CS164: Smoothing and Remeshing

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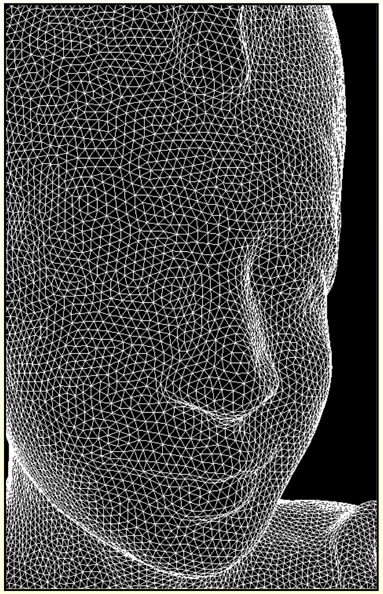
Slides partly by Mario Botsch, Bruno Levy

Mesh Processing Pipeline

Scan

Reconstruct

Clean



Remesh

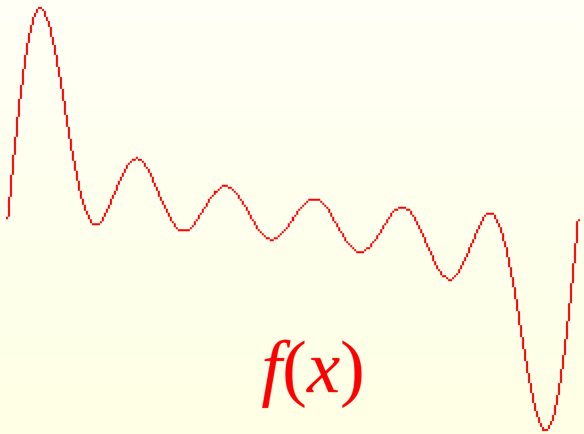
...

Mesh Smoothing
(aka Denoising, Filtering, Fairing)

Input: Noisy mesh (scanned or other)
Output: Smooth mesh
How: Filter out high frequency noise

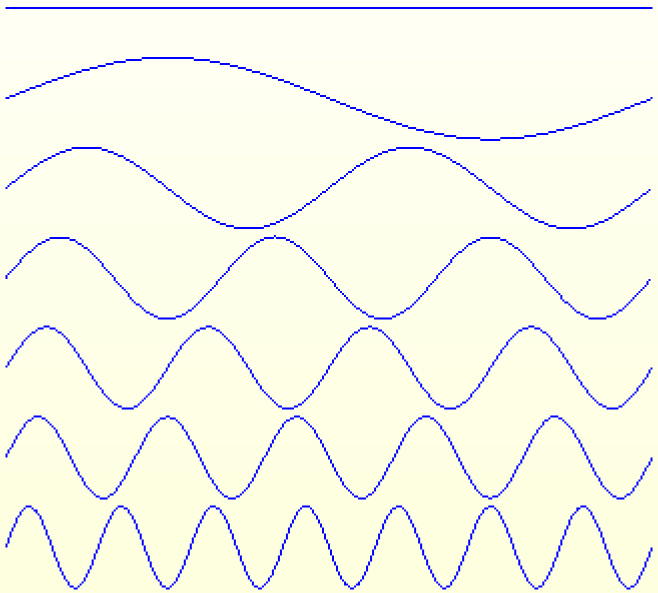
Smoothing by Filtering

Fourier Transform



$f(x)$

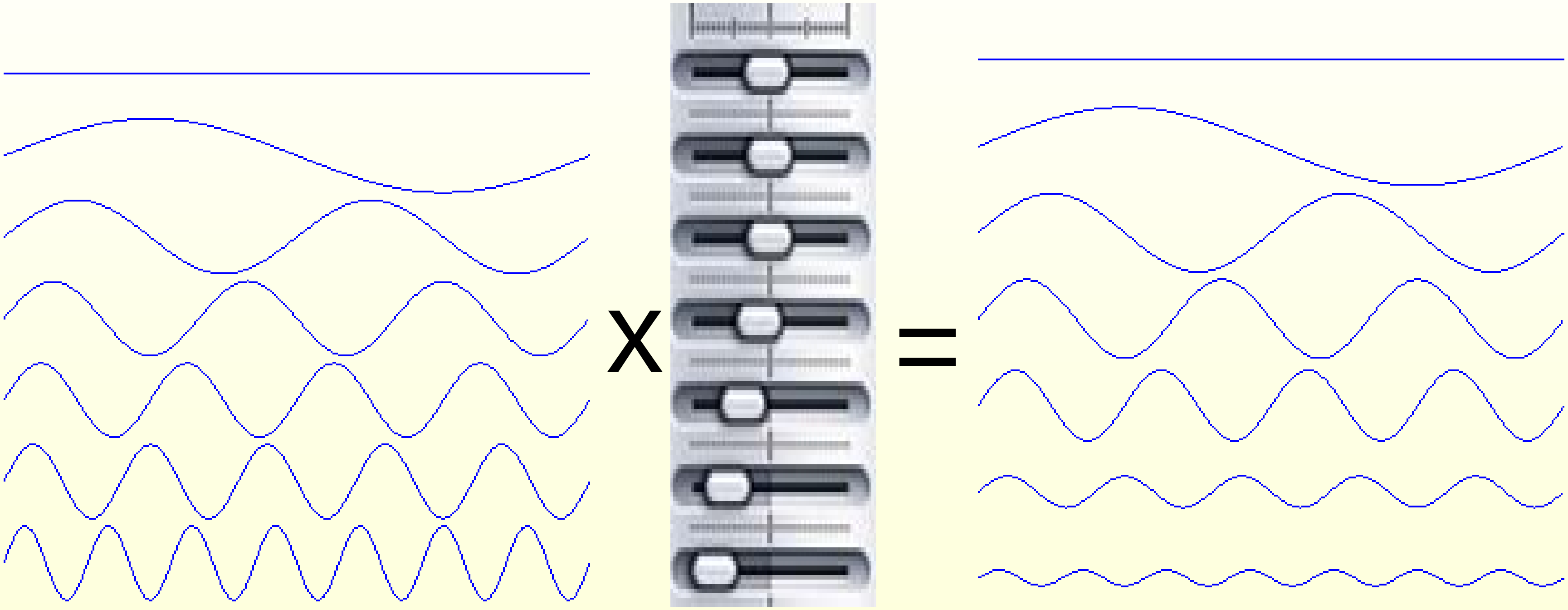
$=\Sigma$



$sin(kx)$

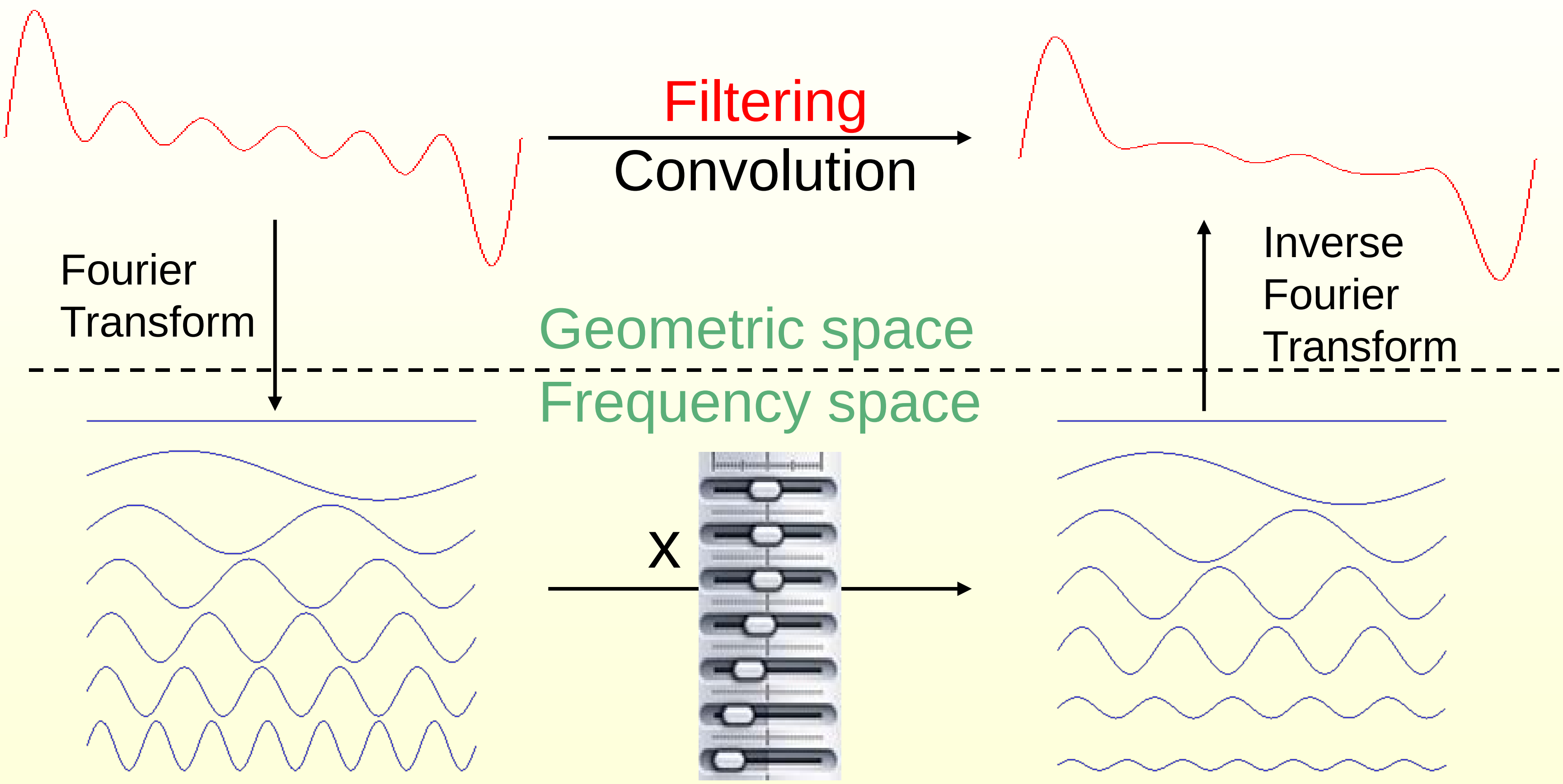
Smoothing by Filtering

Fourier Transform

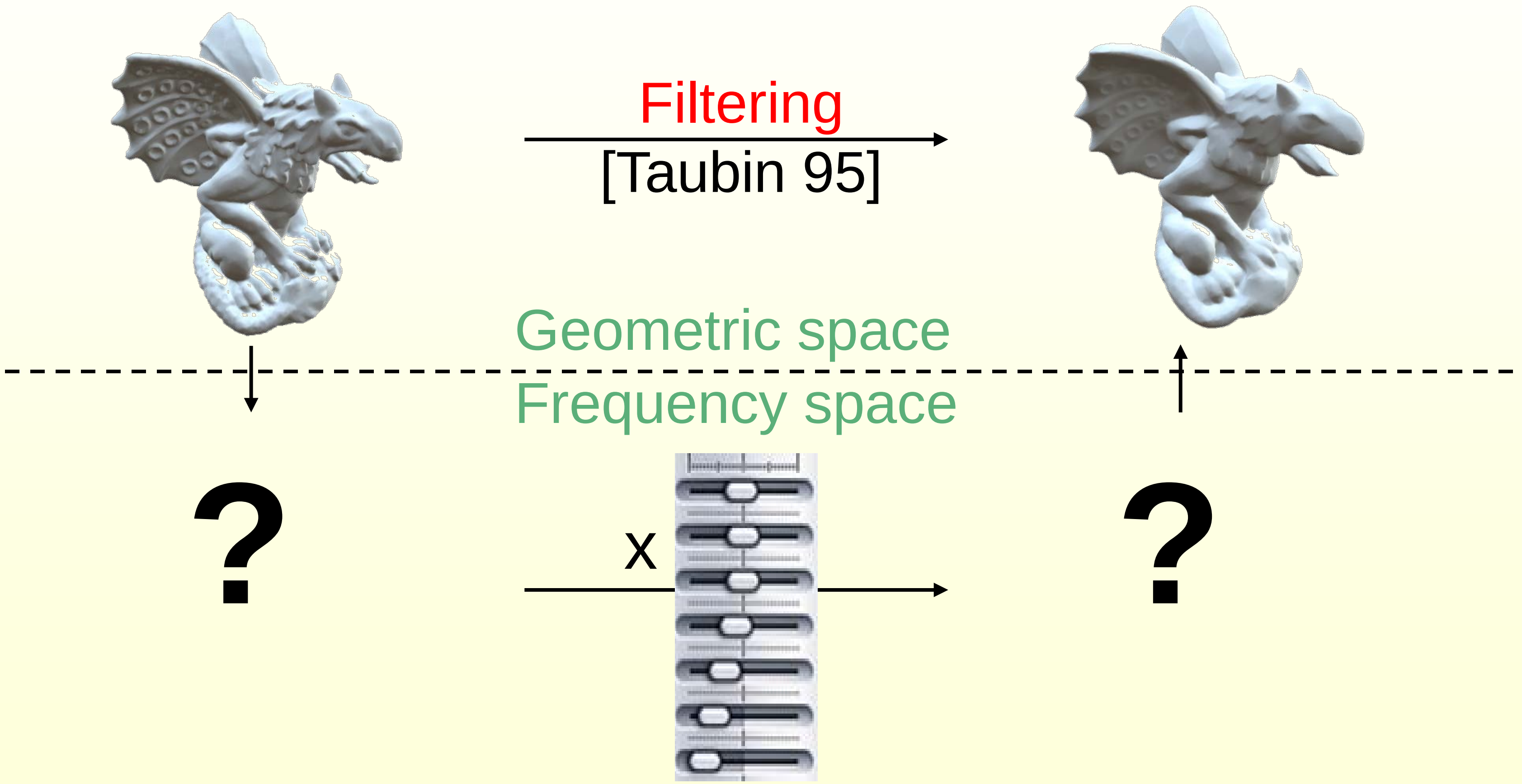


Smoothing by Filtering

Fourier Transform

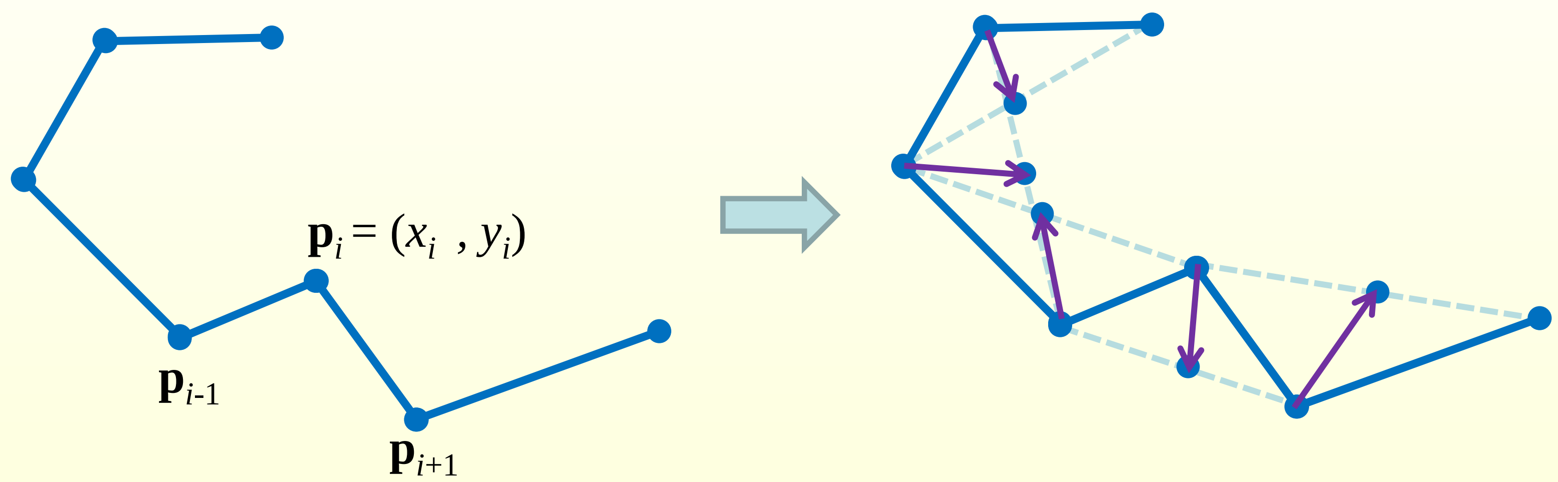


Filtering on a Mesh



Laplacian Smoothing

An easier problem: How to smooth a curve?

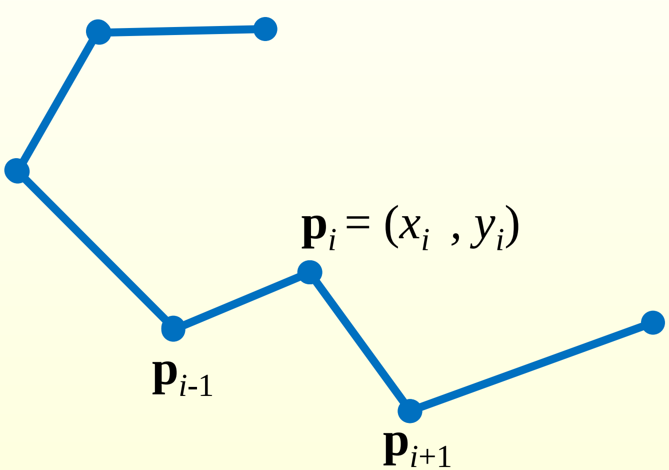


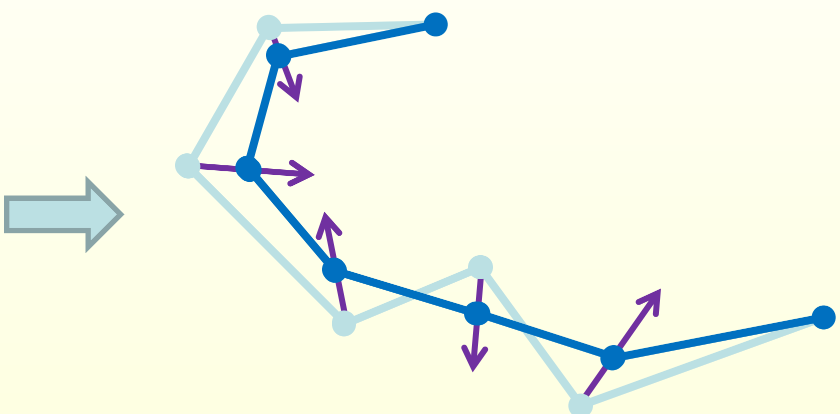
$(\mathbf{p}_{i\text{-}1} + \mathbf{p}_{i+1})/2 - \mathbf{p}_i$

$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i)$$

Laplacian Smoothing

An easier problem: How to smooth a curve?





$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \frac{1}{2} L(\mathbf{p}_i)$$

Finite difference discretization of second derivative
= Laplace operator in one dimension

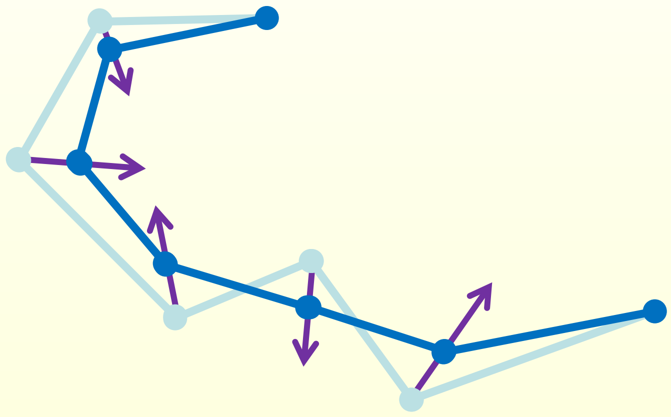
$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i)$$

Laplacian Smoothing

<u>Algorithm:</u>

Repeat for *m* iterations:

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda L(\mathbf{p}_i)$$



For which λ?

$0 < \lambda < 1$

Closed curve converges to?

Single point

Spectral Analysis

Closed Curve

Re-write $\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda L\left(\mathbf{p}_i^{(t)}\right)$

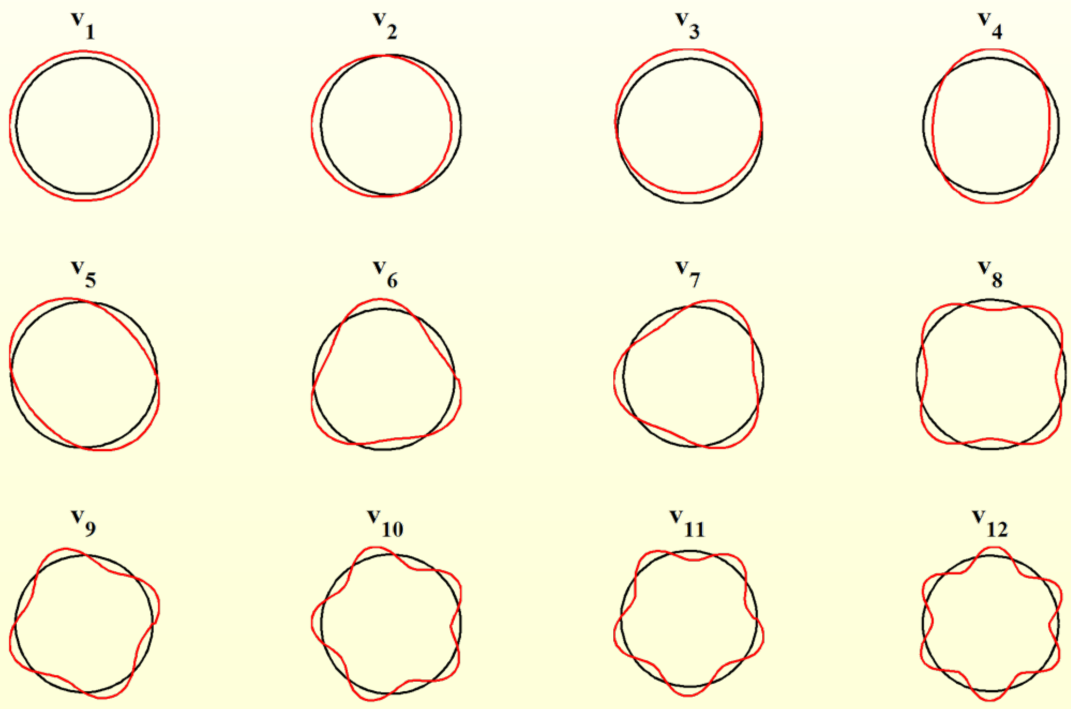
$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i)$$

in matrix notation: $\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)} - \lambda \mathbf{L}\mathbf{P}^{(t)}$

$$\mathbf{P} = \begin{pmatrix} x_1 & y_2 \\ \dots & \dots \\ x_n & y_n \end{pmatrix} \in \mathbb{R}^{n\times 2} \qquad \mathbf{L} = \frac{1}{2}\begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & & \dots & & & \\ & & & -1 & 2 & -1 \\ -1 & & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

The Eigenvectors of $\mathbf{L}$

$$\mathbf{L} \quad \mathbf{V}\mathbf{D}\mathbf{V}^T \qquad \mathbf{V} \quad \begin{matrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{matrix} \quad , \quad \mathbf{D} \quad \begin{matrix} k_1 & & & \\ & k_2 & & \\ & & \cdots & \\ & & & k_n \end{matrix}$$



Spectral Analysis

Then: $\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)} - \lambda \mathbf{L} \mathbf{P}^{(t)} = (\mathbf{I} - \lambda \mathbf{L}) \mathbf{P}^{(t)}$

After m iterations: $\mathbf{P}^{(m)} = (\mathbf{I} - \lambda \mathbf{L})^{m} \mathbf{P}^{(0)}$

Can be described using eigen-decomposition of $\mathbf{L}$

$$\mathbf{L} = \mathbf{V}\mathbf{D}\mathbf{V}^{T}$$

$$\mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \cdots & \\ & & & k_n \end{pmatrix}$$

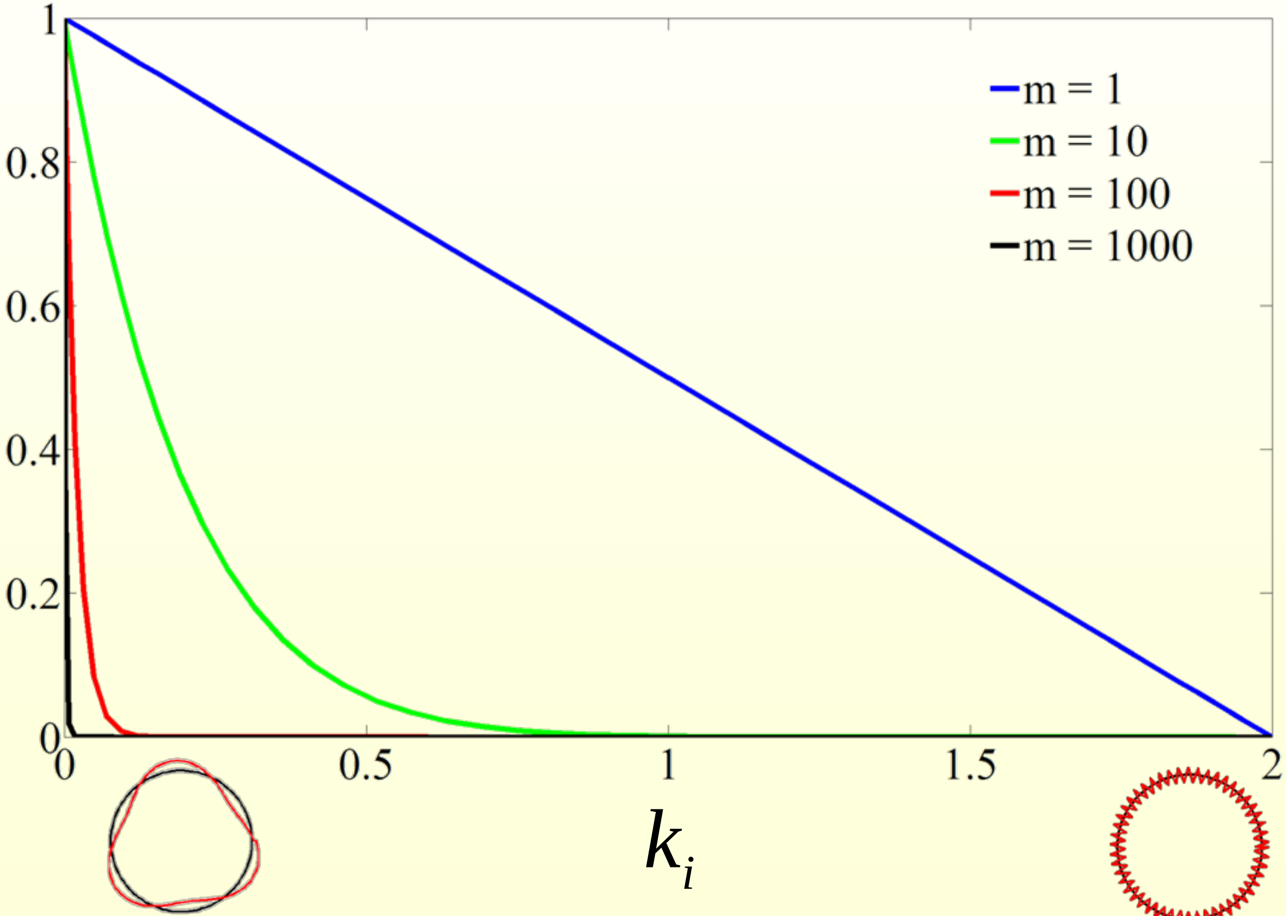
$$\Rightarrow \quad \mathbf{P}^{(m)} = \mathbf{V} (\mathbf{I} - \lambda \mathbf{D})^{m} \mathbf{V}^{T} \mathbf{P}^{(0)}$$

Filtering high frequencies

Spectral Analysis

Laplacian Smoothing

$1 \quad 0.5k_i \quad {}^{m}$



Laplacian Smoothing on Meshes

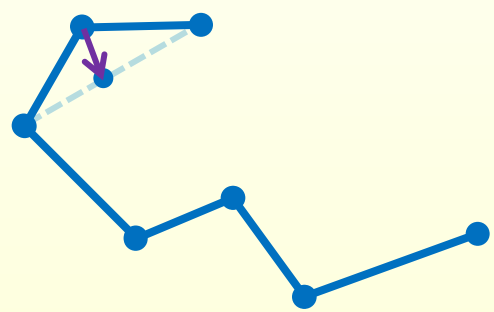
Same as for curves:

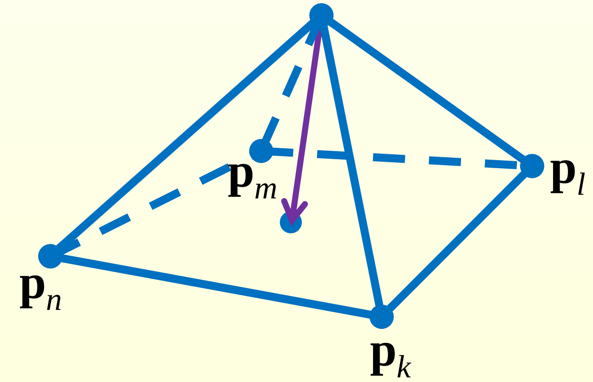
$$\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$$

$N_i = \{k, l, m, n\}$

$\mathbf{p}_i = (x_i, y_i, z_i)$

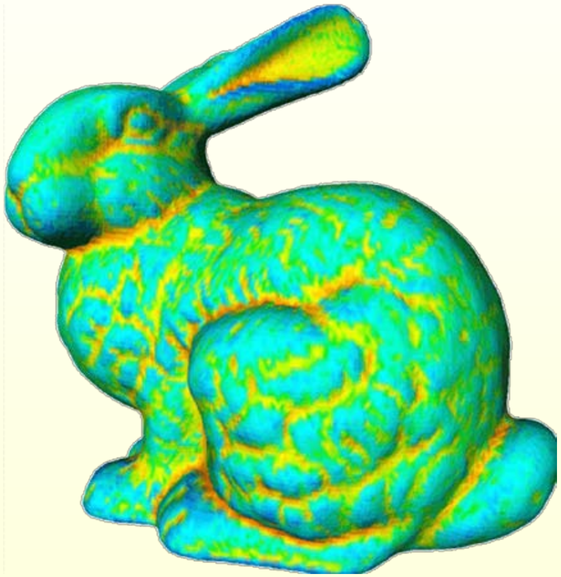
What is $\Delta \mathbf{p}_i$?



