using color-coding. Purple denotes the smallest neighborhood and turns blue as the neighborhood size increases

The End