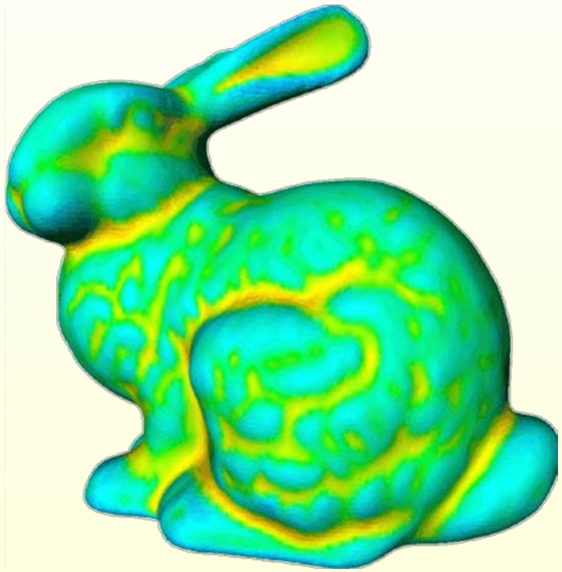


$$\frac{1}{2}\left(\mathbf{p}_{i+1} + \mathbf{p}_{i-1}\right) - \mathbf{p}_i \qquad \frac{1}{|N_i|}\left(\sum_{j \in N_i} \mathbf{p}_j\right) - \mathbf{p}_i$$

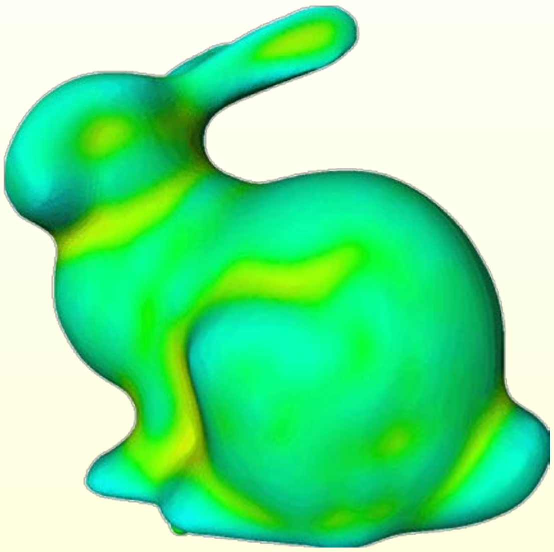
Laplacian Smoothing on Meshes



0 Iterations



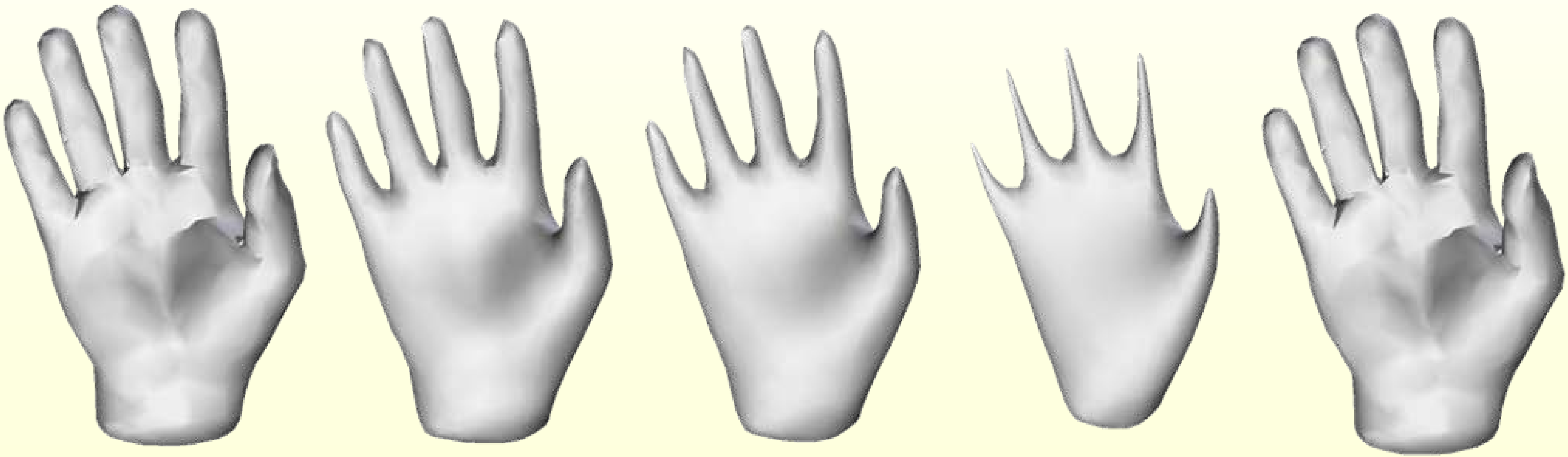
5 Iterations



20 Iterations

Problem - Shrinkage

Repeated iterations of Laplacian smoothing shrinks the mesh



original 3 steps 6 steps 18 steps original

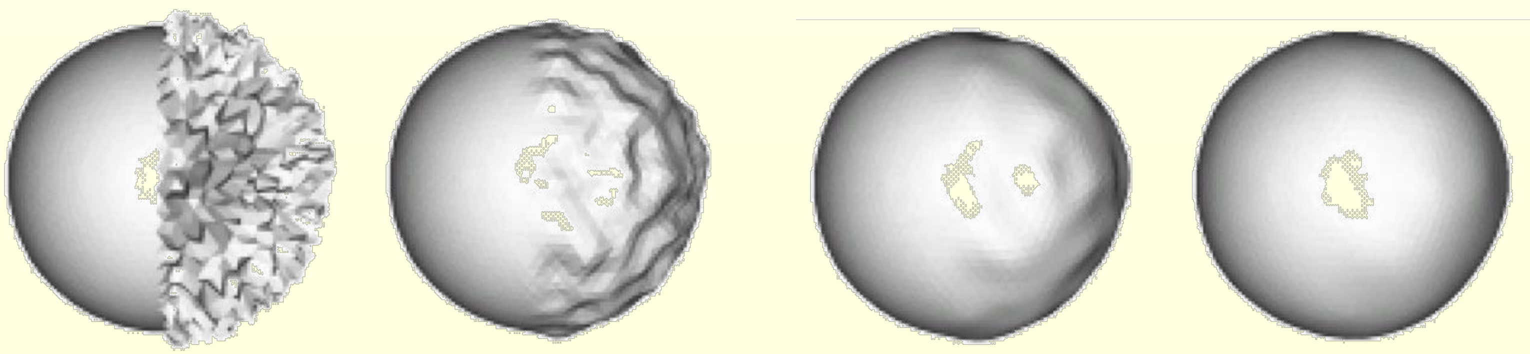
Taubin Smoothing

Iterate:

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i \qquad \text{Shrink}$$

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \mu \Delta \mathbf{p}_i \qquad \text{Inflate}$$

with $\lambda > 0$ and $\mu < 0$

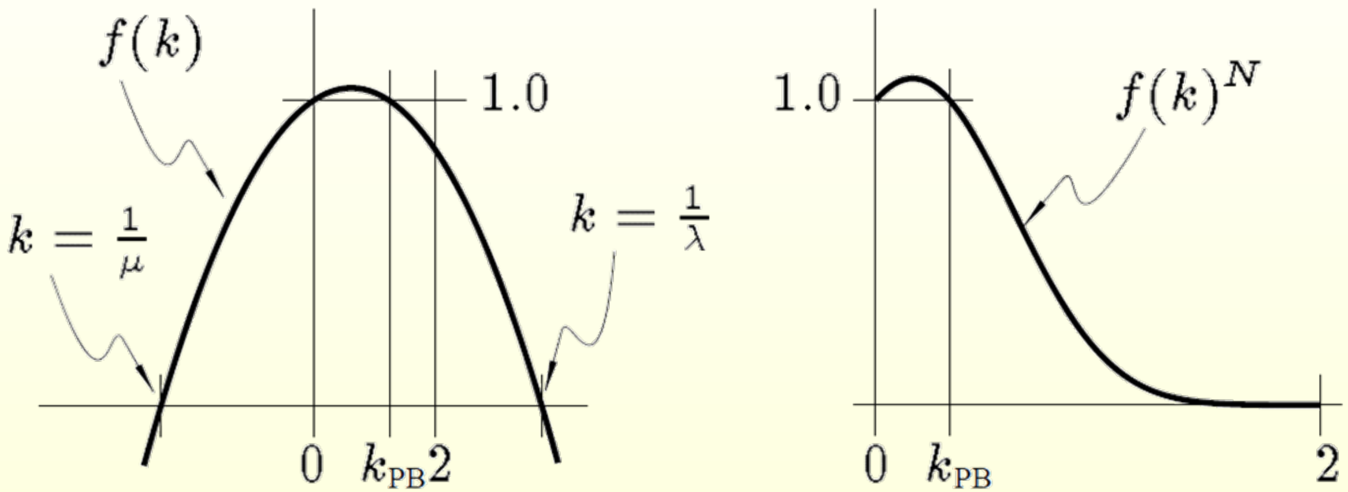


original 10 steps 50 steps 200 steps

From Taubin, Siggraph 1995

Spectral Analysis

Taubin Smoothing



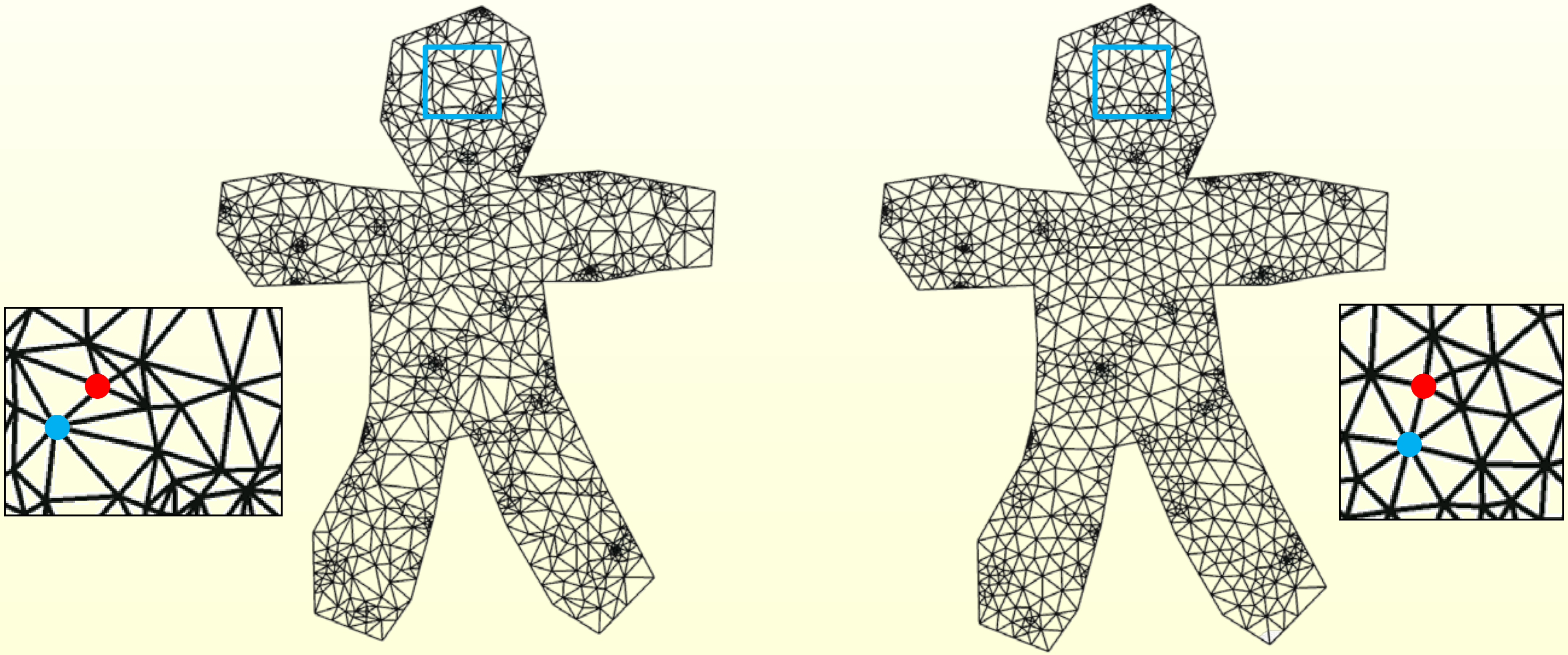
$$f \quad k_i \qquad 1 \qquad k_i \quad 1 \qquad k_i$$

From Taubin, Siggraph 1995

Laplace Operator Discretization

The Problem

Sanity check – what should happen if the mesh lies in the plane: $\mathbf{p}_i = (x_i, y_i, 0)$?

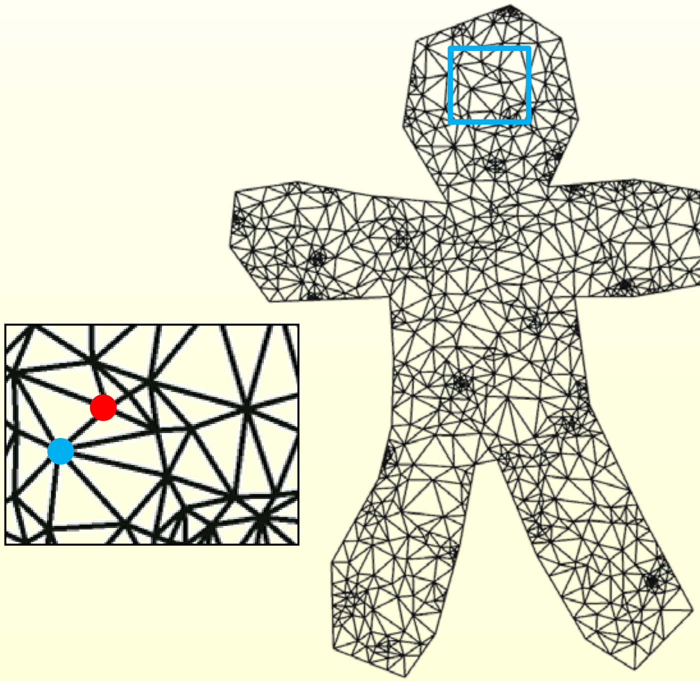


0 Iterations 5 Iterations

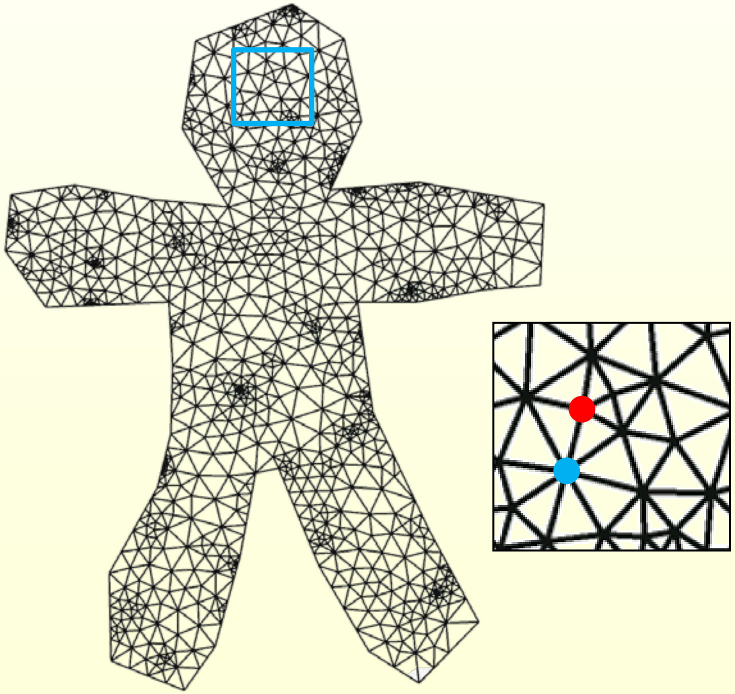
Laplace Operator Discretization

The Problem

Not good – A flat mesh is smooth, should stay the same after smoothing



0 Iterations

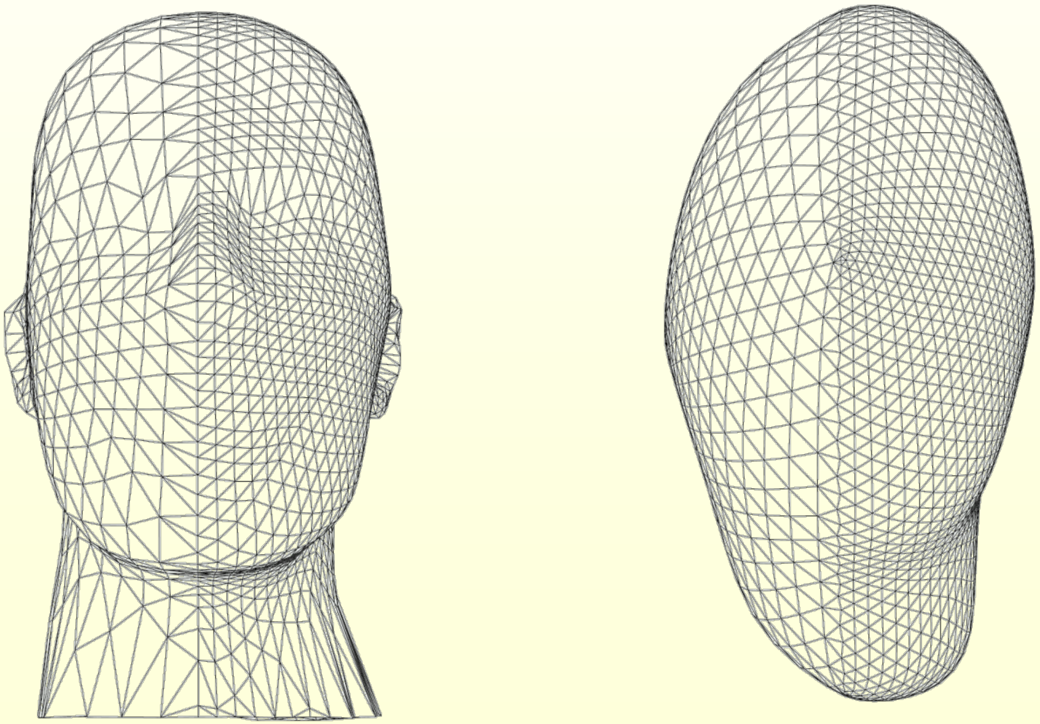


5 Iterations

Laplace Operator Discretization

The Problem

Not good – The result should not depend on triangle sizes



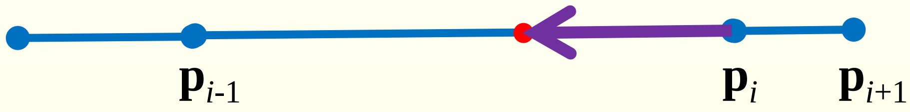
From Desbrun et al., Siggraph 1999

Laplace Operator Discretization

What Went Wrong?

Back to curves:

$$\frac{1}{2}\left(\mathbf{p}_{i+1}+\mathbf{p}_{i-1}\right)-\mathbf{p}_i$$



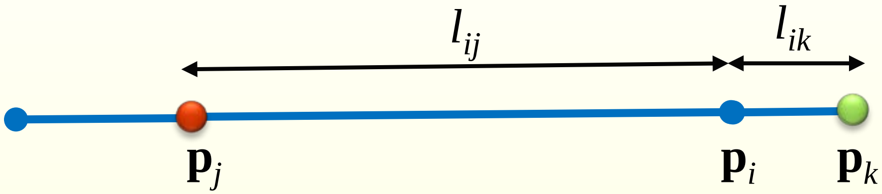
Same weight for both neighbors, although one is closer

Laplace Operator Discretization

The Solution

Use a weighted average to define Δ

Which weights?



$$w_{ij} = \frac{1}{l_{ij}} \qquad w_{ik} = \frac{1}{l_{ik}}$$

$$L(\mathbf{p}_i) = \frac{w_{ij}\mathbf{p}_j + w_{ik}\mathbf{p}_k}{w_{ij} + w_{ik}} - \mathbf{p}_i$$

Straight curves will be invariant to smoothing

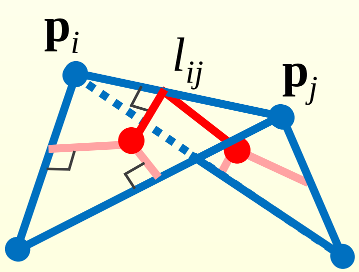
Laplace Operator Discretiztion

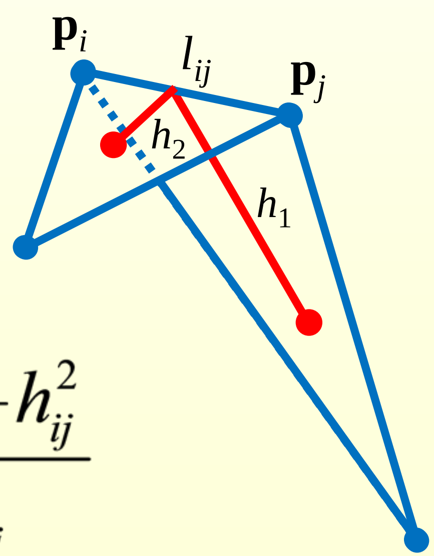
Cotangent Weights

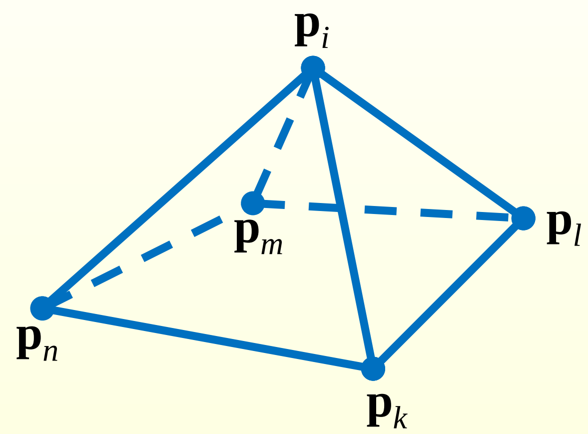
Use a weighted average to define Δ

$N_i = \{k, l, m, n\}$

Which weights? $w_{ij} = \dfrac{1}{l_{ij}}$?







$$L(\mathbf{p}_i) = \frac{1}{\sum_{j \in N_i} w_{ij}} \left(\sum_{j \in N_i} w_{ij} \mathbf{p}_j \right) - \mathbf{p}_i$$

$$w_{ij} = \frac{h_{ij}^1 + h_{ij}^2}{l_{ij}}$$

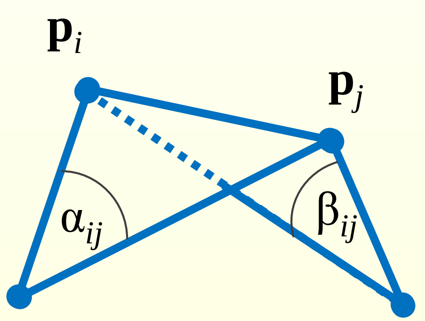
Laplace Operator Discretiztion

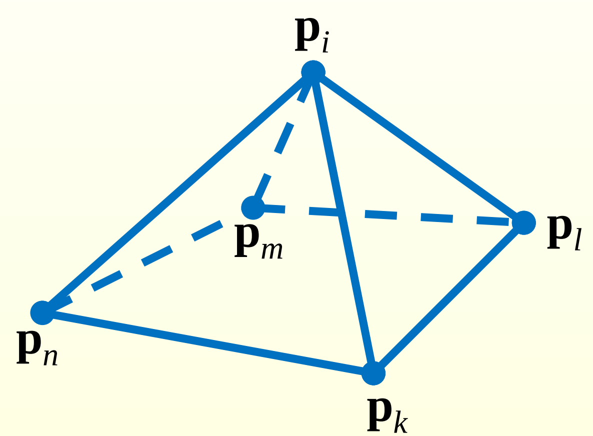
Cotangent Weights

Use a weighted average to define Δ

Which weights?

$N_i = \{k,l,m,n\}$



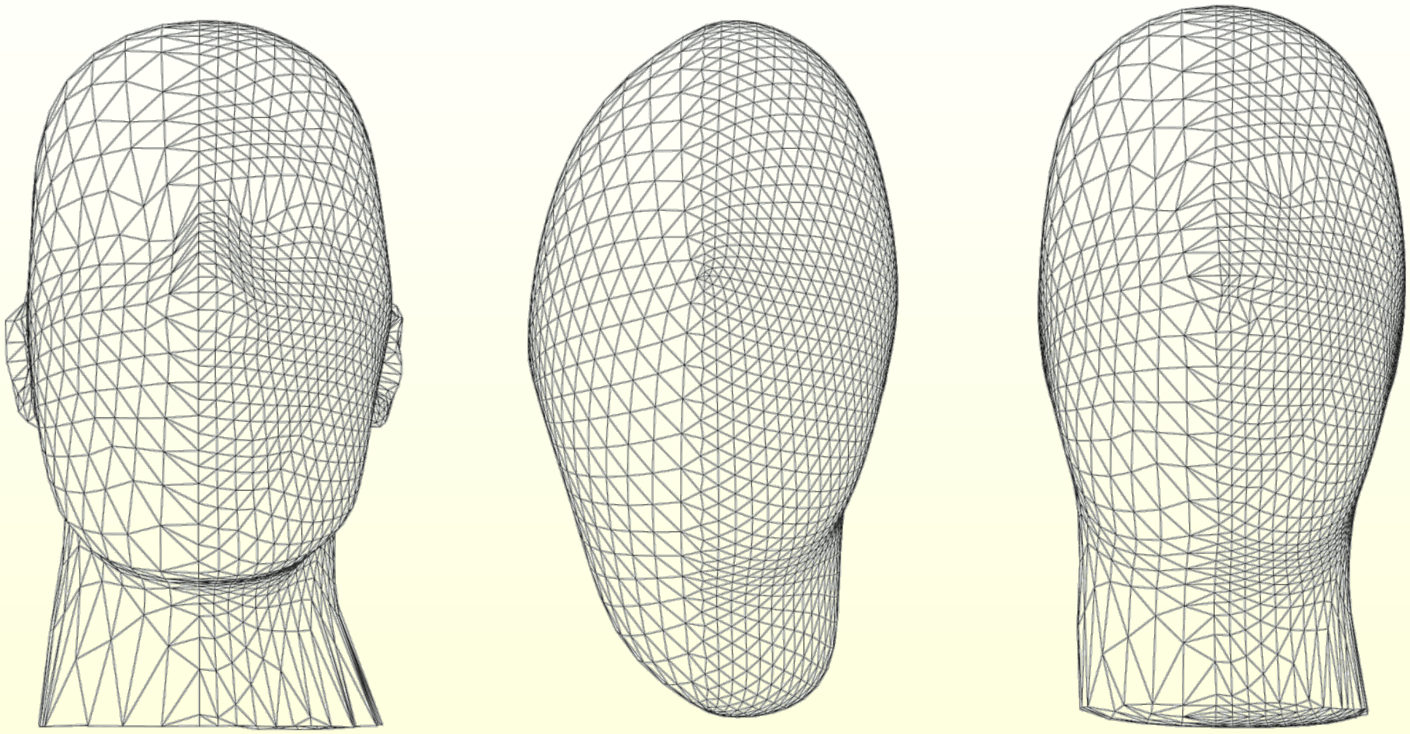


$$w_{ij} \quad \frac{h_{ij}^1 \quad h_{ij}^2}{l_{ij}} \quad \frac{1}{2} \cot \quad {}_{ij} \quad \cot \quad {}_{ij}$$

$$L(\mathbf{p}_i) \quad \frac{1}{w_{ij} \;{}_{j \;\; N_i}} \, w_{ij} \;\; \mathbf{p}_j \;\; \mathbf{p}_i$$

Planar meshes will be invariant to smoothing

Smoothing with the Cotangent Laplacian



original | uniform weights | cotangent weights

From Desbrun et al., Siggraph 1999

Spectral Analysis
Weighted Laplacian

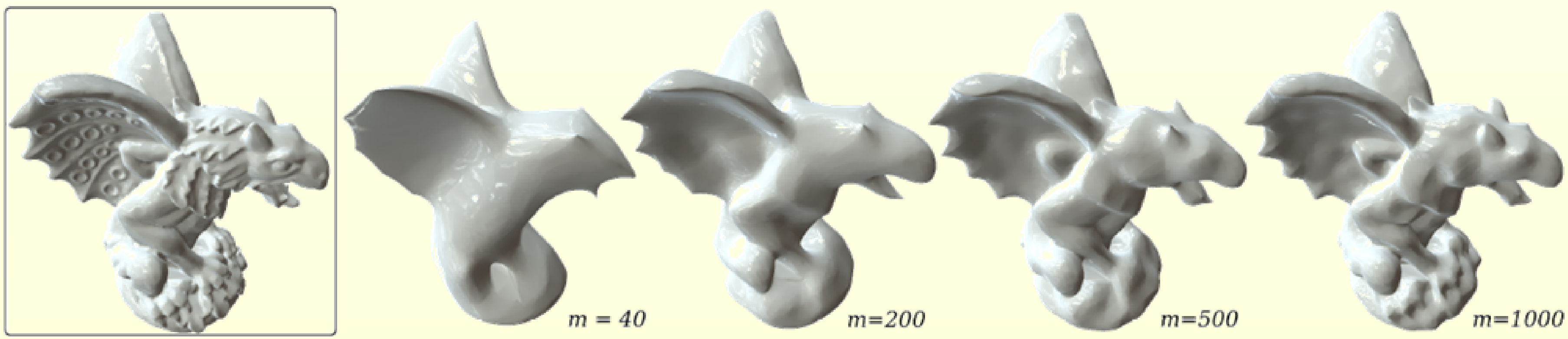
$$\mathbf{L} \quad \mathbf{V}\mathbf{D}\mathbf{V}^T \qquad \mathbf{V} \quad \begin{matrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \\ | & | & & | \end{matrix} \quad , \quad \mathbf{D} \quad \begin{matrix} k_1 & & & \\ & k_2 & & \\ & & \ldots & \\ & & & k_n \end{matrix}$$

$\mathbf{v}_2$ $\mathbf{v}_{50}$

From Vallet et al., Eurographics 2008

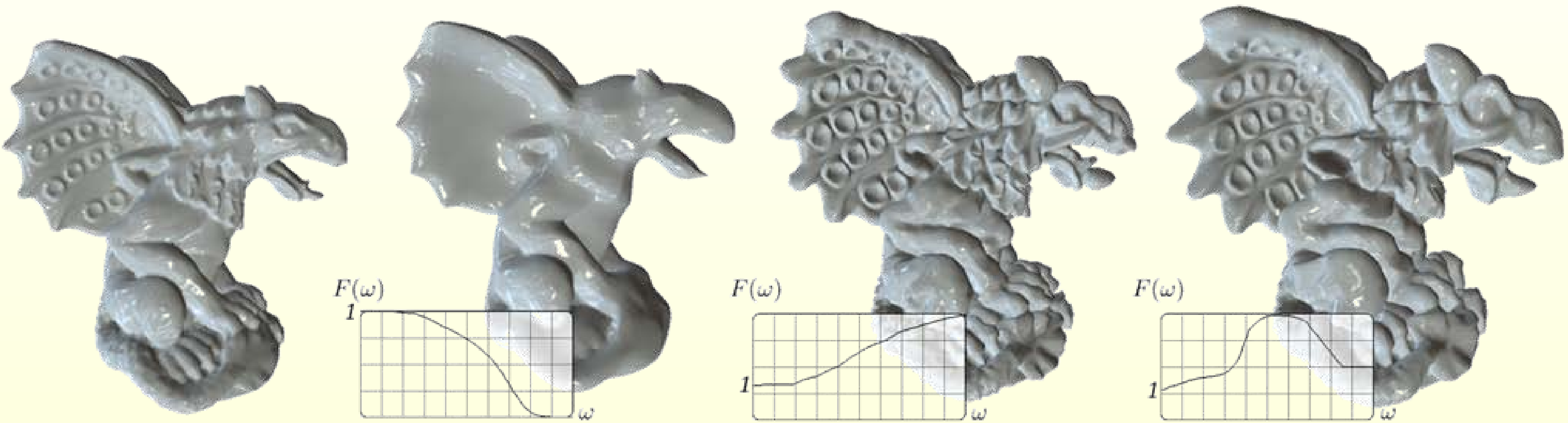
Smoothing using the
Laplacian Eigen-decomposition

$$\mathbf{P}^{smooth} \quad \mathbf{V} \; \mathbf{D}_m \; \mathbf{V}^T \mathbf{P} \quad , \quad \mathbf{D}_m \quad \begin{matrix} k_1 & & & \\ & \cdots & & \\ & & k_m & \\ & & & 0 \end{matrix}$$



From Vallet et al., Eurographics 2008

Geometry Filtering



Demo: http://alice.loria.fr/index.php/software/9-demo/39-manifold-harmonics-demo.html

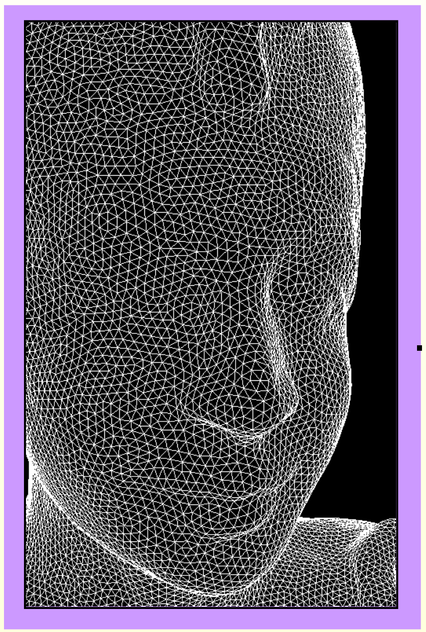
From Vallet et al., Eurographics 2008

Mesh Processing Pipeline

Scan

Reconstruct

Clean



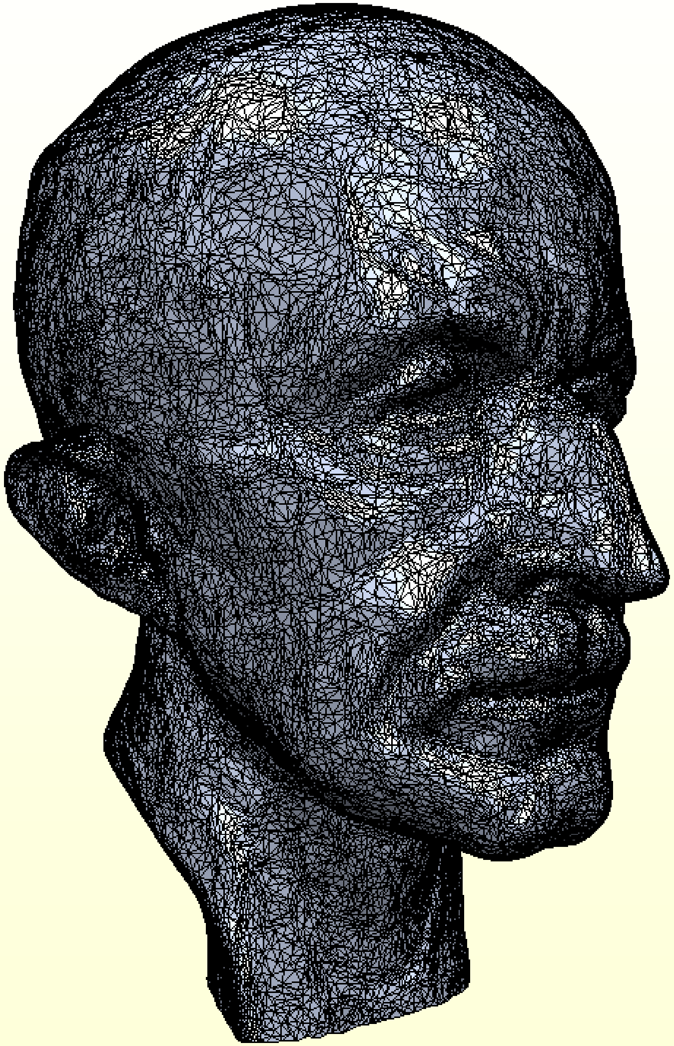
Remesh

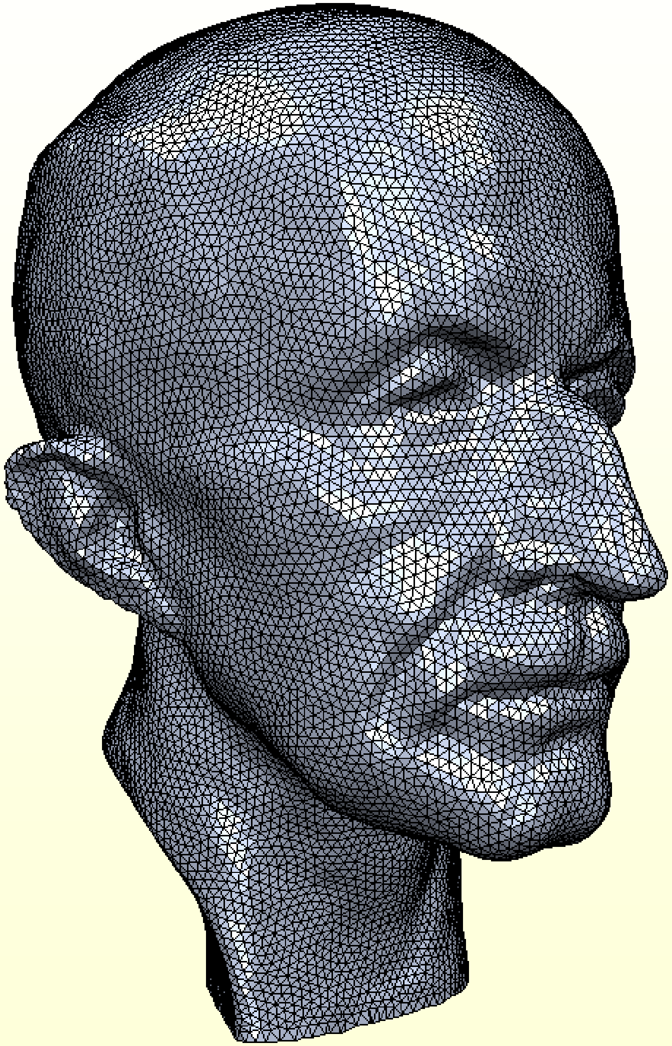
...

Remeshing

"Given a 3D mesh, improve its triangulation While preserving its geometry."

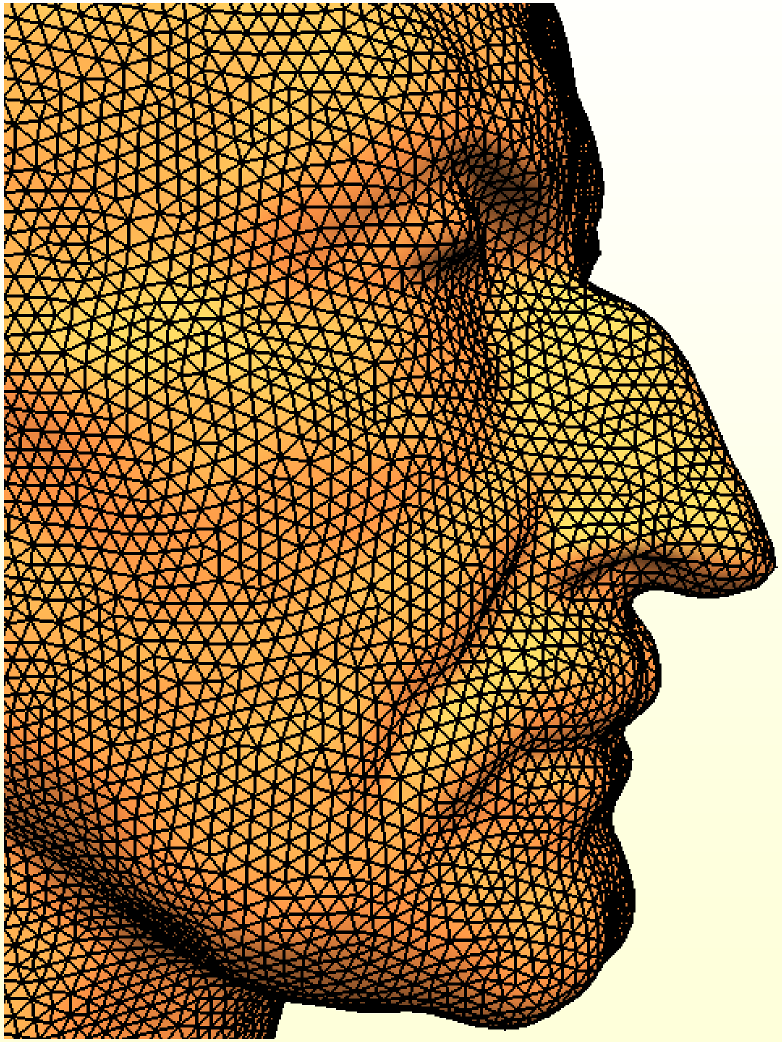
What is a good mesh?





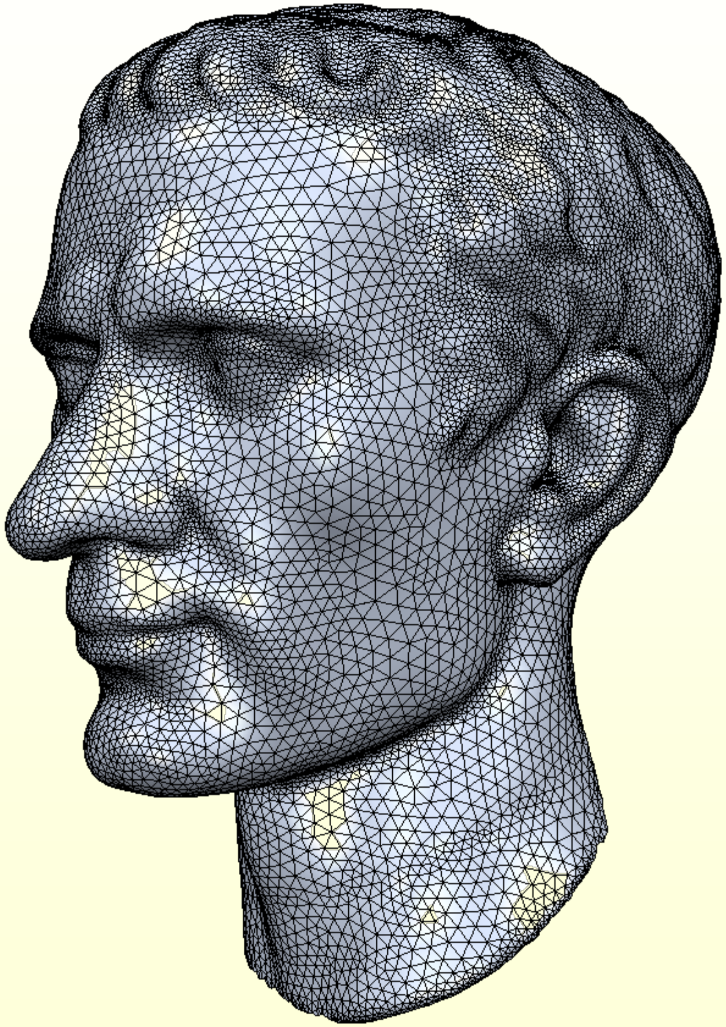
What is a good mesh?

- Equal edge lengths
- Equilateral triangles
- Valence close to 6



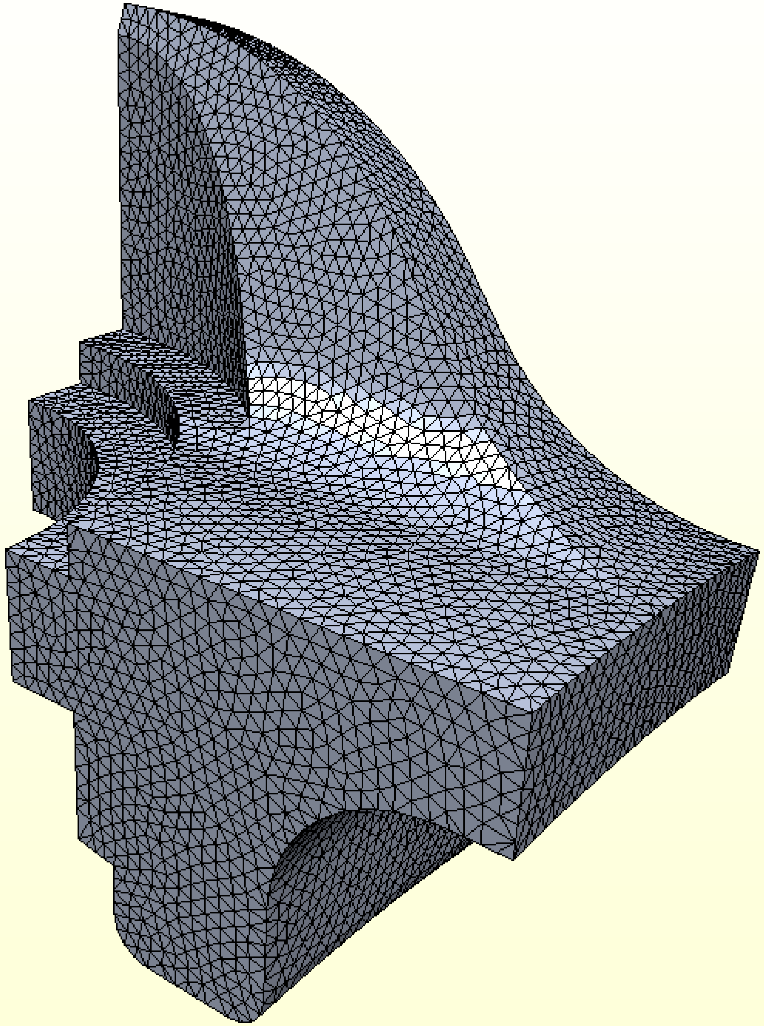
What is a good mesh?

- Equal edge lengths
- Equilateral triangles
- Valence close to 6
- Uniform vs. adaptive sampling



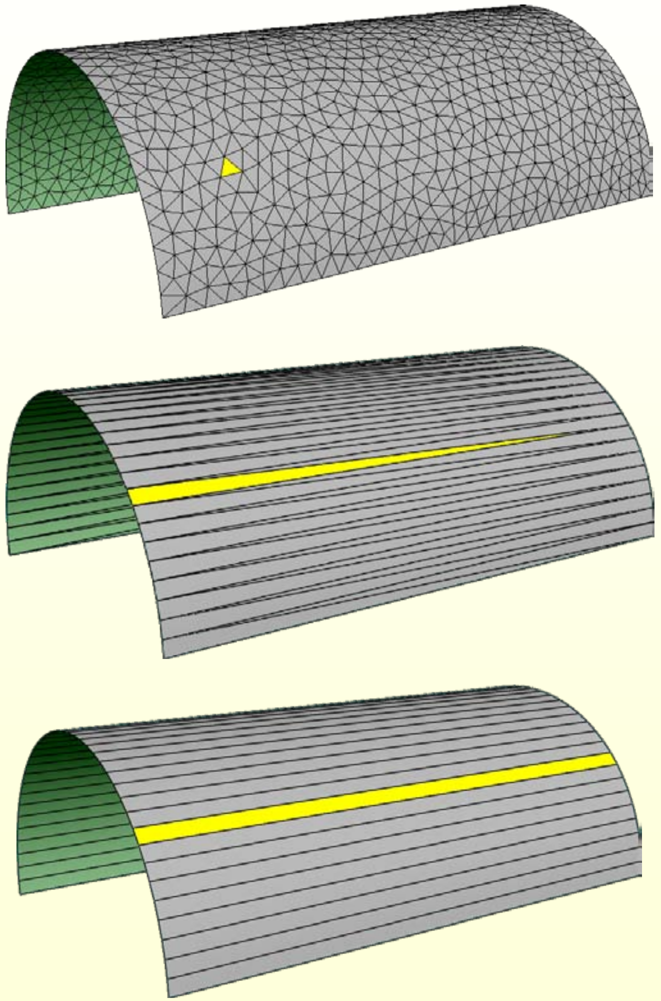
What is a good mesh?

- Equal edge lengths
- Equilateral triangles
- Valence close to 6
- Uniform vs. adaptive sampling
- Feature preservation



What is a good mesh?

- Equal edge lengths
- Equilateral triangles
- Valence close to 6
- Uniform vs. adaptive sampling
- Feature preservation
- Alignment to curvature lines
- Isotropic vs. anisotropic
- Triangles vs. quadrangles



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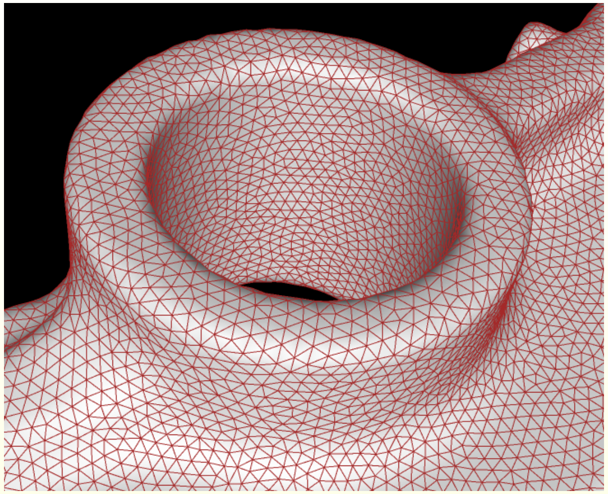
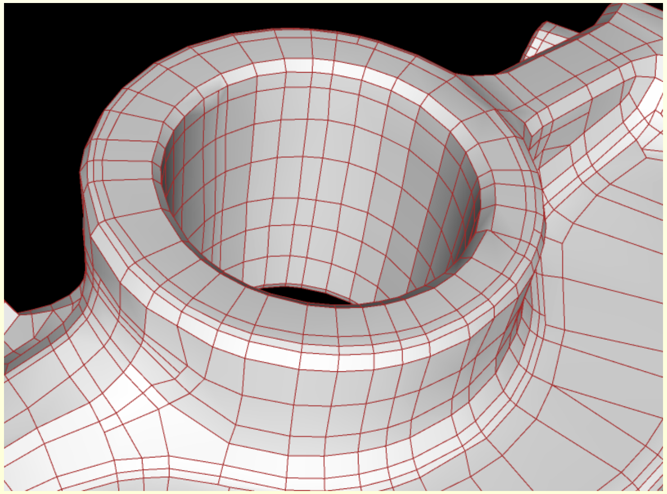


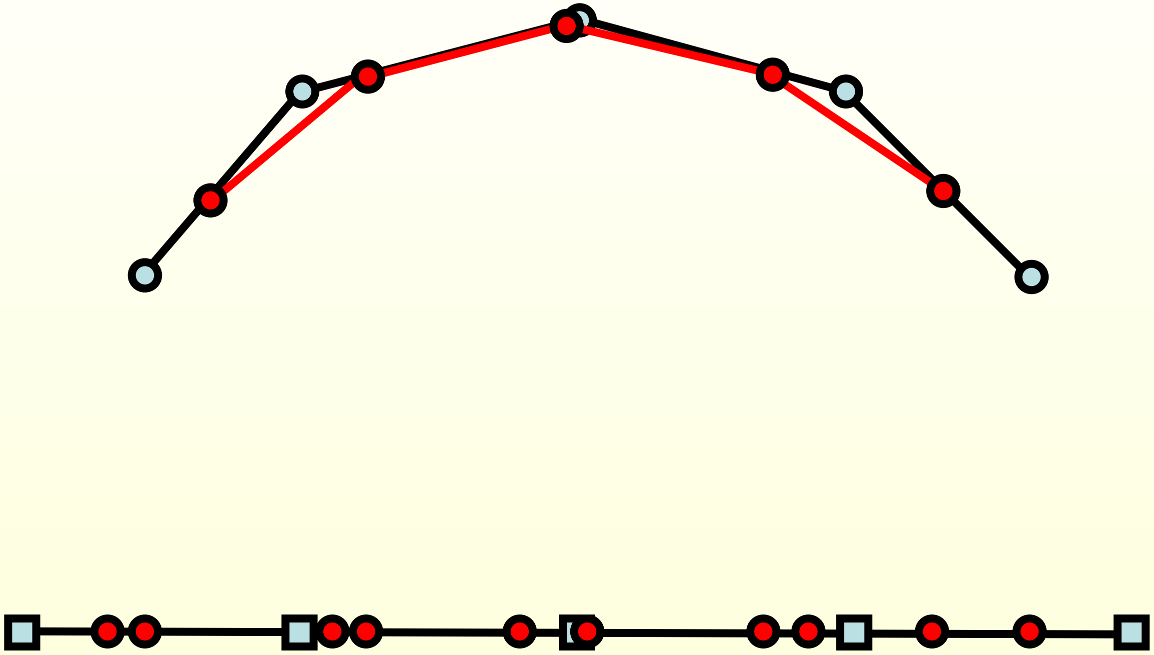


Two Fundamental Approaches

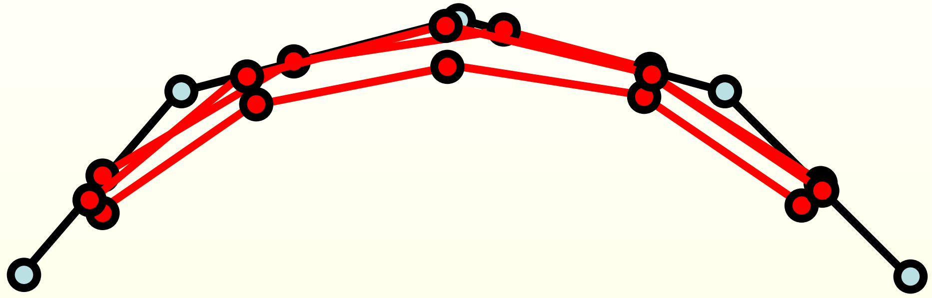
- Parametrization based
  - map to 2D domain / 2D problem
  - computationally more expensive
  - works even for coarse resolution remeshing

- Surface oriented
  - operate directly of the surface
  - treat surface as a set of points/polygons in space
  - efficient for high resolution remeshing

Parametrization Based



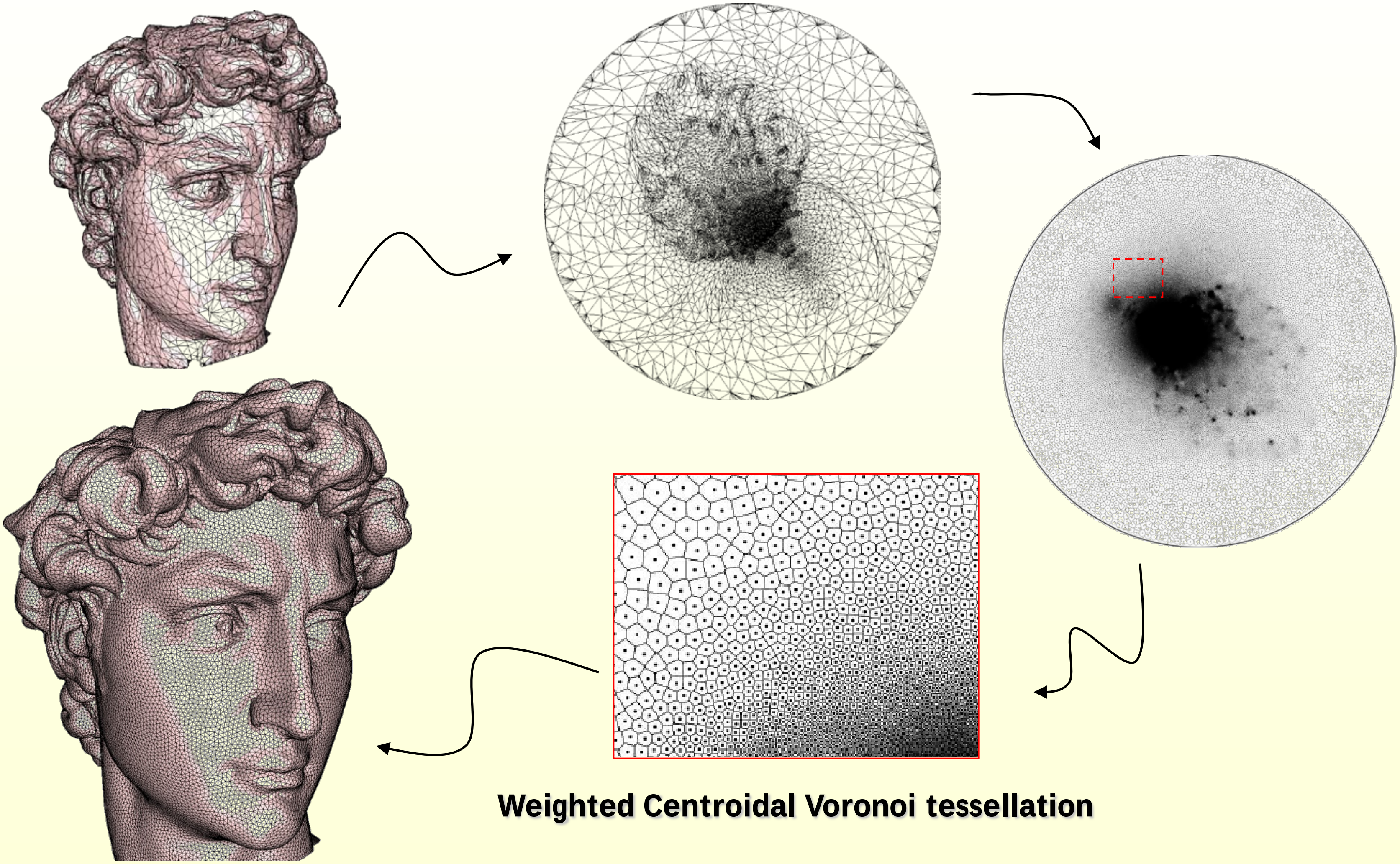
Surface Oriented



Overview

- Isotropic Remeshing
  - **Parameterization-based** **[Alliez 2002]**
  - Surface-based [Botsch 2004]
- Anisotropic Remeshing
  - Parameterization-based [Alliez 2003]
  - Surface-based [Marinov 2004]

Alliez: Isotropic Remeshing



Weighted Centroidal Voronoi tessellation

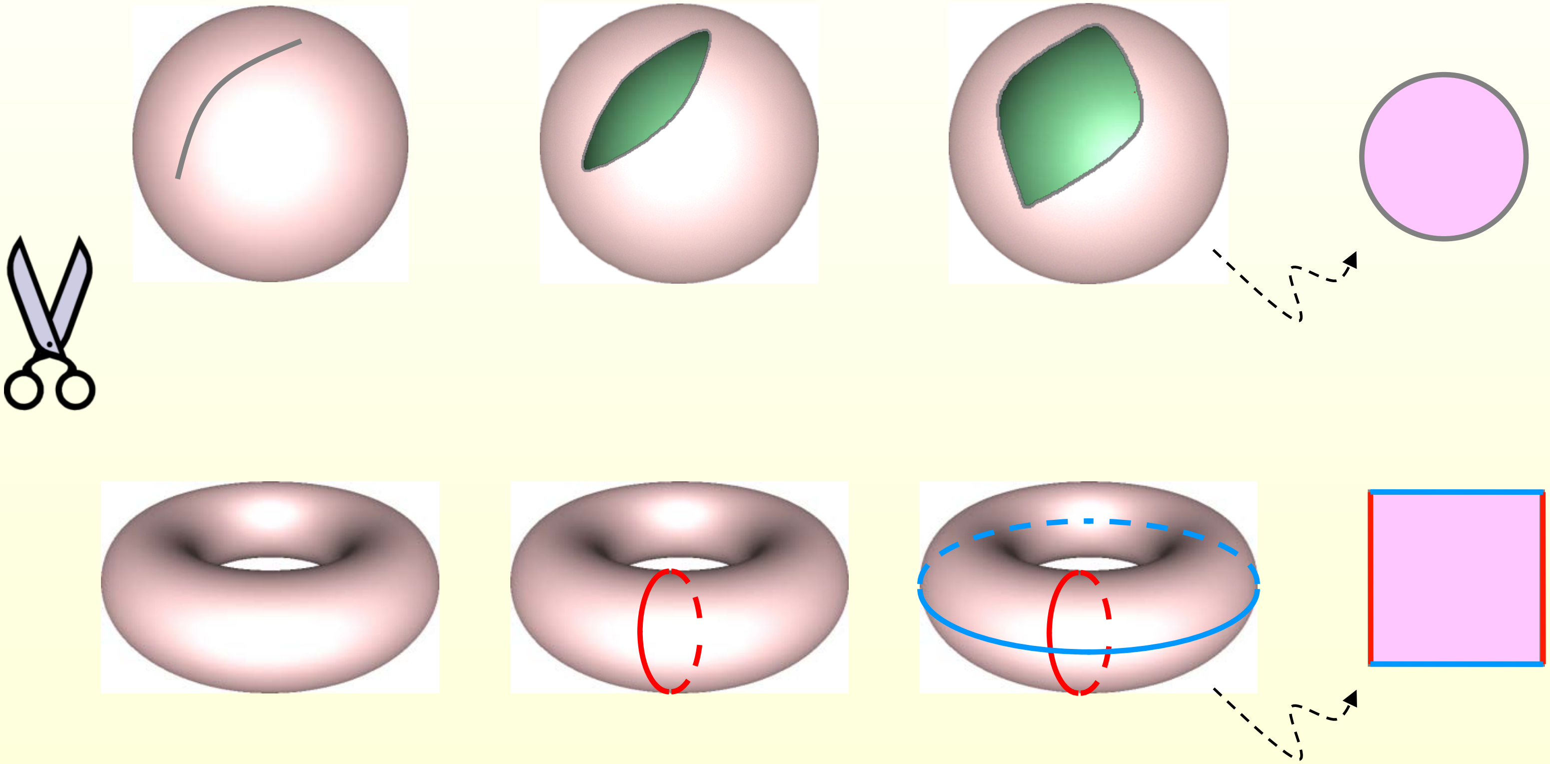
Use a Global Parameterization

- Motivation: 2D remeshing is much easier
  - Sample distribution
  - Delaunay triangulation
  - Centroidal Voronoi diagram

- Which parameterization method to choose?

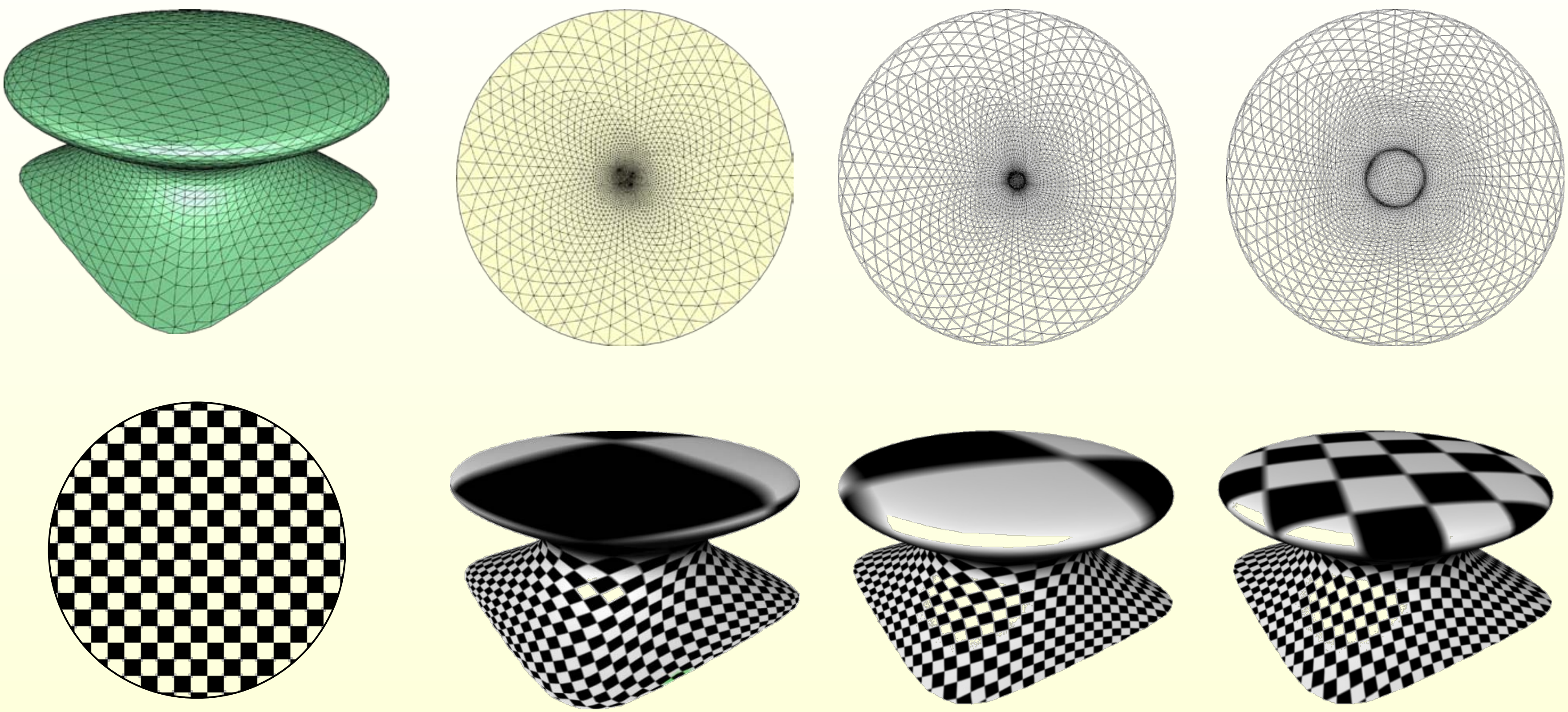
Need disk-like topology

- Introduce cuts on the mesh



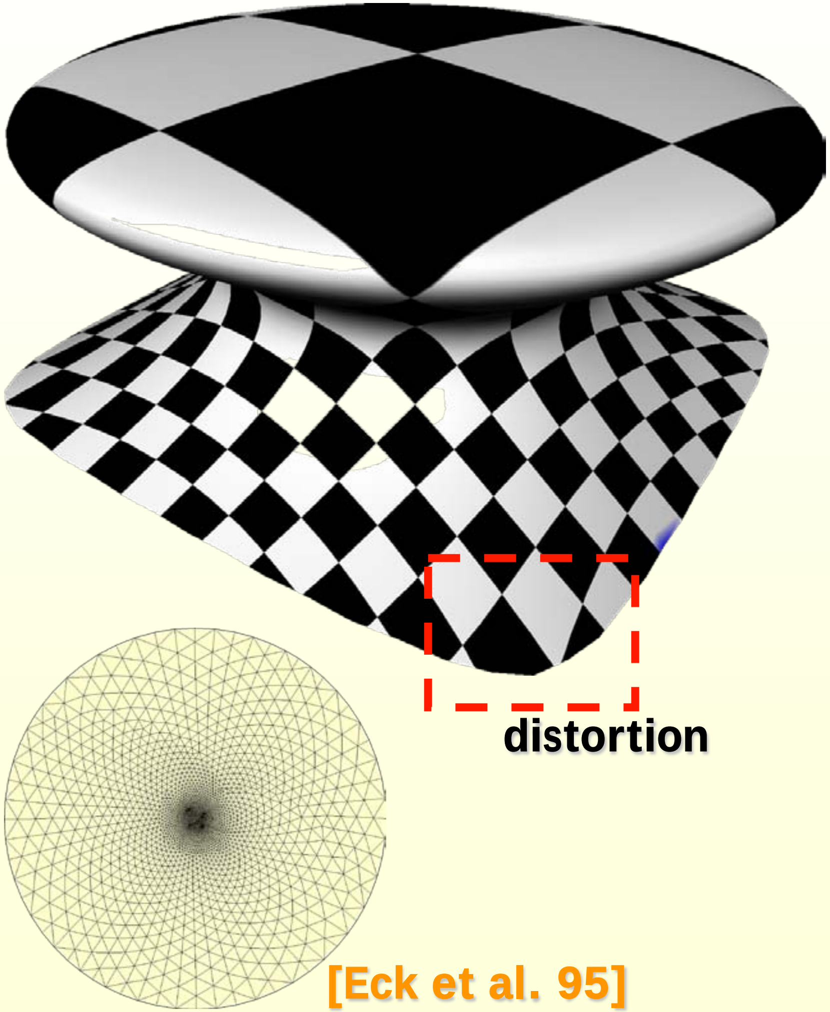
Parameterization Methods

Tutte | **Shape-preserving** | **Conformal**

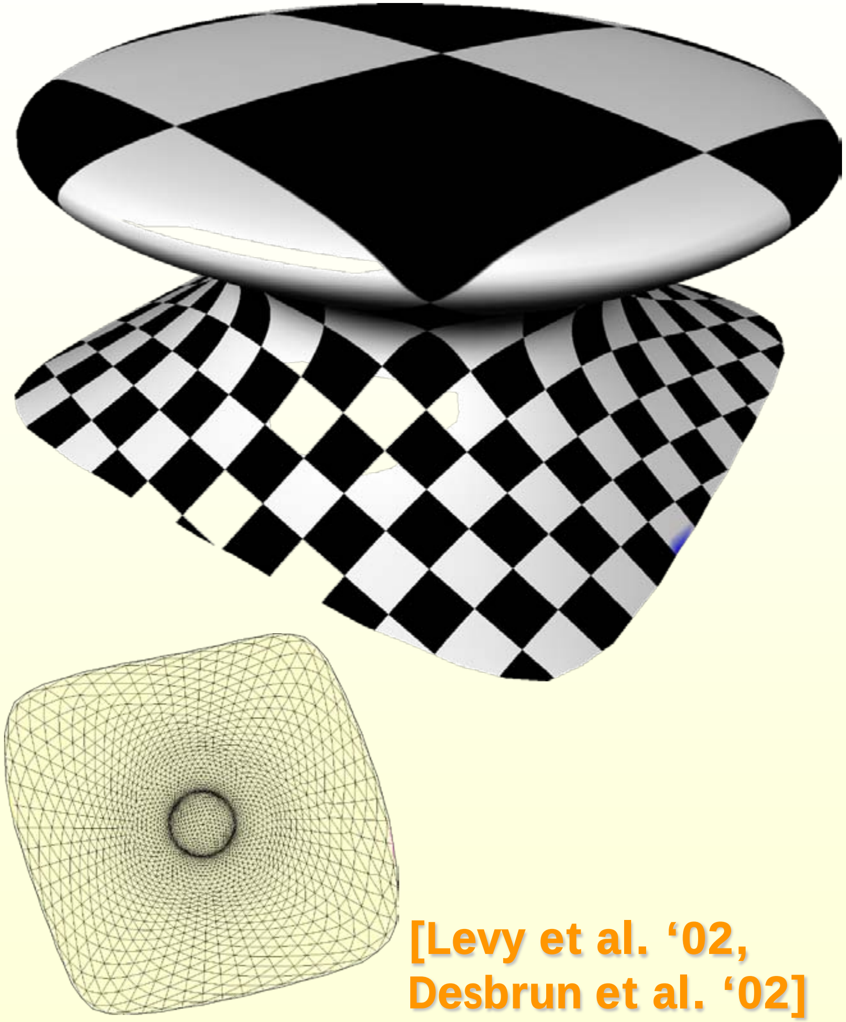


Texture map | [Tutte 63] | [Floater 97] | [Eck et al. 95]

Conformal: Fixed vs. Free Boundary

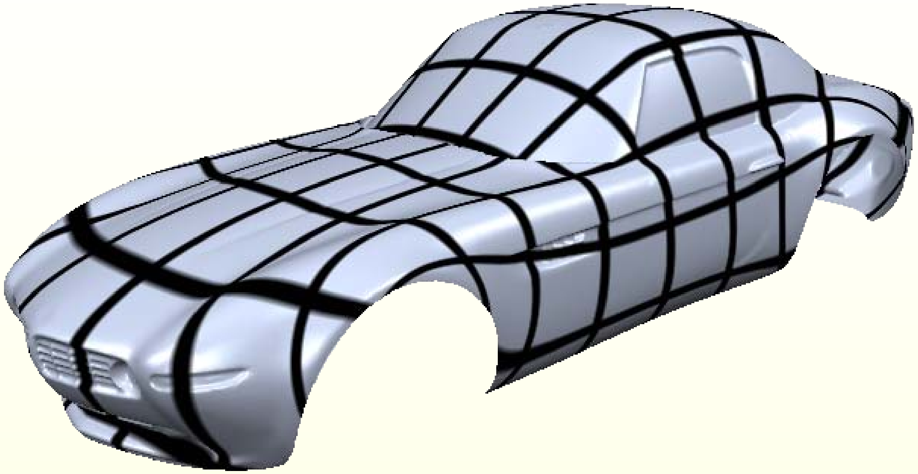


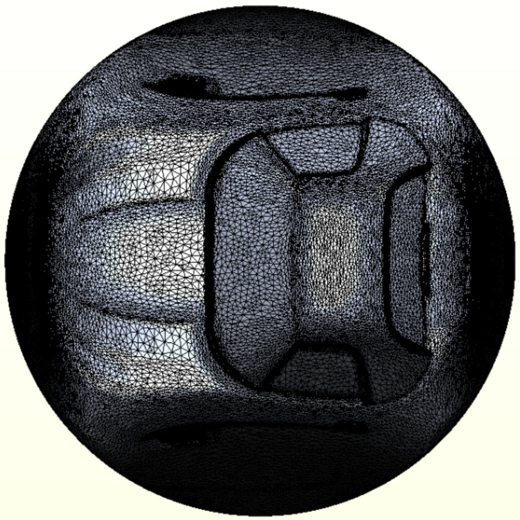
[Eck et al. 95]

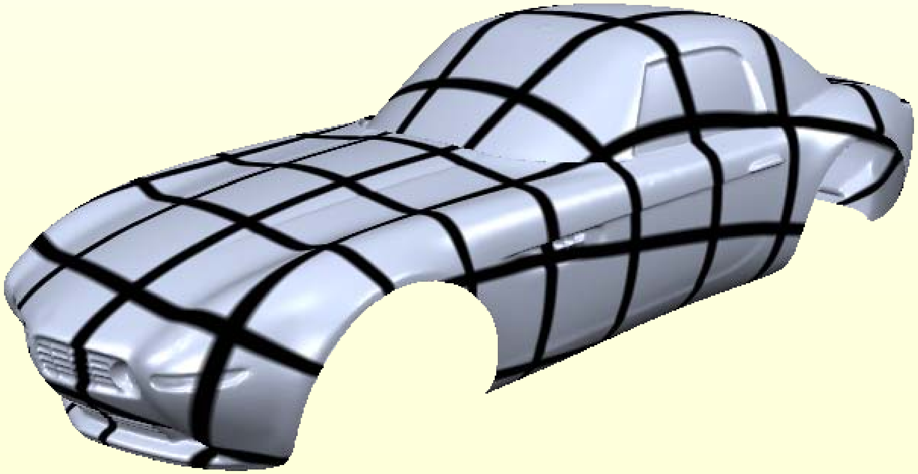


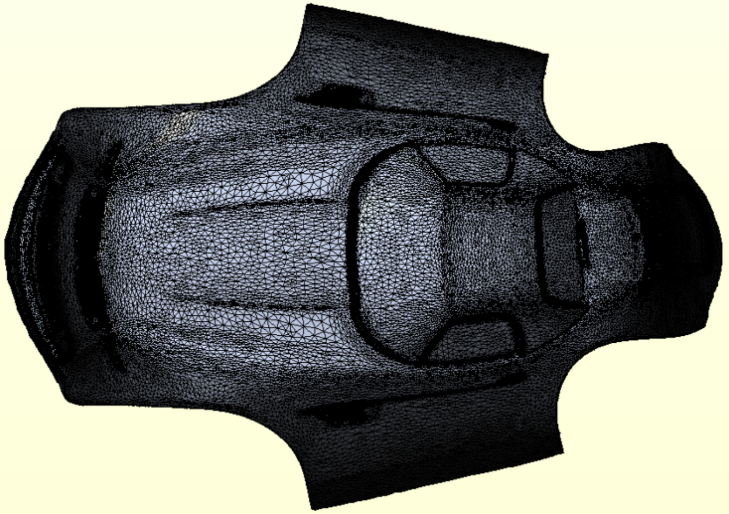
[Levy et al. ‘02, Desbrun et al. ‘02]

Conformal: Fixed vs. Free Boundary







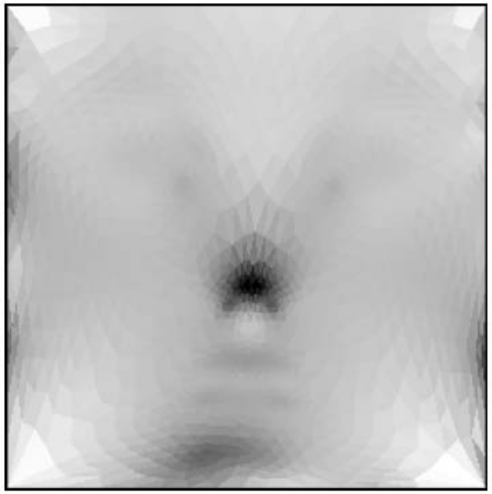


Initial Sampling

- Randomly sample triangles
  - Weighted by area and density
  - Density: curvature or user-defined sizing field
- Compensate area distortion when sampling in the parameter domain
  - Distortion = 3D area / 2D area

Initial Sampling [Alliez02]

- Compose importance map



Area stretch

•

Mean curvature

=

Importance map

Initial Sampling [Alliez02]

- 2D error diffusion on importance map
  - Half-toning, dithering
  - Can also be done on 3D meshes [Alliez03]

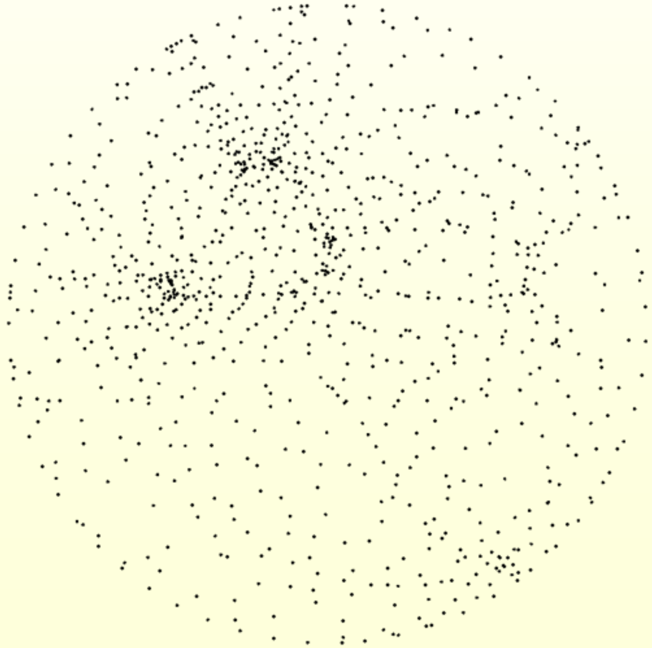

Importance map

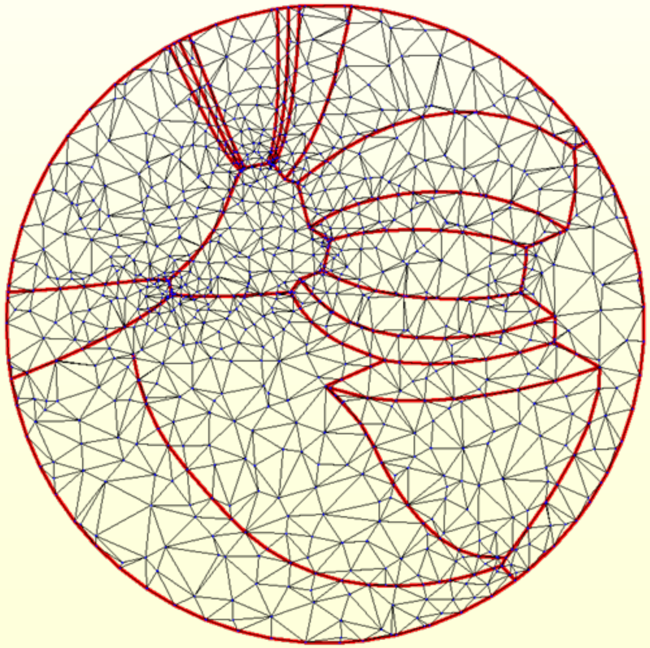

8k samples


30k samples

Initial Meshing

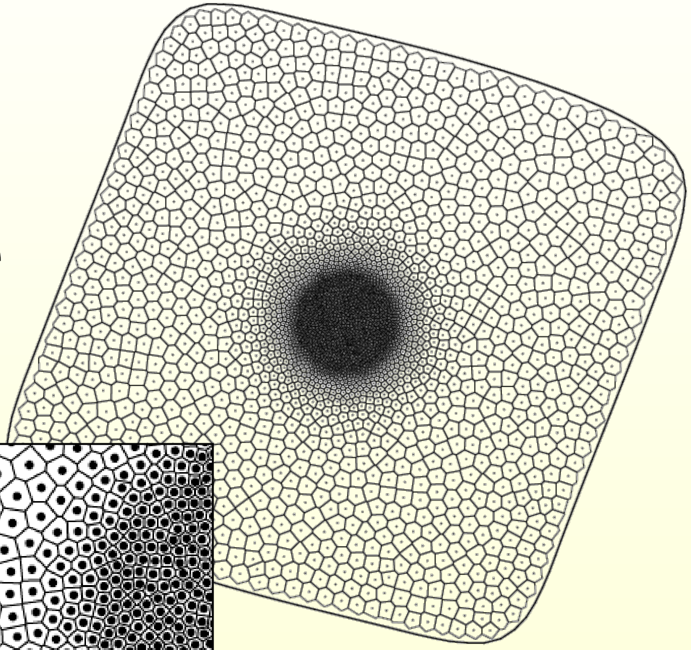
- 2D constrained Delaunay triangulation
- CGAL library provides robust implementation

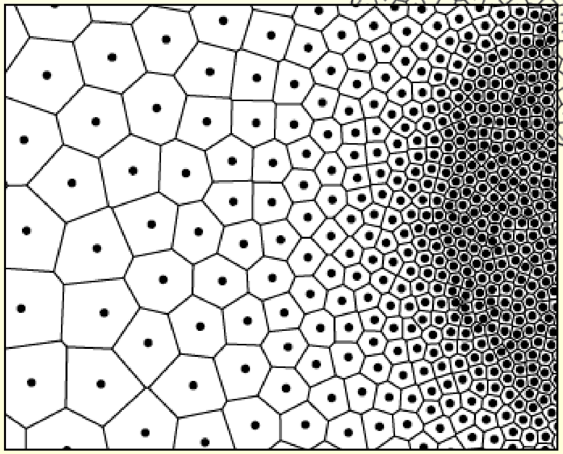




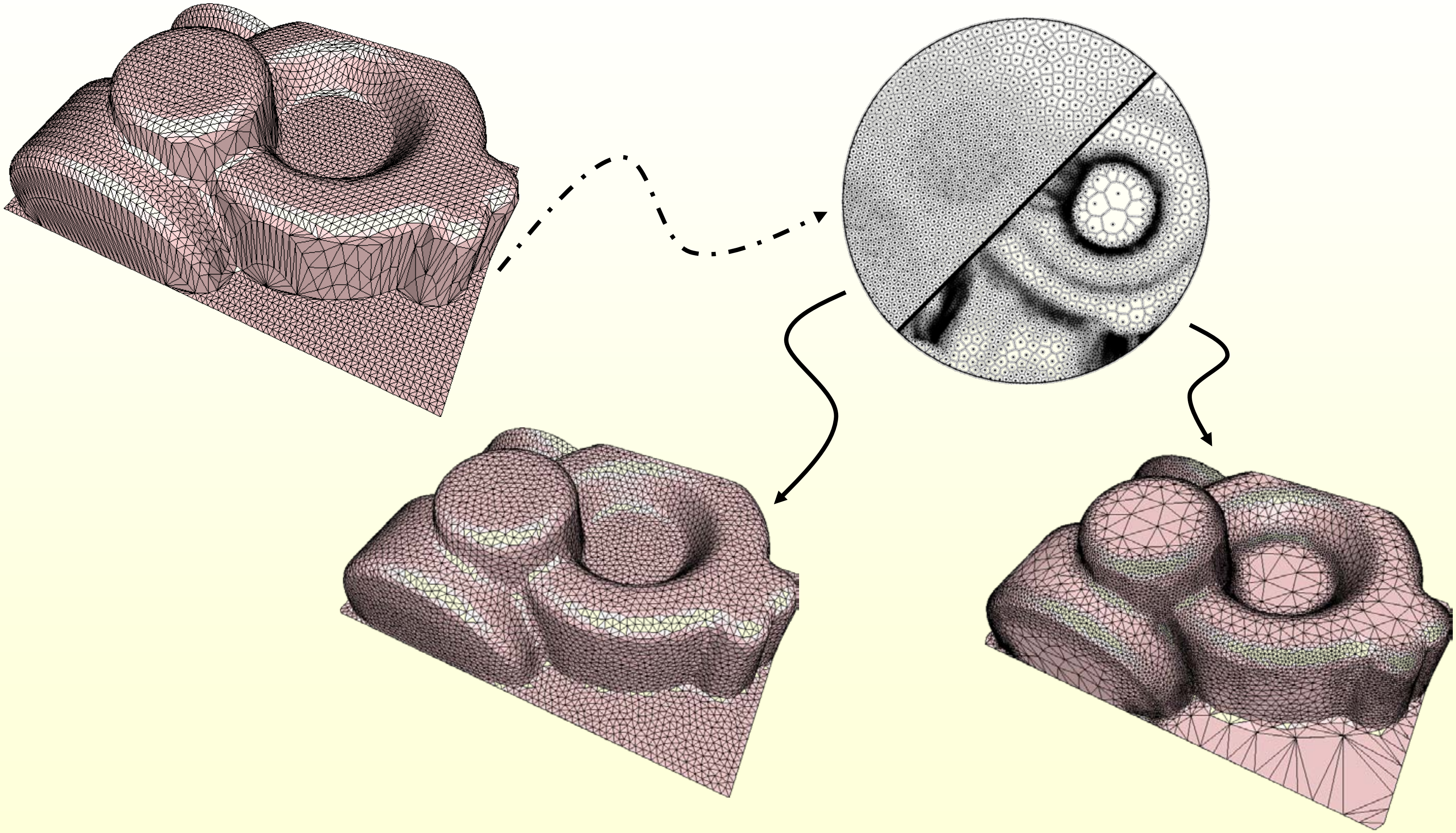
Optimize Sampling / Meshing

- Density-weighted centroidal Voronoi diagram
  - Equal mass enclosing
  - Tiles as compact as possible
  - Highly isotropic sampling
  - Lloyd clustering

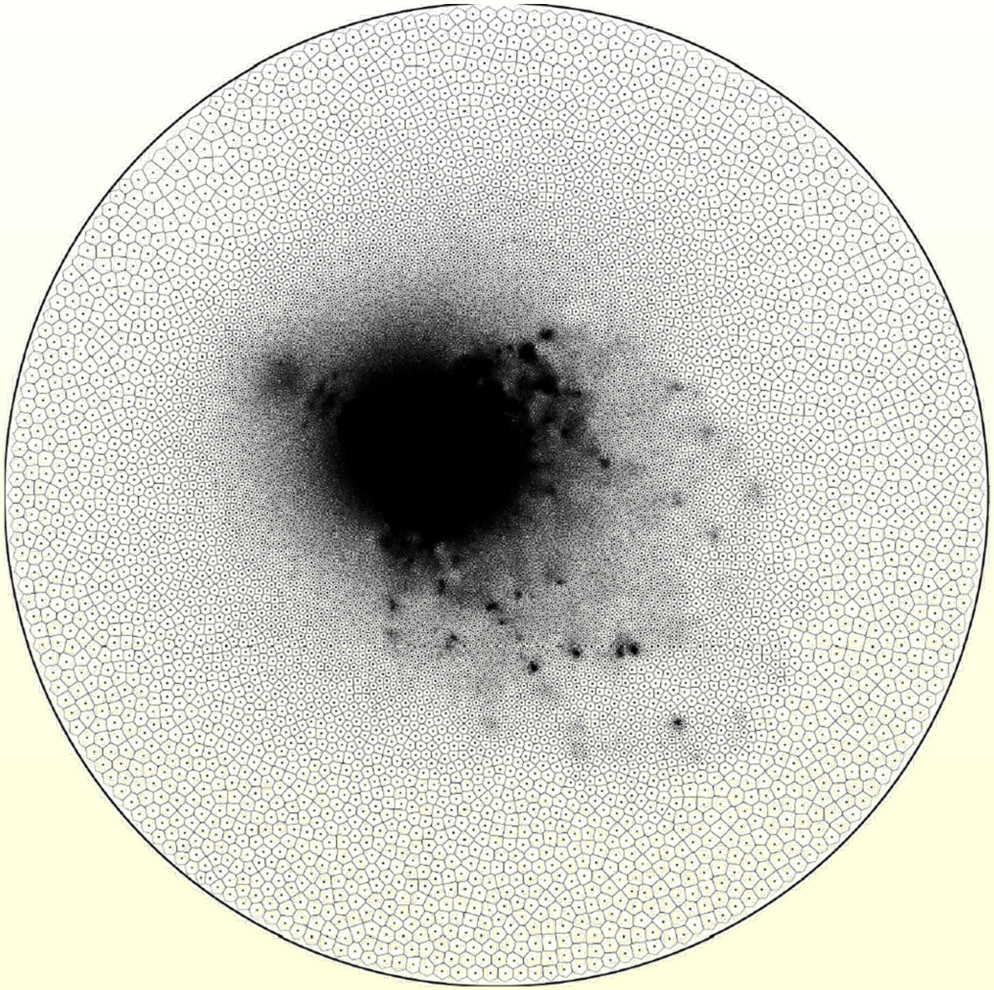




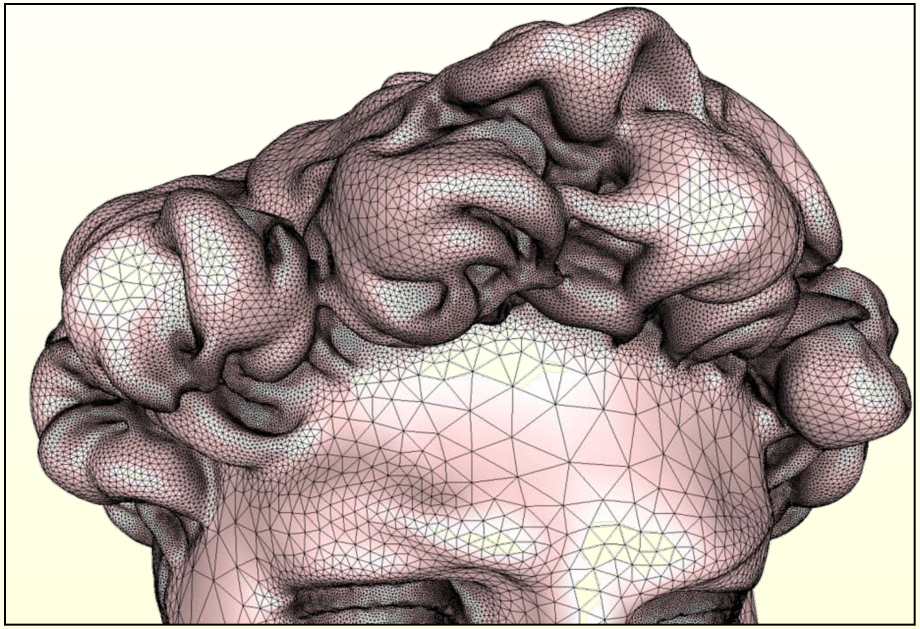
Uniform vs. Adaptive

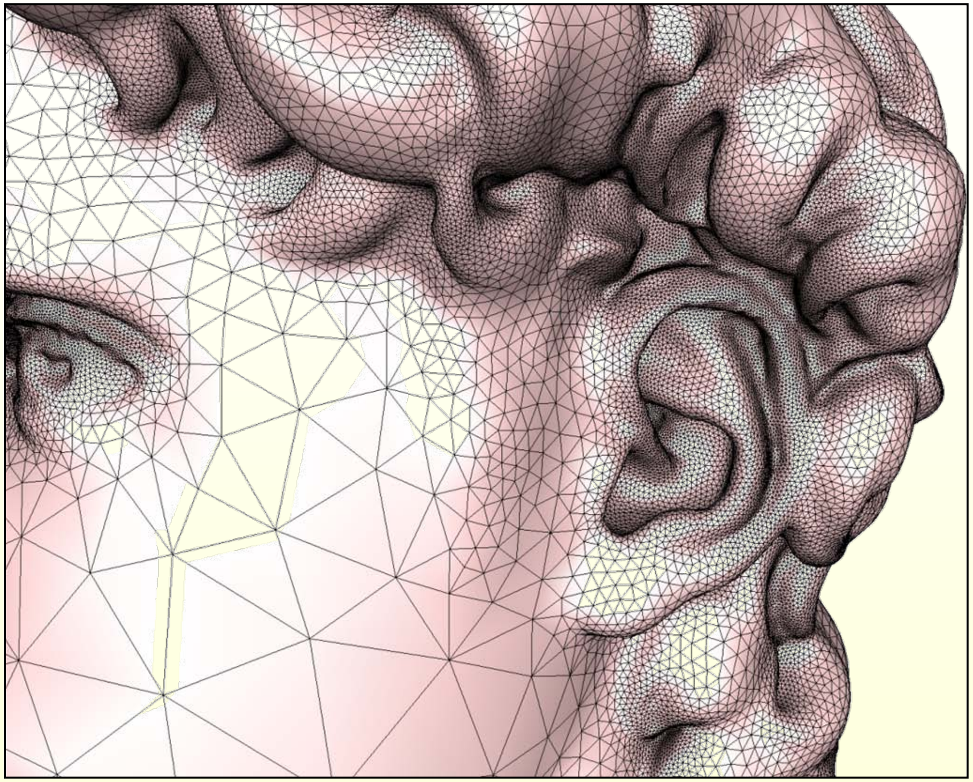


Uniform Sampling



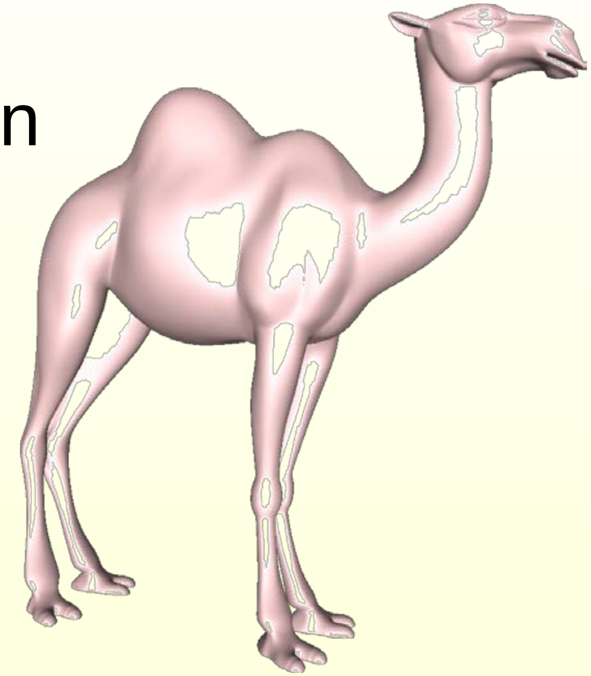
Adaptive Sampling



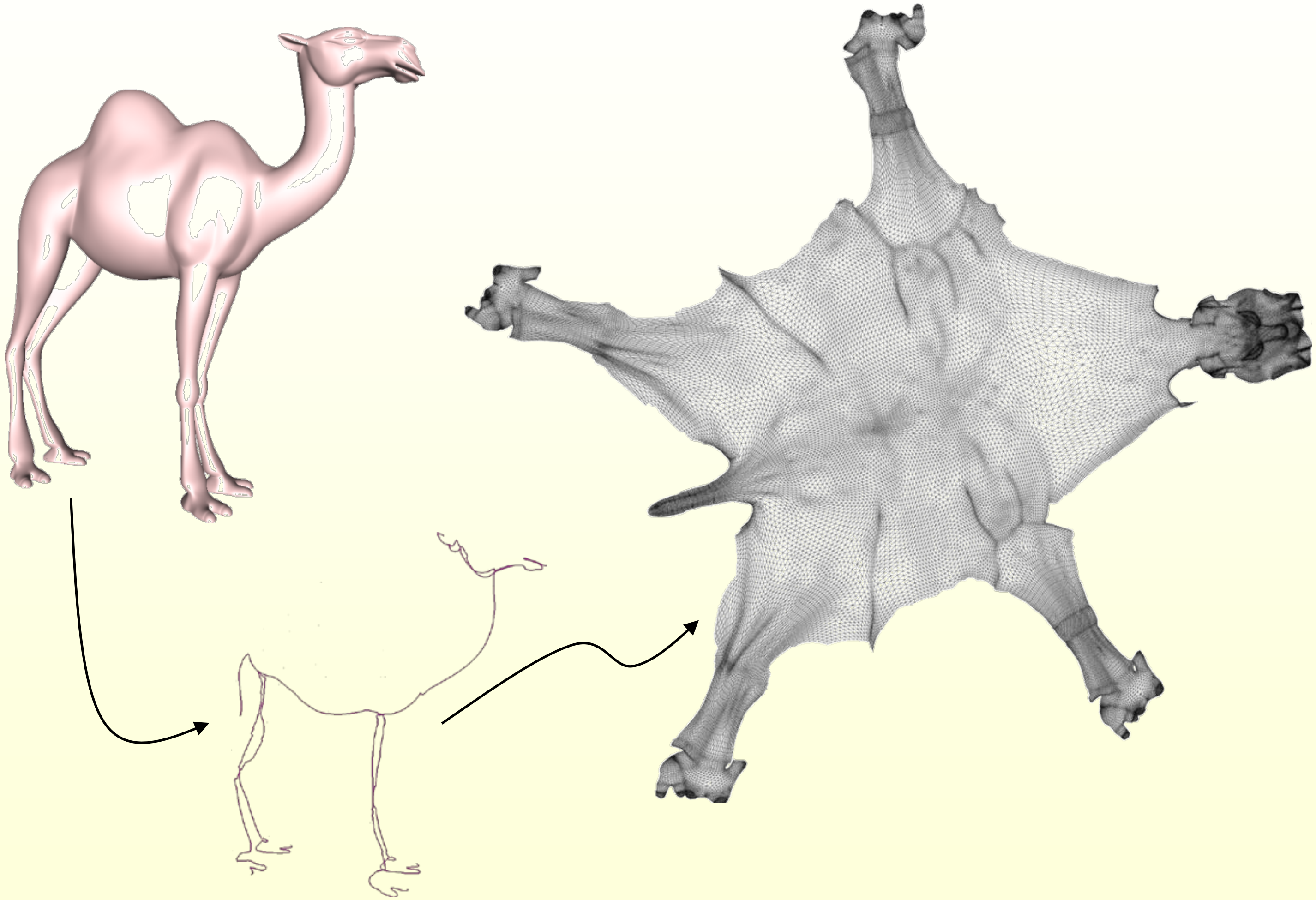


Limitations

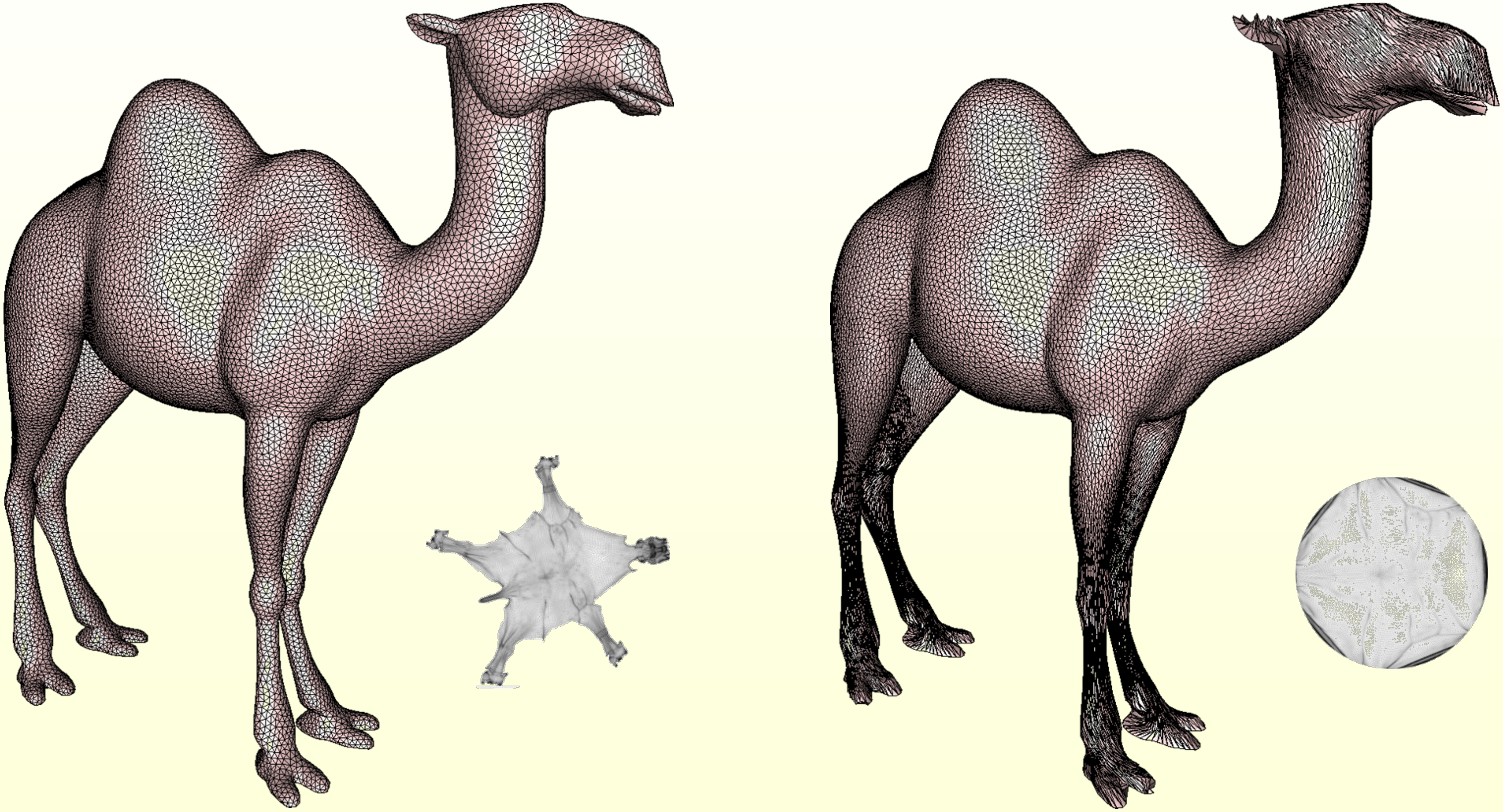
- Closed genus 0
  - Need a good cut
  - Free boundary parameterization
  - Stitch seams afterwards
- Protruding legs
  - Sampling
  - Numerical problems



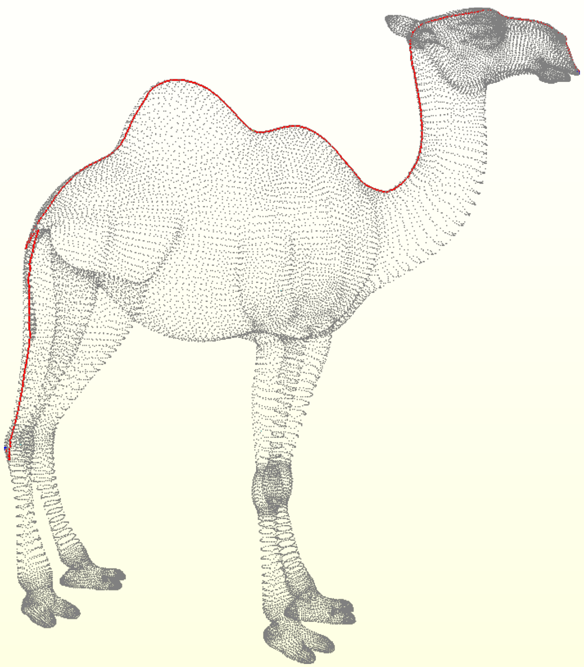
Smart Cut, Free Boundary

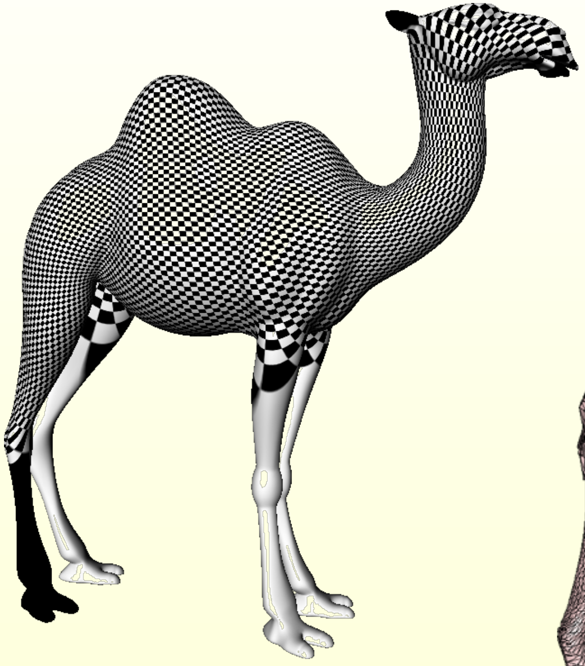


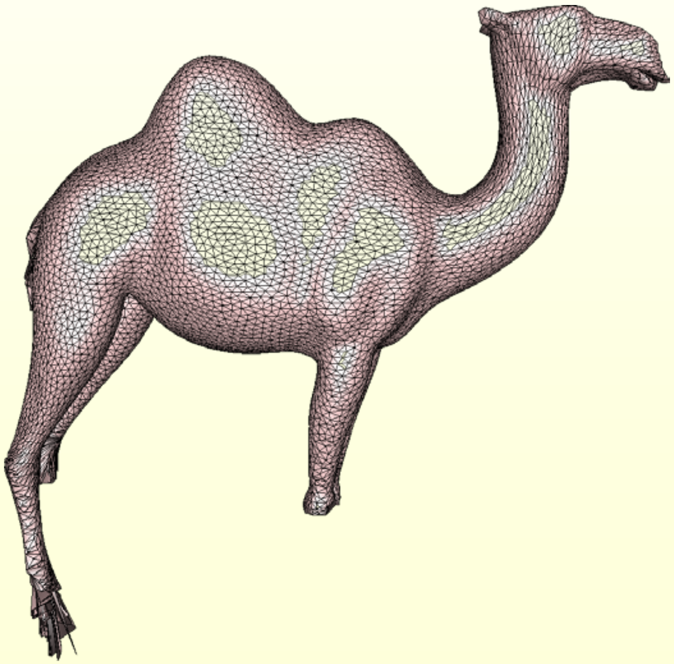
Free vs. Fixed Boundary



Naive Cut, Numerical Problems





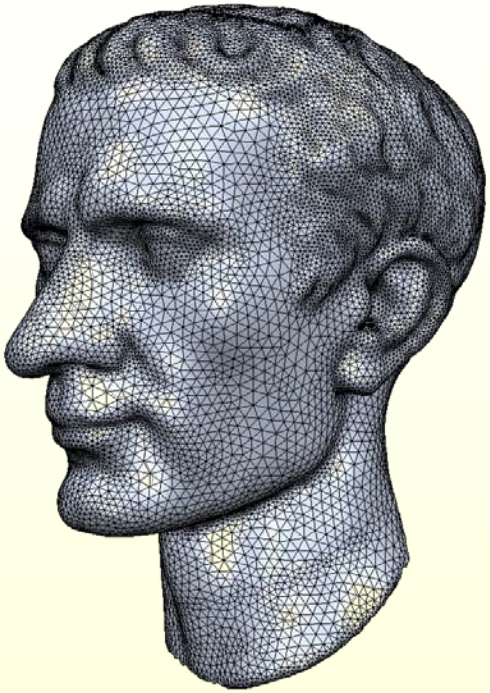


Overview

- Isotropic Remeshing
  - Parameterization-based [Alliez 2002]
  - **Surface-based [Botsch 2004]**
- Anisotropic Remeshing
  - Parameterization-based [Alliez 2003]
  - Surface-based [Marinov 2004]

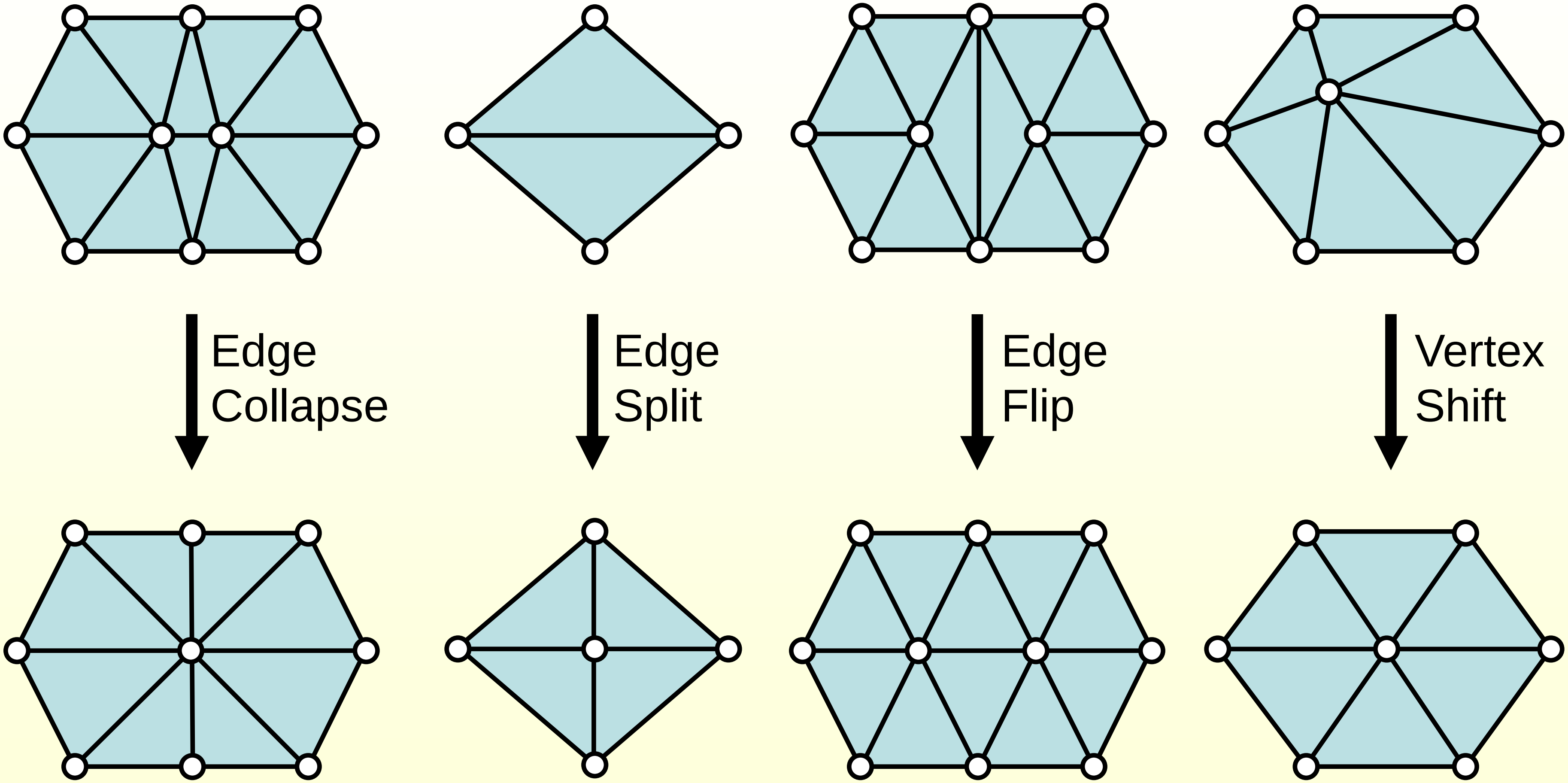
Direct Surface Remeshing

- Avoid global parameterization
  - Numerically very sensitive
  - Topological restrictions



- Avoid local parameterizations
  - Expensive computations
- Use local operators & back-projections
  - Resampling of 100k triangles in < 5s

Local Remeshing Operators



Isotropic Remeshing

Specify target edge length L

Iterate:

1. Split edges longer than L_{max}
2. Collapse edges shorter than L_{min}
3. Flip edges to get closer to valence 6
4. Vertex shift by tangential relaxation
5. Project vertices onto reference mesh

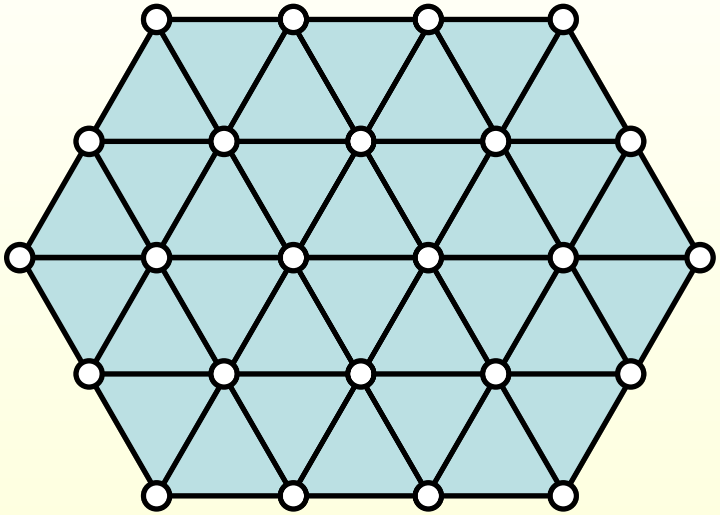
Edge Flip

Improve valences

- Avg. valence is 6 (Euler)
- Reduce variation

Optimal valence is

- 6 for interior vertices
- 4 for boundary vertices



Edge Flip

Improve valences

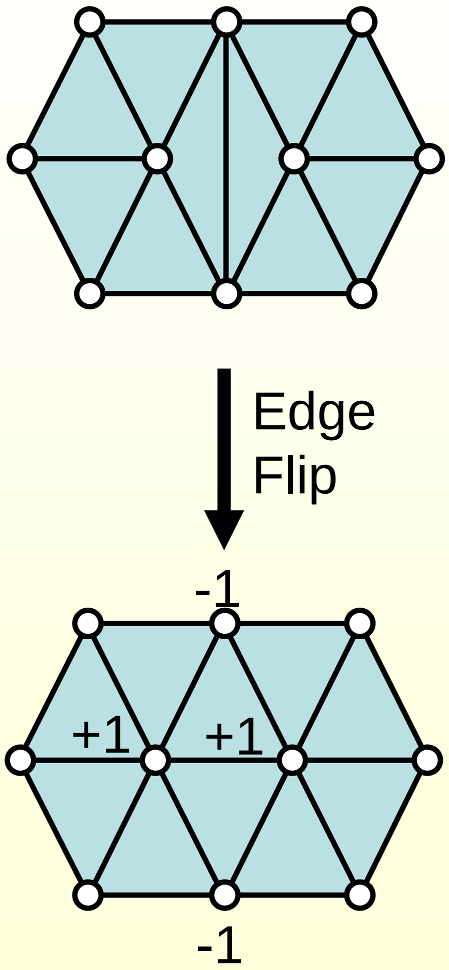
- Avg. valence is 6 (Euler)
- Reduce variation

Optimal valence is

- 6 for interior vertices
- 4 for boundary vertices

Minimize valence excess

$$\sum_{i=1}^{4} (\text{valence}(v_i) - \text{opt\_valence}(v_i))^2$$

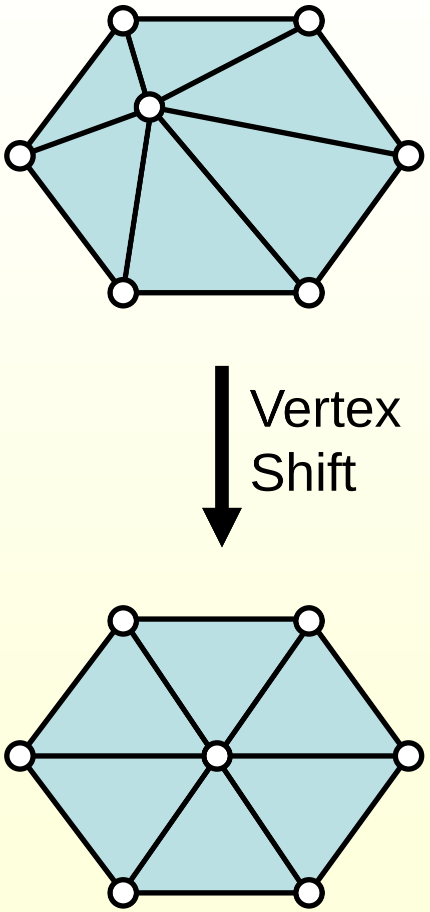


Vertex Shift

Local “spring” relaxation

- Uniform Laplacian smoothing
- Barycenter of one-ring neighbors

$$\mathbf{c}_i = \frac{1}{\mathrm{valence}\,(v_i)} \sum_{j \in N(v_i)} \mathbf{p}_j$$



Vertex Shift

Local “spring” relaxation

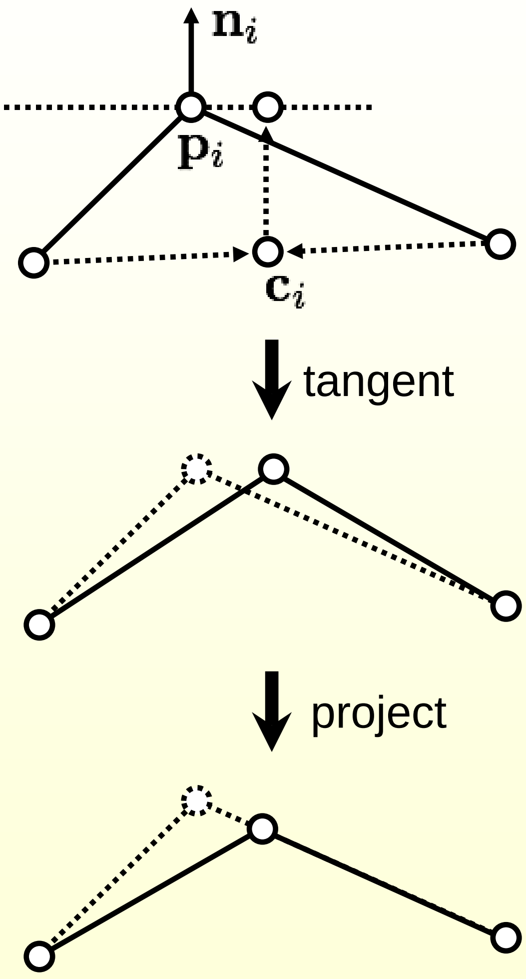
- Uniform Laplacian smoothing
- Barycenter of one-ring neighbors

$$\mathbf{c}_i = \frac{1}{\text{valence}(v_i)} \sum_{j \in N(v_i)} \mathbf{p}_j$$

Keep vertex (approx.) of surface

- Restrict movement to tangent plane

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \left(I - \mathbf{n}_i \mathbf{n}_i^T \right) (\mathbf{c}_i - \mathbf{p}_i)$$



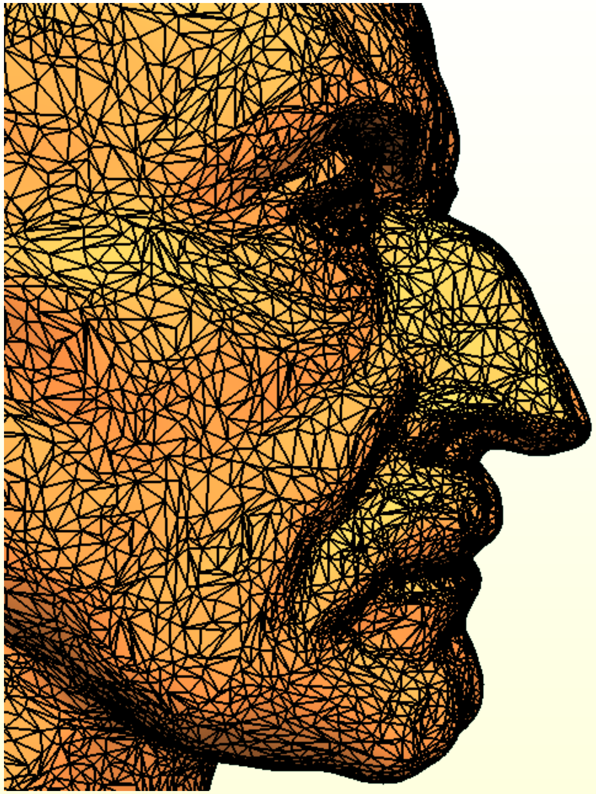
Isotropic Remeshing

Specify target edge length L

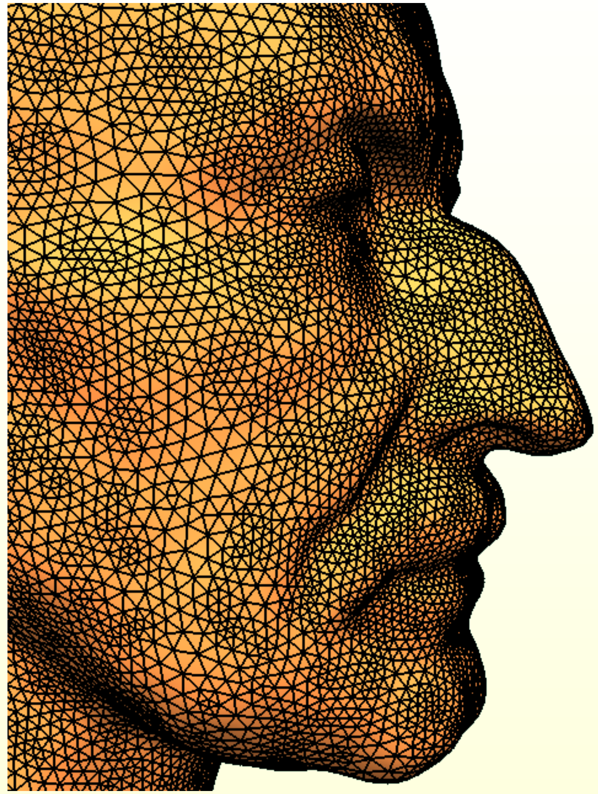
Iterate:

1. Split edges longer than L_{max}
2. Collapse edges shorter than L_{min}
3. Flip edges to get closer to valence 6
4. Vertex shift by tangential relaxation
5. Project vertices onto reference mesh

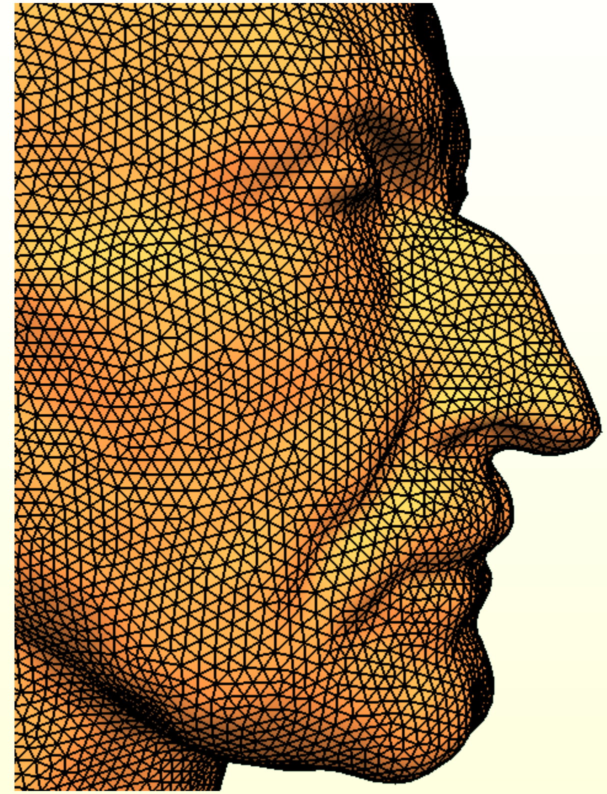
Remeshing Results



Original

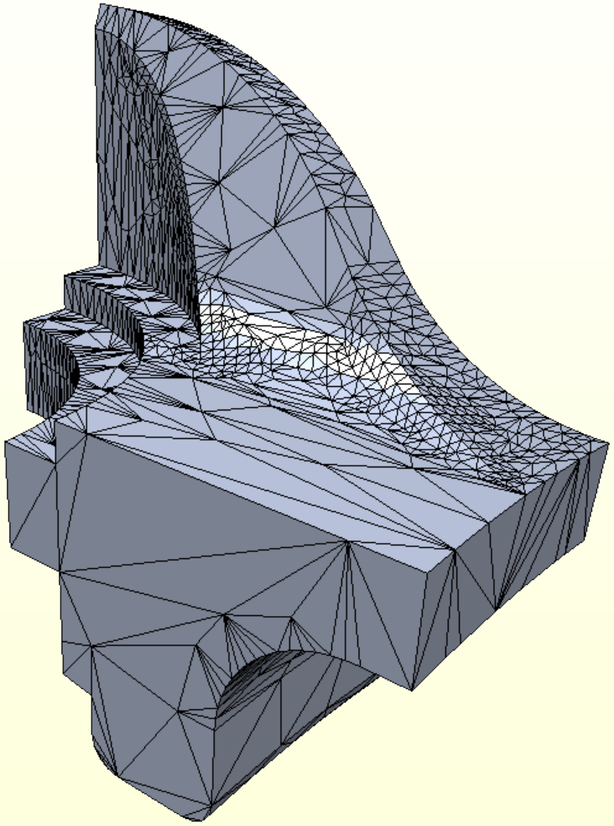


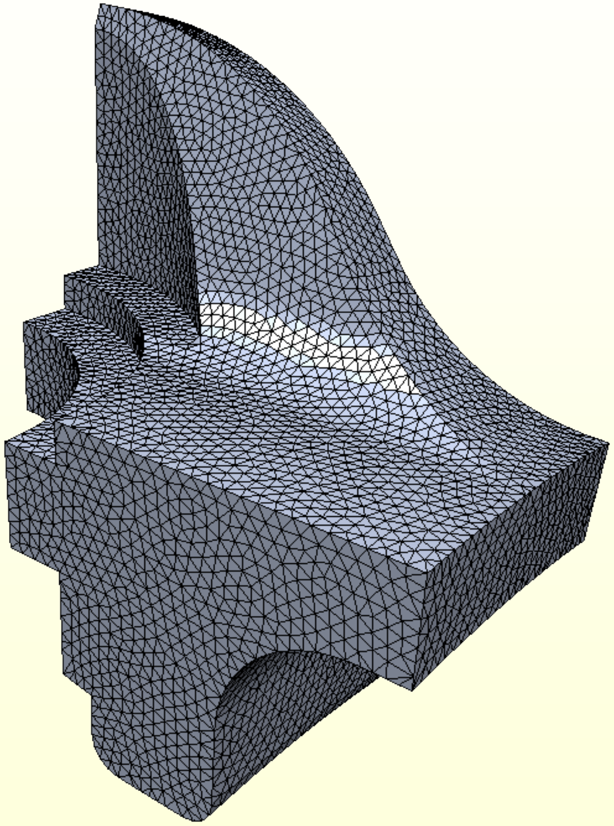
$\left(\frac{1}{2}, 2\right)$



$\left(\frac{4}{5}, \frac{4}{3}\right)$

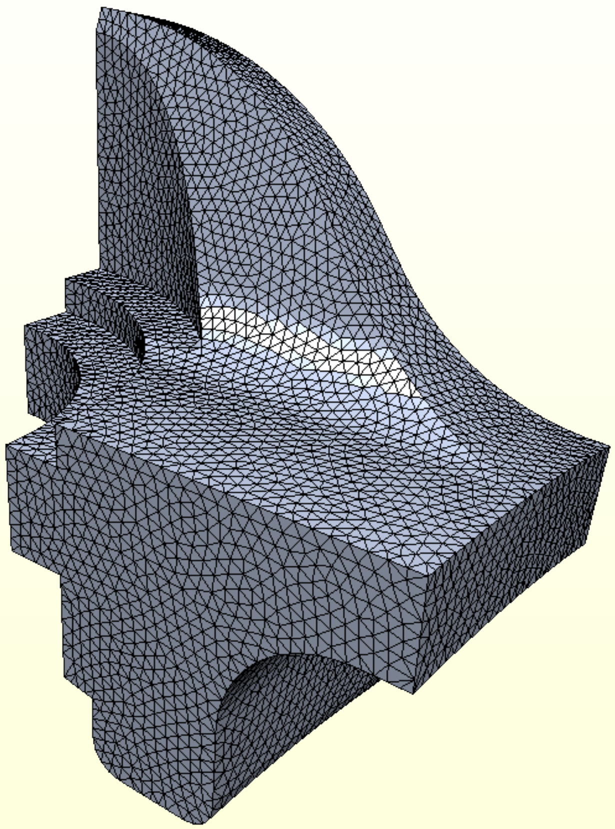
Feature Preservation?





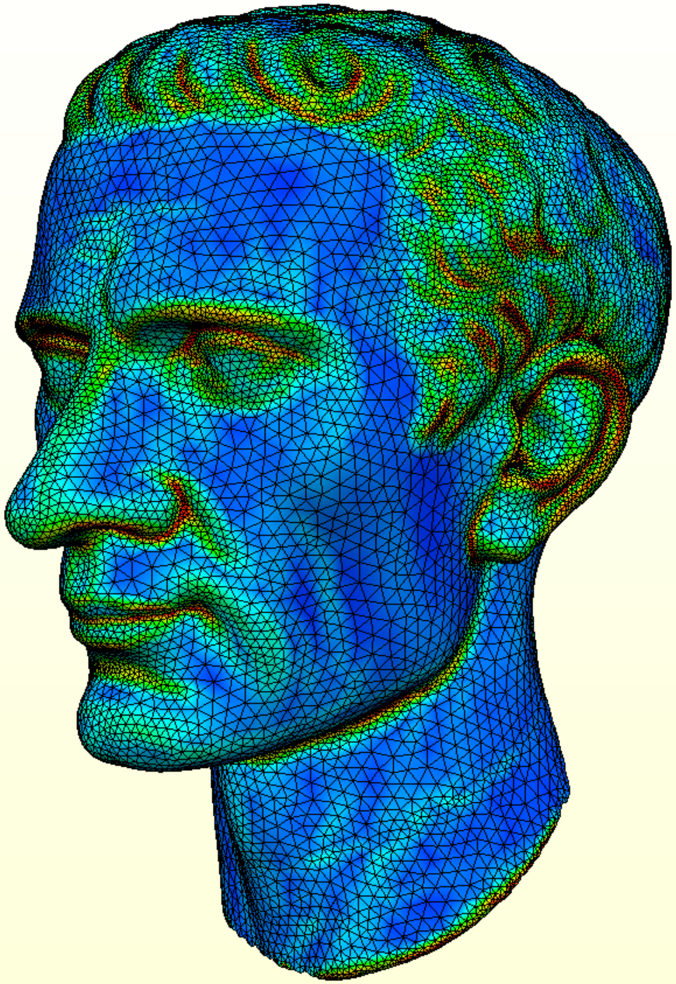
Feature Preservation

- Define features
  - Sharp edges
  - Material boundaries
- Adjust local operators
  - Don't flip
  - Collapse only along features
  - Univariate smoothing
  - Project to feature curves

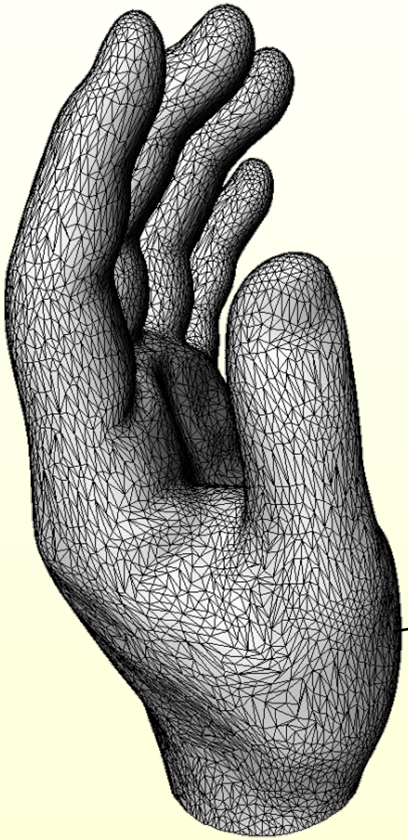


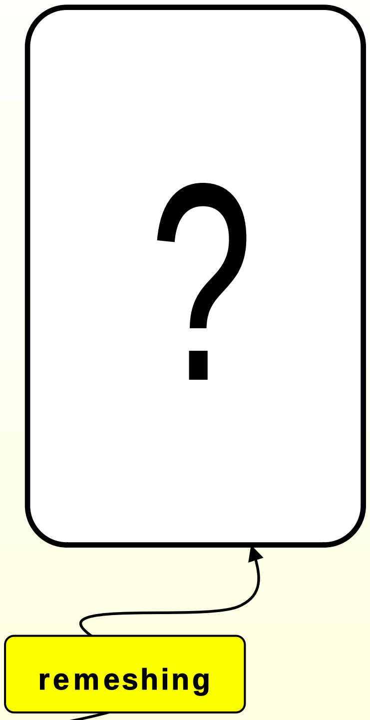
Adaptive Remeshing

- Precompute max. curvature on reference mesh
- Target edge length locally determined by curvature
- Adjust split / collapse criteria

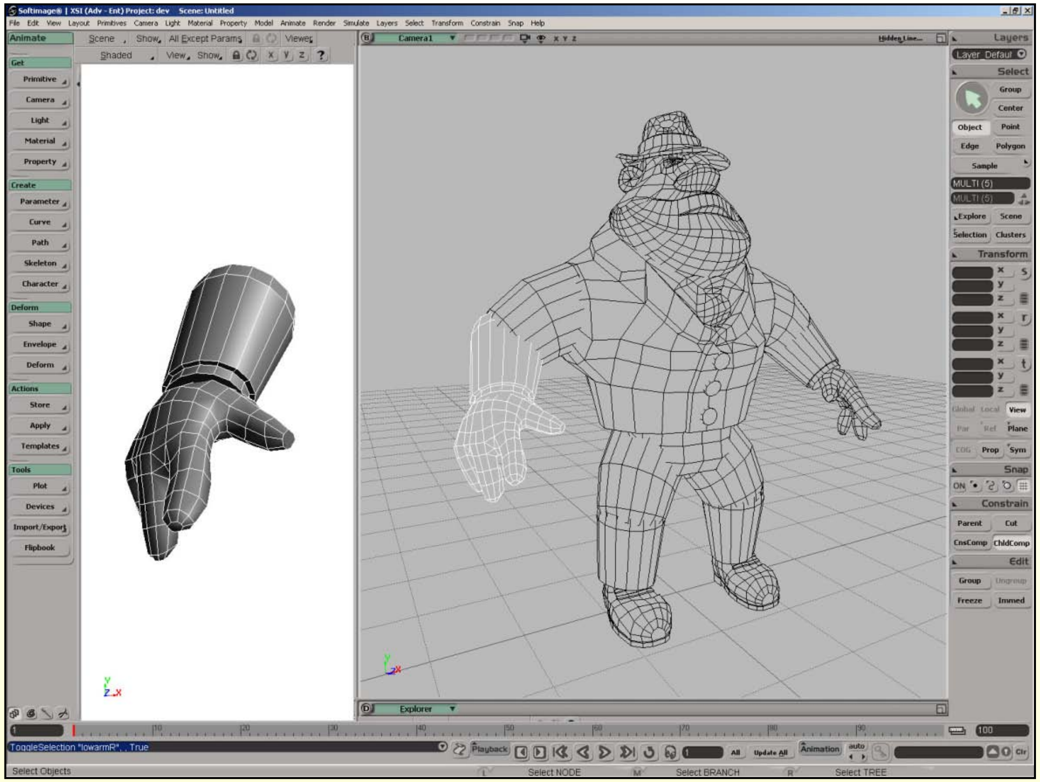


Anisotropic Remeshing



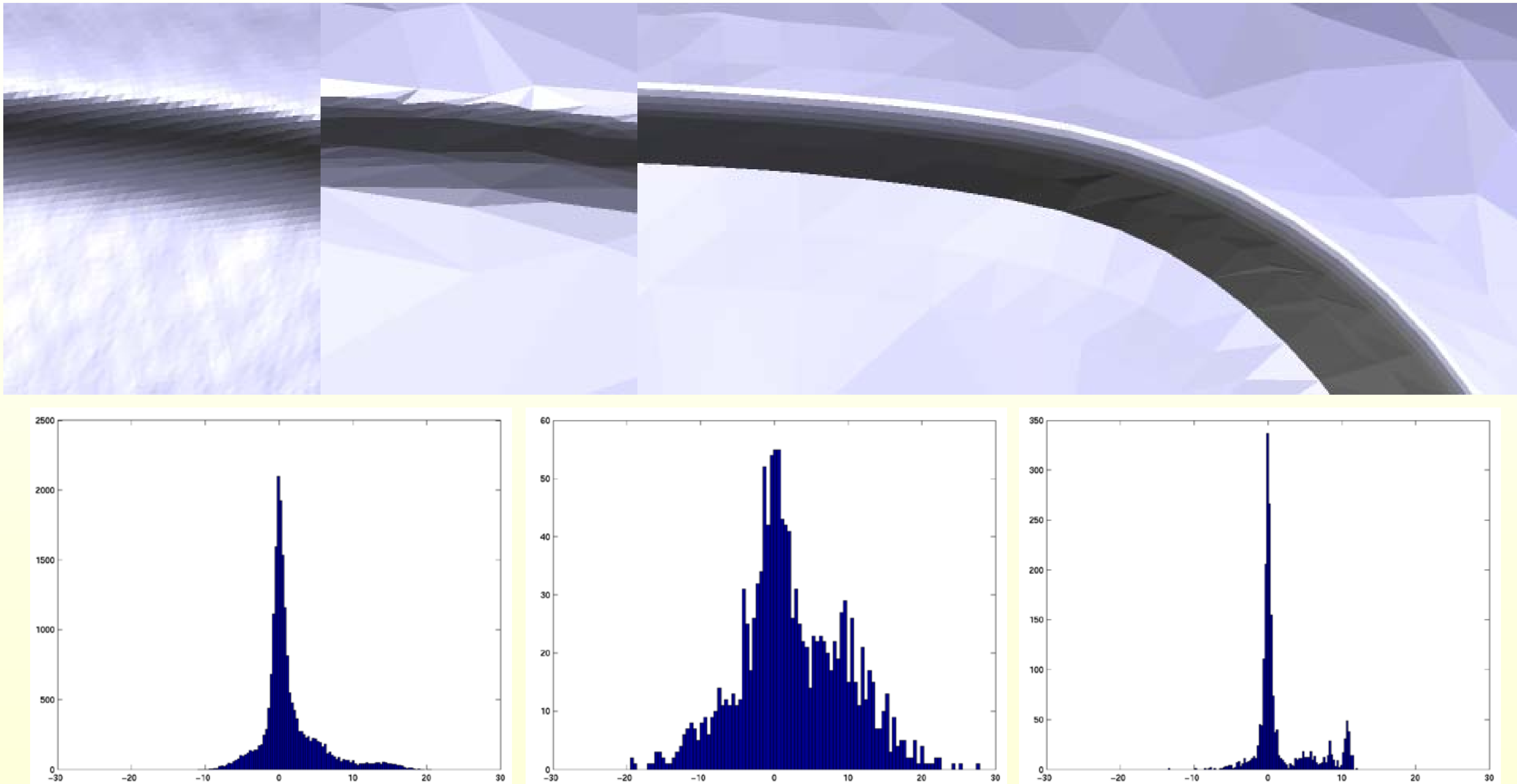


scanned
surface
geometry



carefully designed model
www.viewpoint.com

Normal Noise

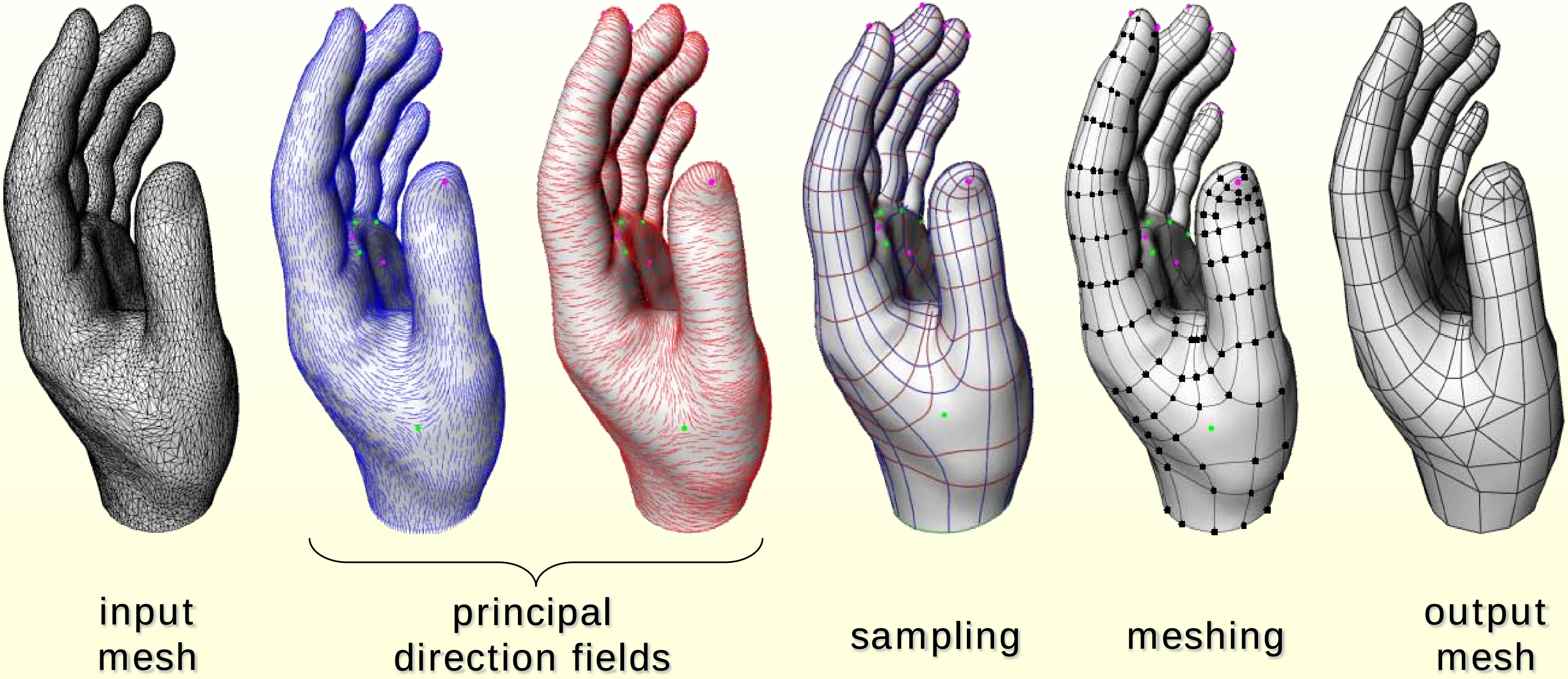


Anisotropic Remeshing

Overview

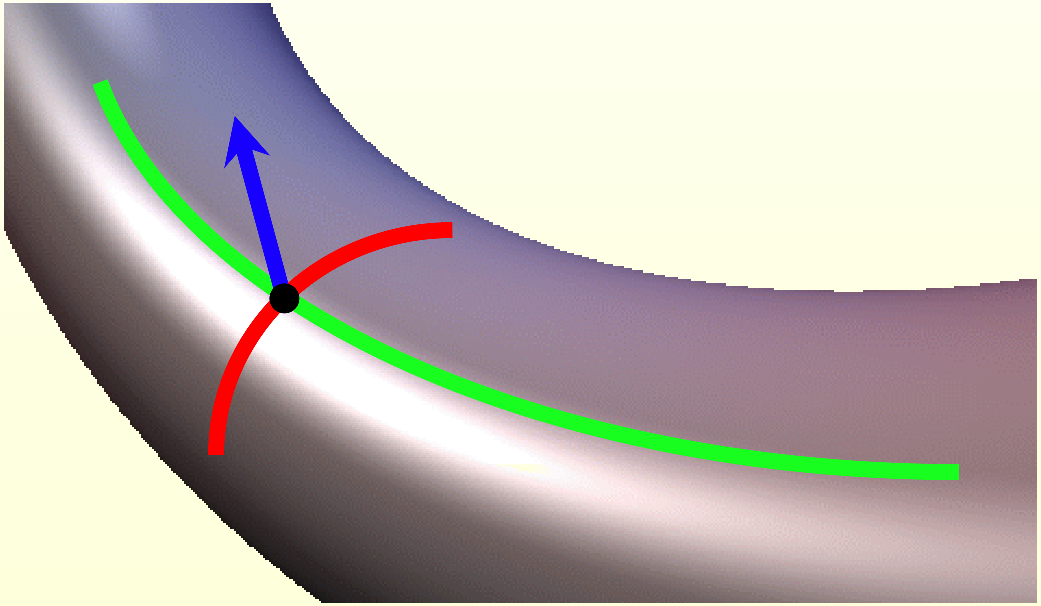
- Isotropic Remeshing
  - Parameterization-based [Alliez 2002]
  - Surface-based [Botsch 2004]
- Anisotropic Remeshing
  - **Parameterization-based [Alliez 2003]**
  - Surface-based [Marinov 2004]

Alliez: Anistropic Remeshing



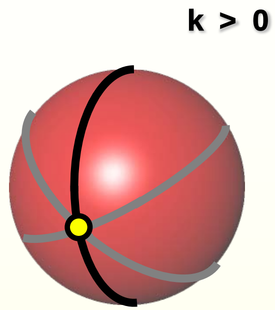
Anisotropy

- Differential geometry
  - 2nd fundamental form defines a local **orthogonal** frame (min- / max-curvature directions and the normal)

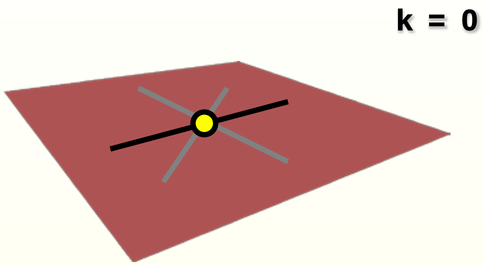


3D Curvature Tensor

Isotropic



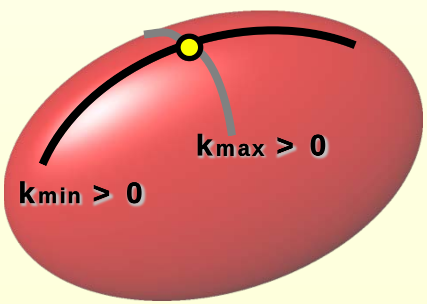
spherical



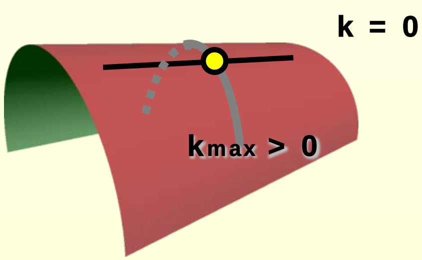
planar

Anisotropic

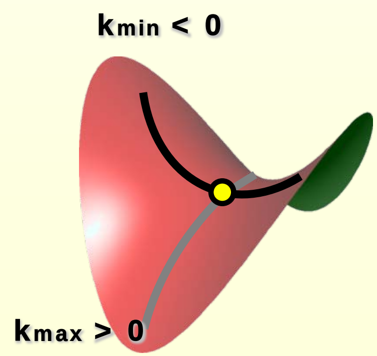
2 principal directions



elliptic

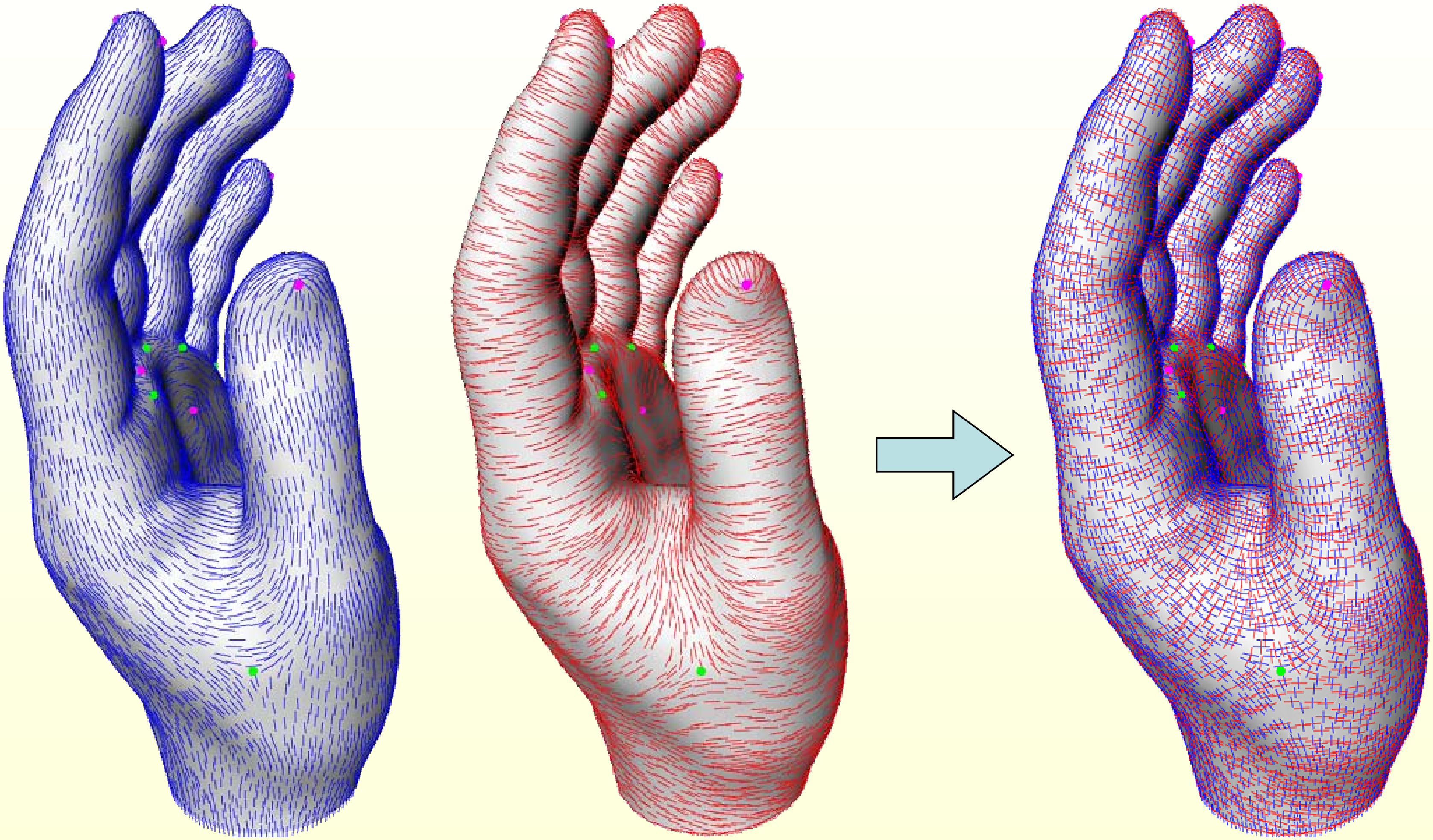


parabolic



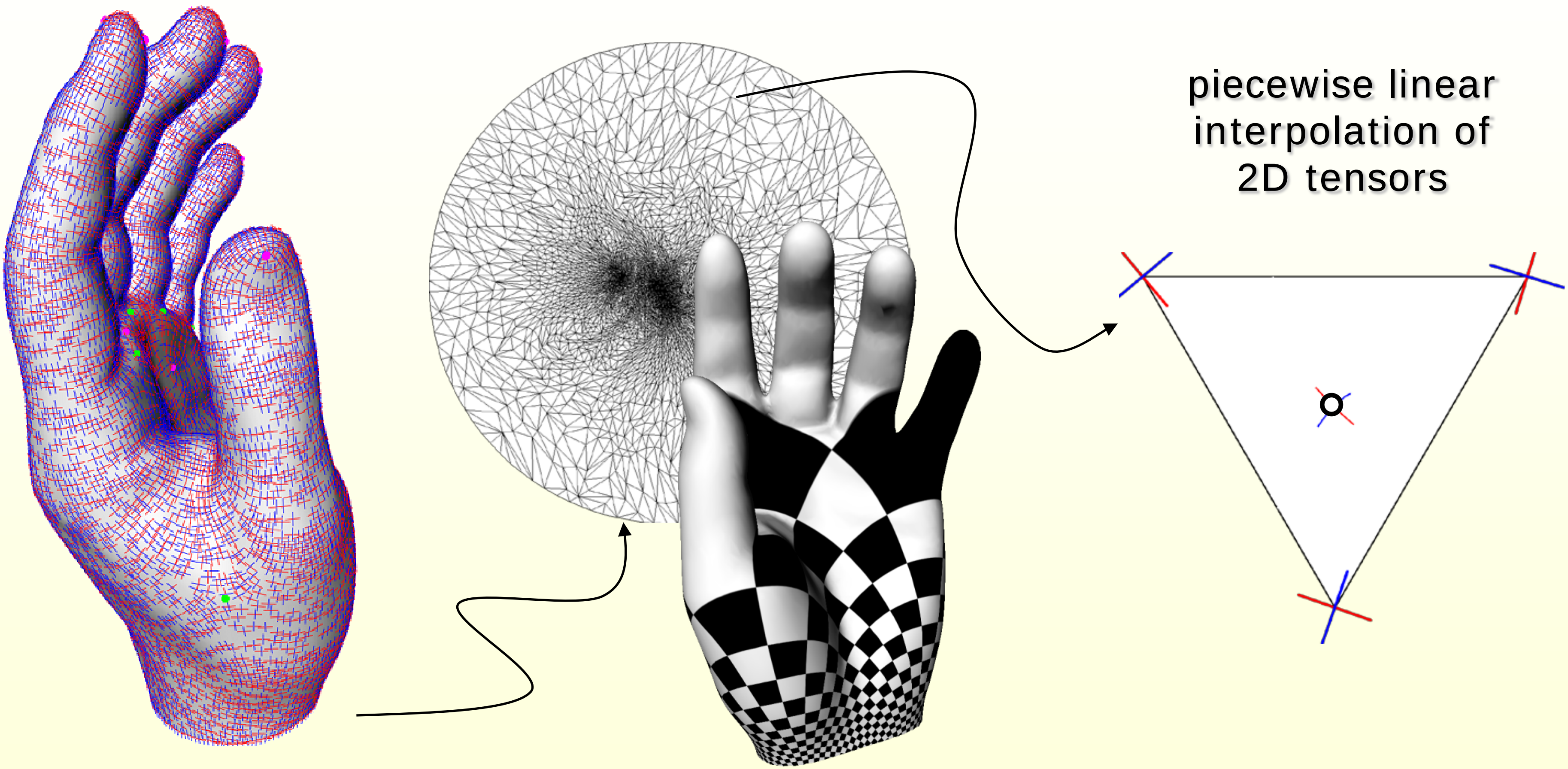
hyperbolic

Principal Directions Fields



min curvature   max curvature

Flattening to 2D



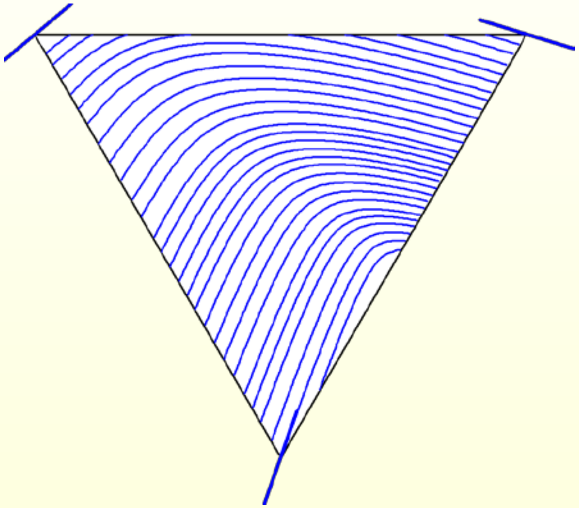
one 3D tensor
per vertex

discrete conformal
parameterization

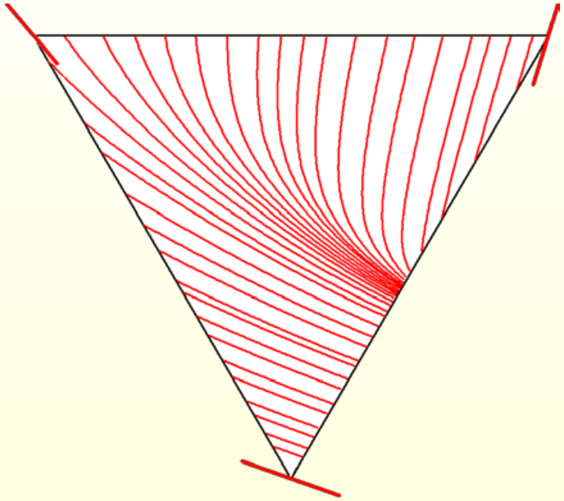
2D tensor **field**
using barycentric
coordinates

2D Direction Fields

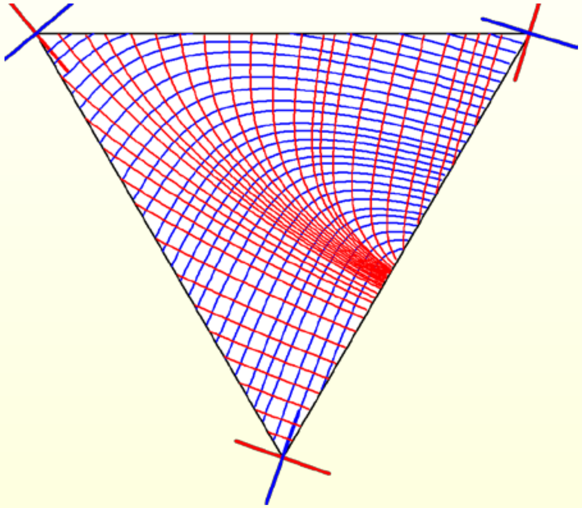
- Regular case



minor foliation



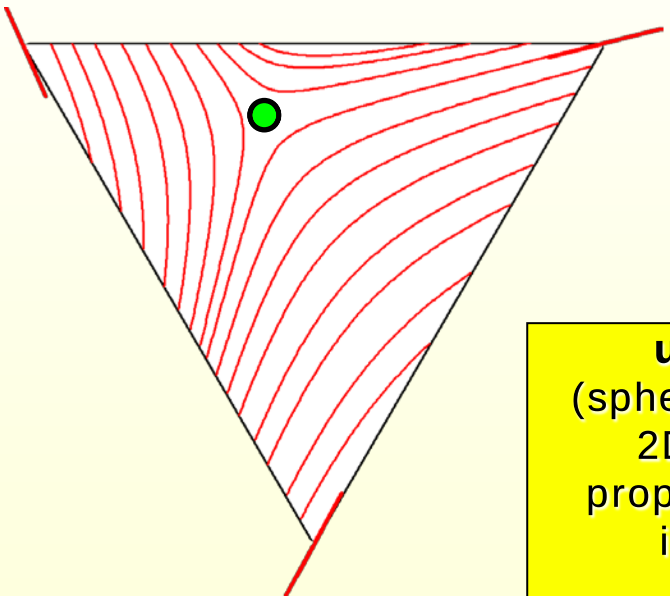
major foliation



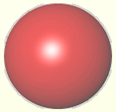
principal foliations

2D Direction Fields

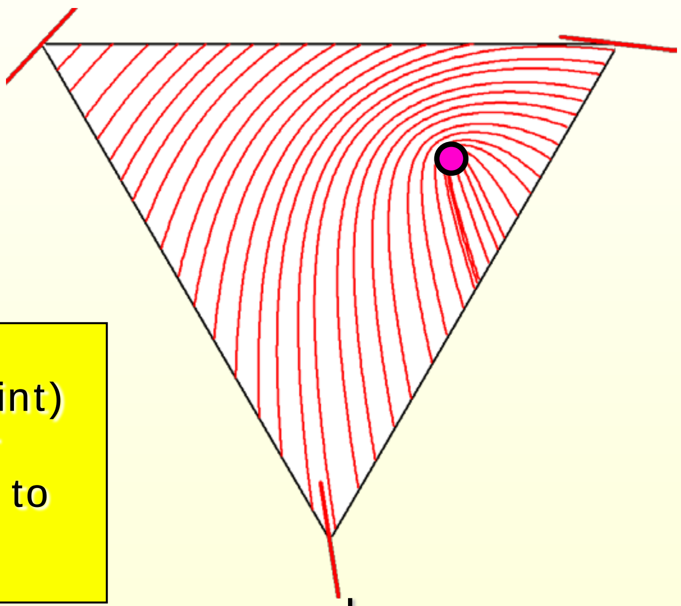
- Singularities



trisector

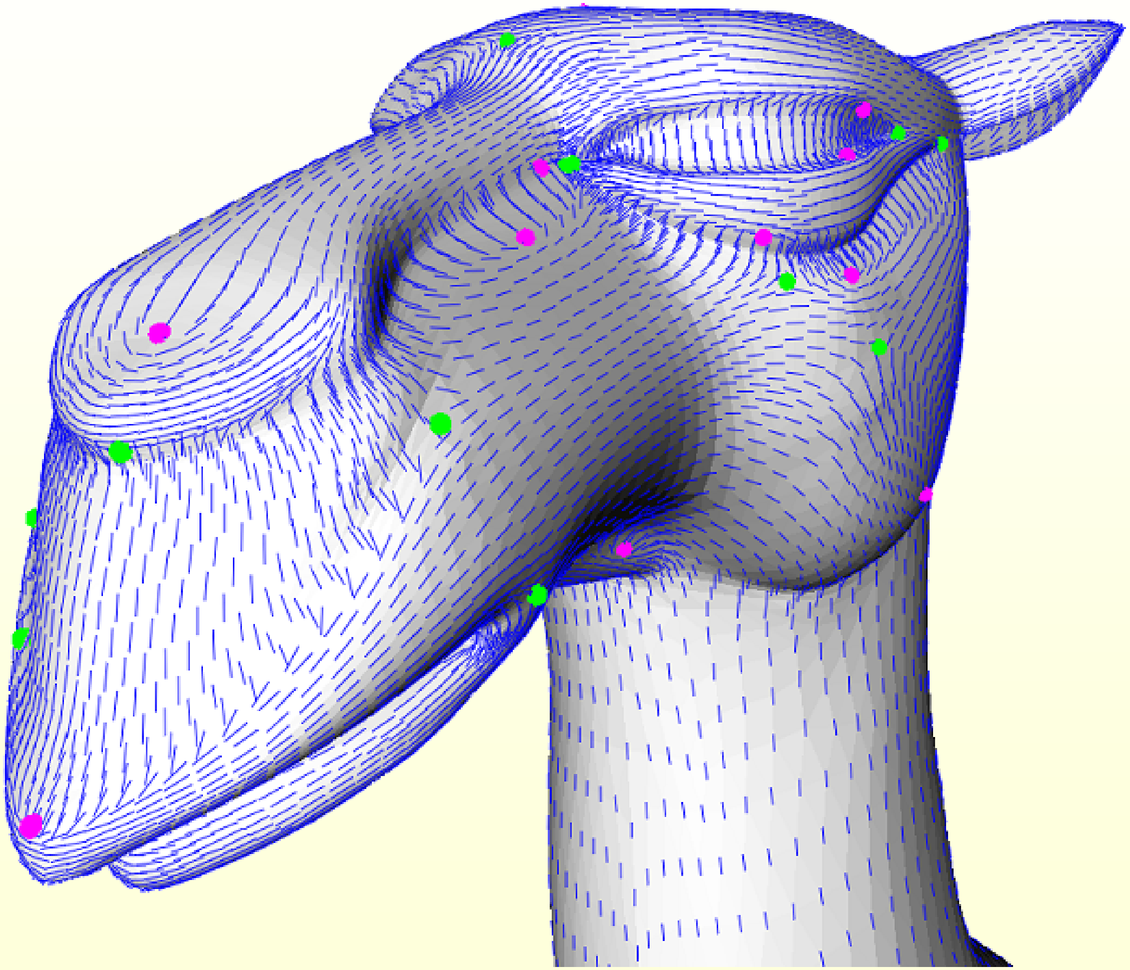


umbilic
(spherical point)
2D tensor
proportional to
identity

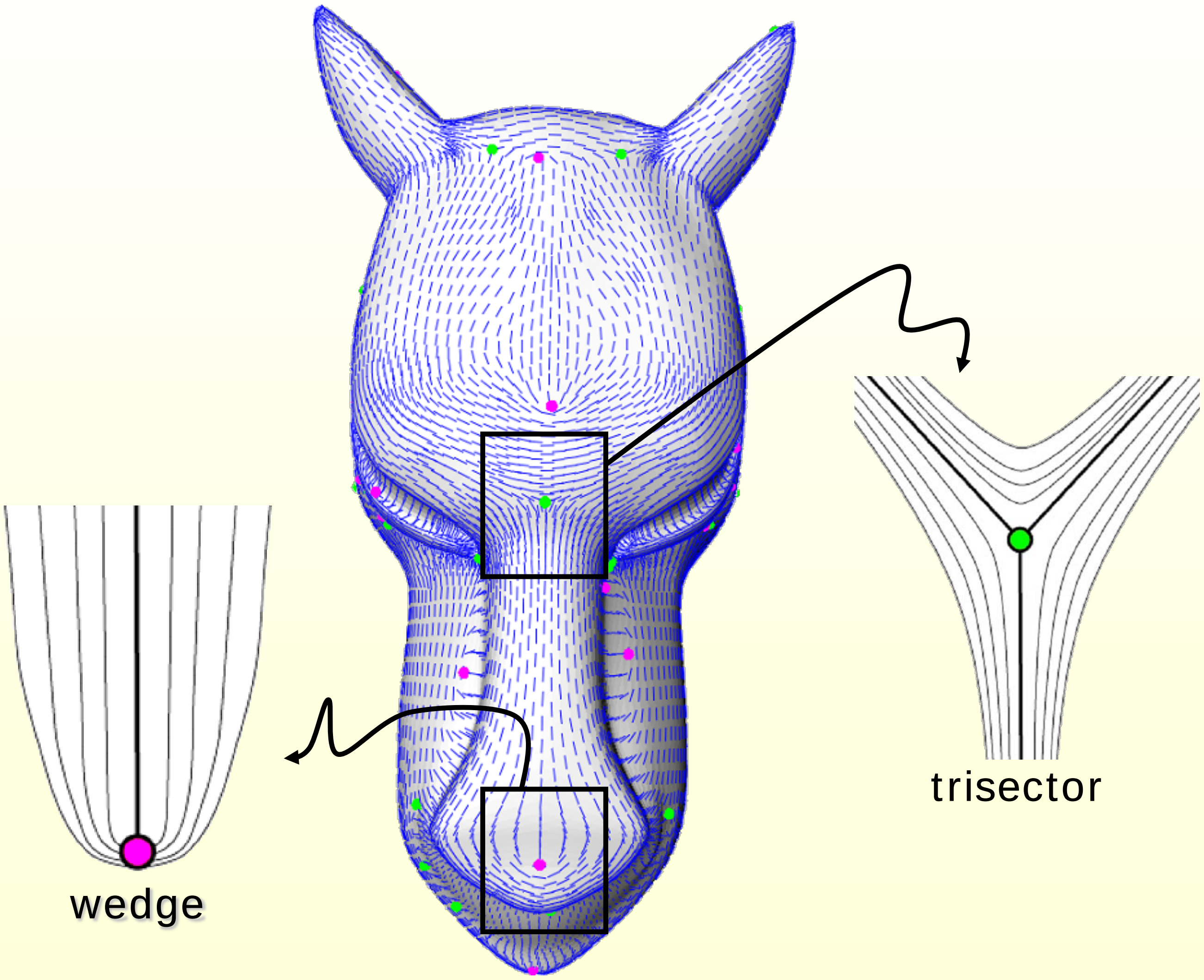


wedge

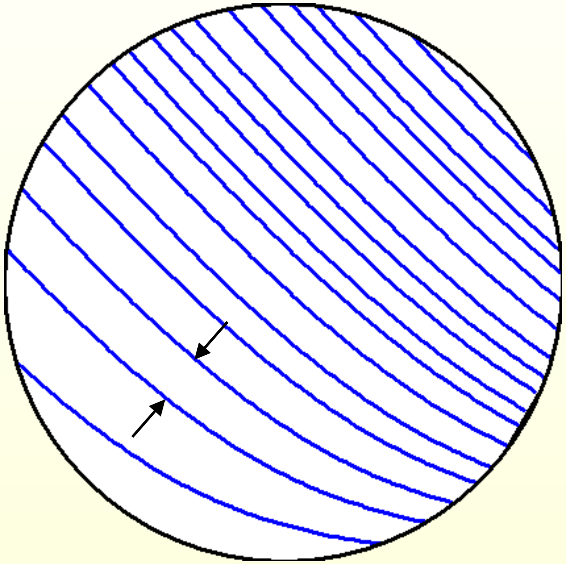
Umbilics



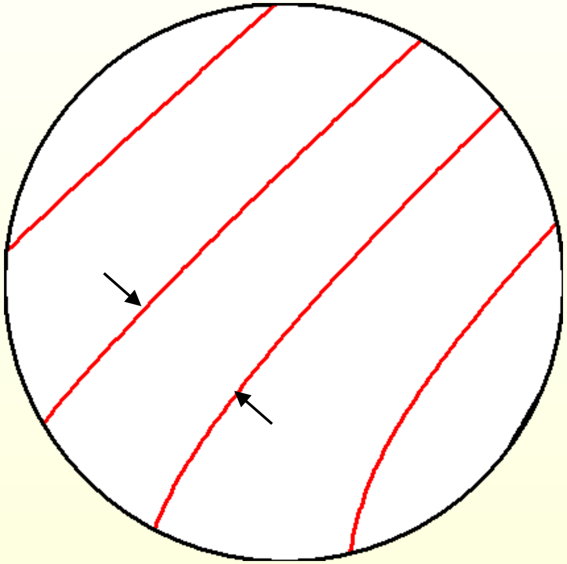
Umbilics



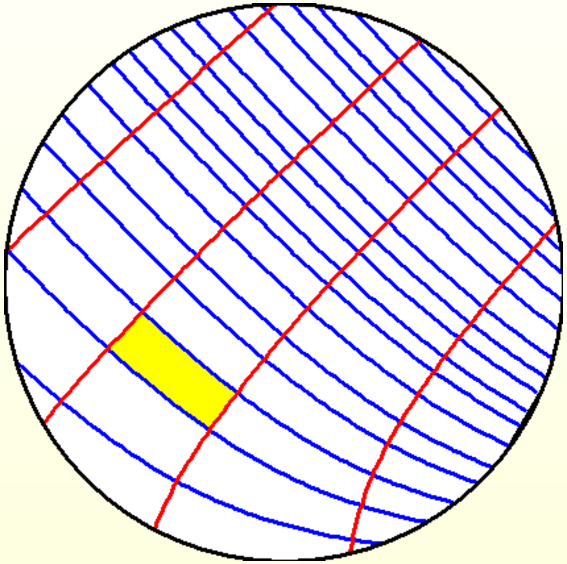
Lines of Curvature



minor net

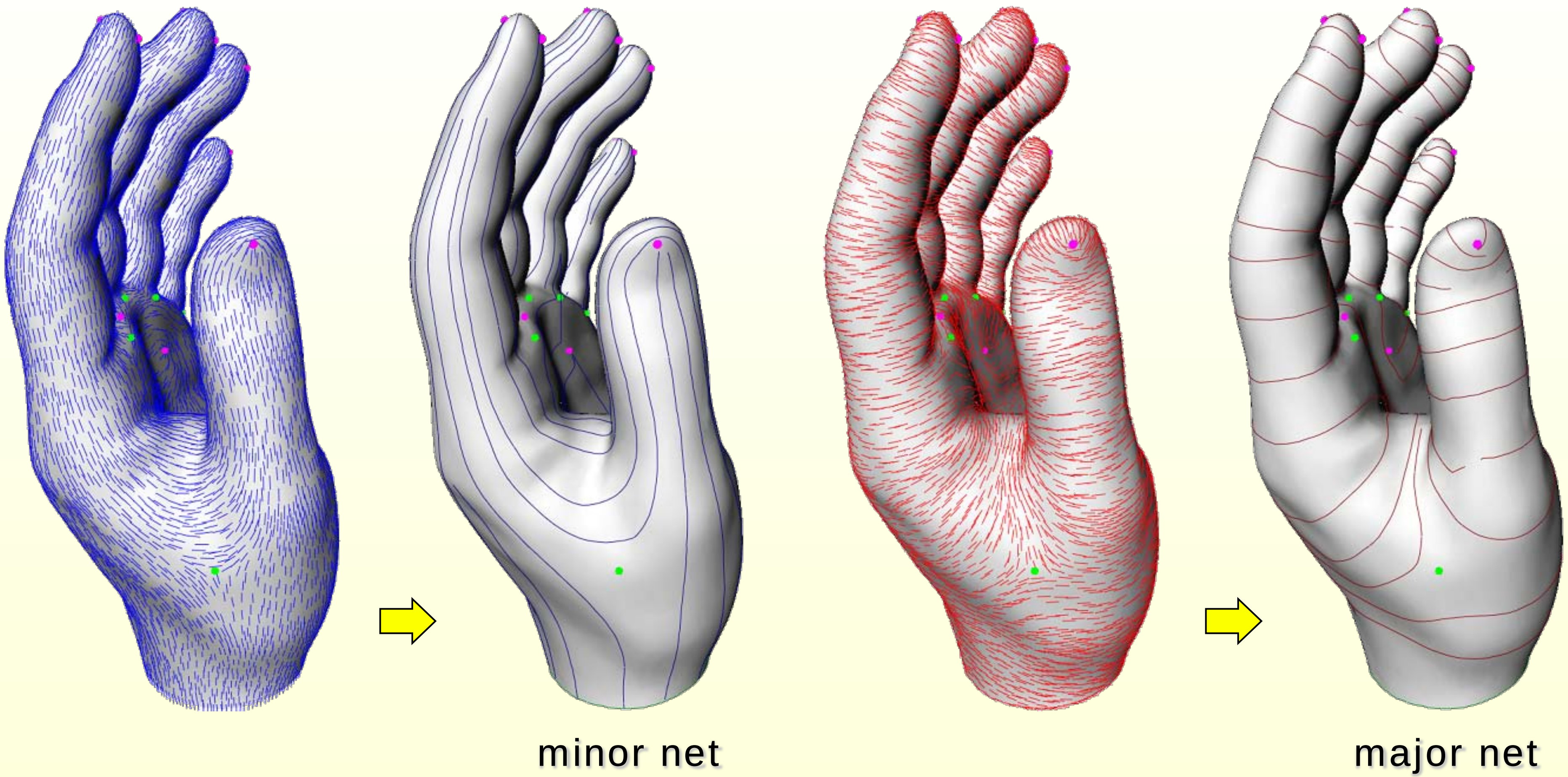


major net



overlay

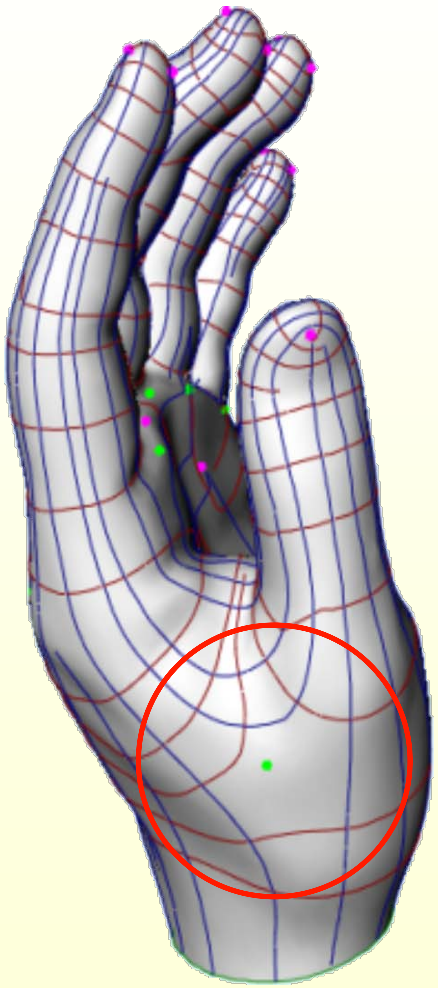
Lines of Curvature



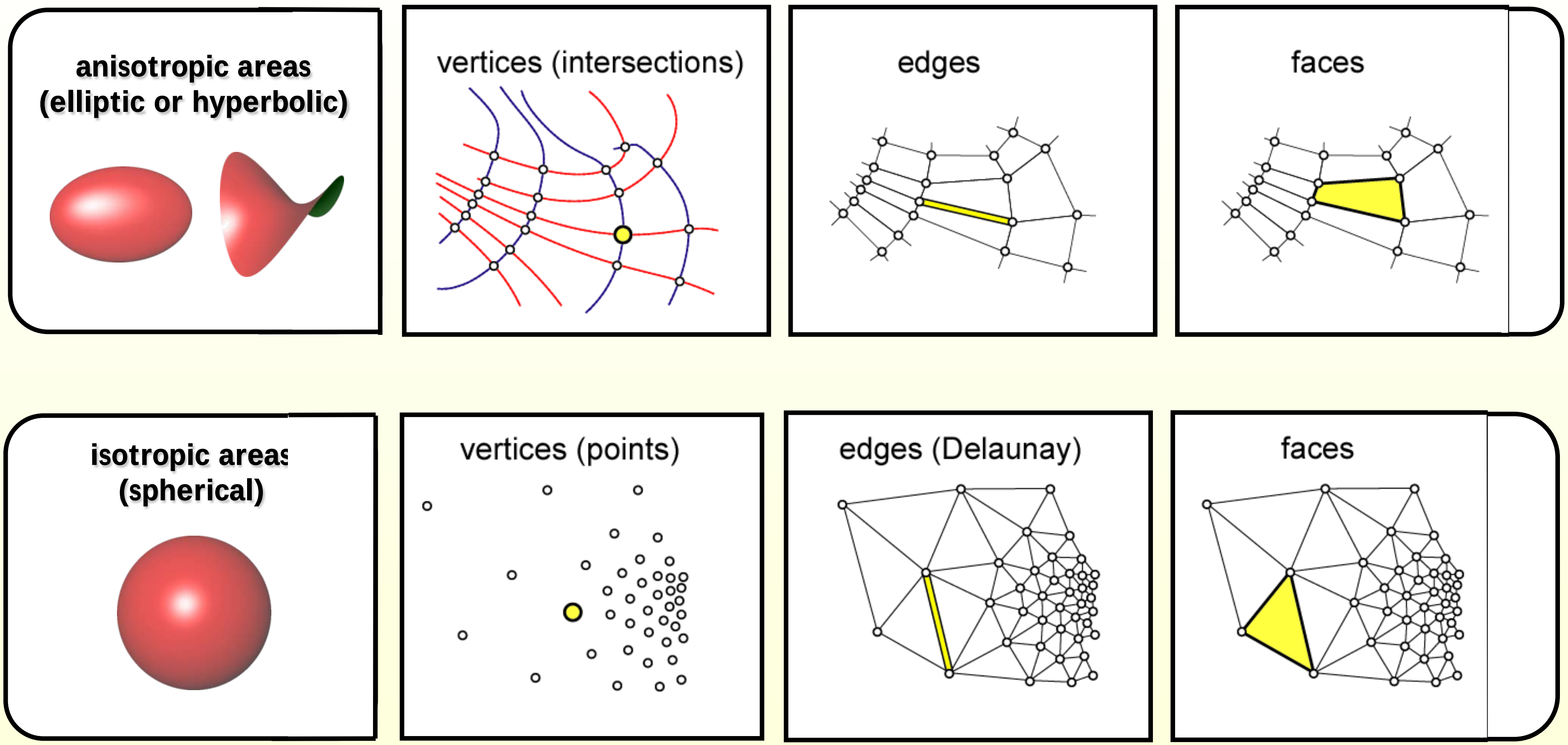
Overlay

- Overlay curvature lines in anisotropic regions

- Add umbilic points in isotropic regions

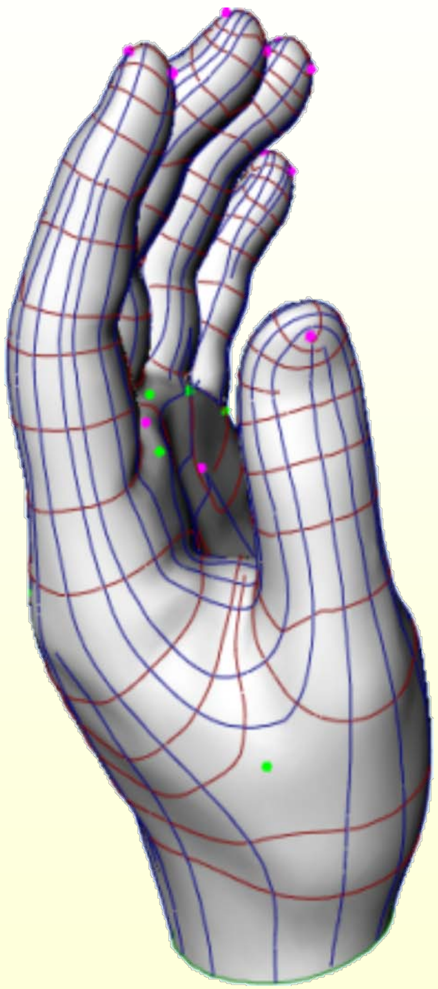


Meshing

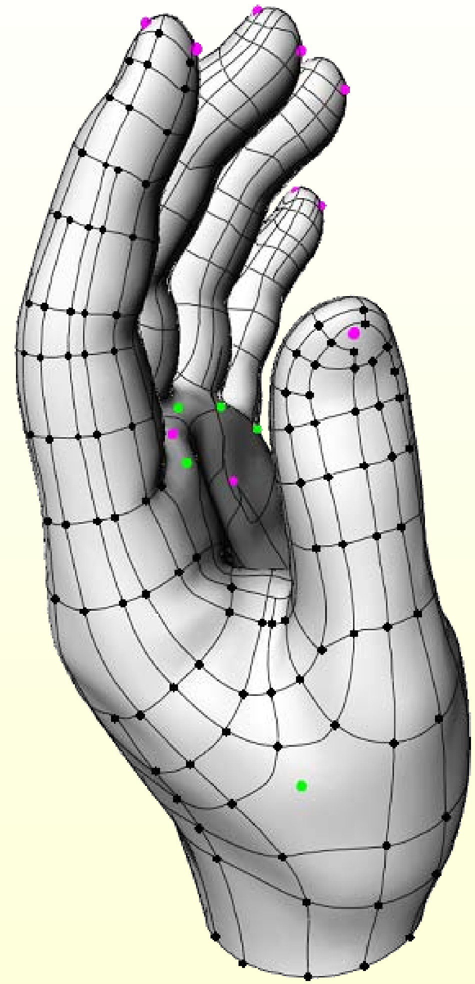


[CGAL] **www.cgal.org**

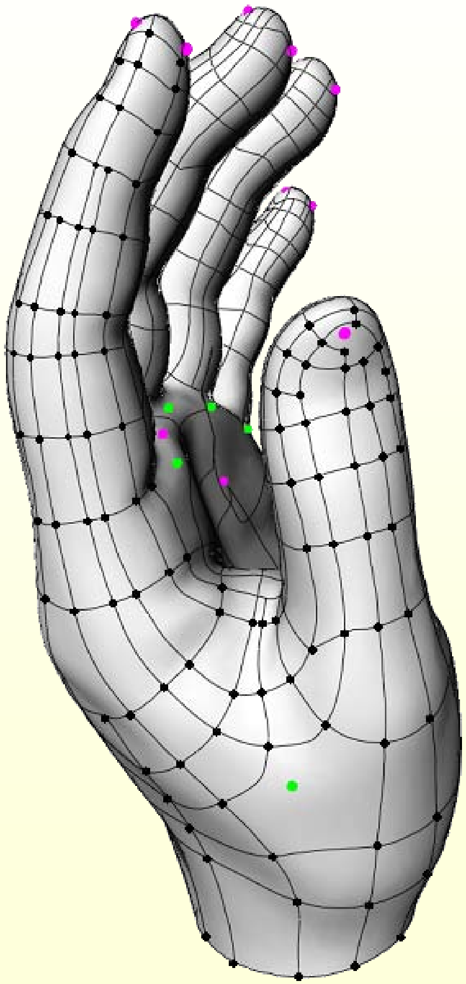
Vertices



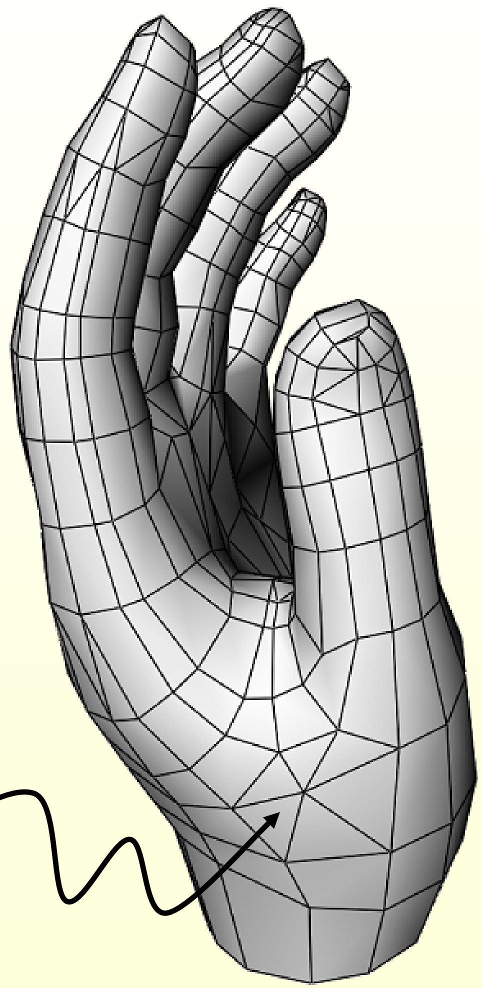
intersect lines of curvatures



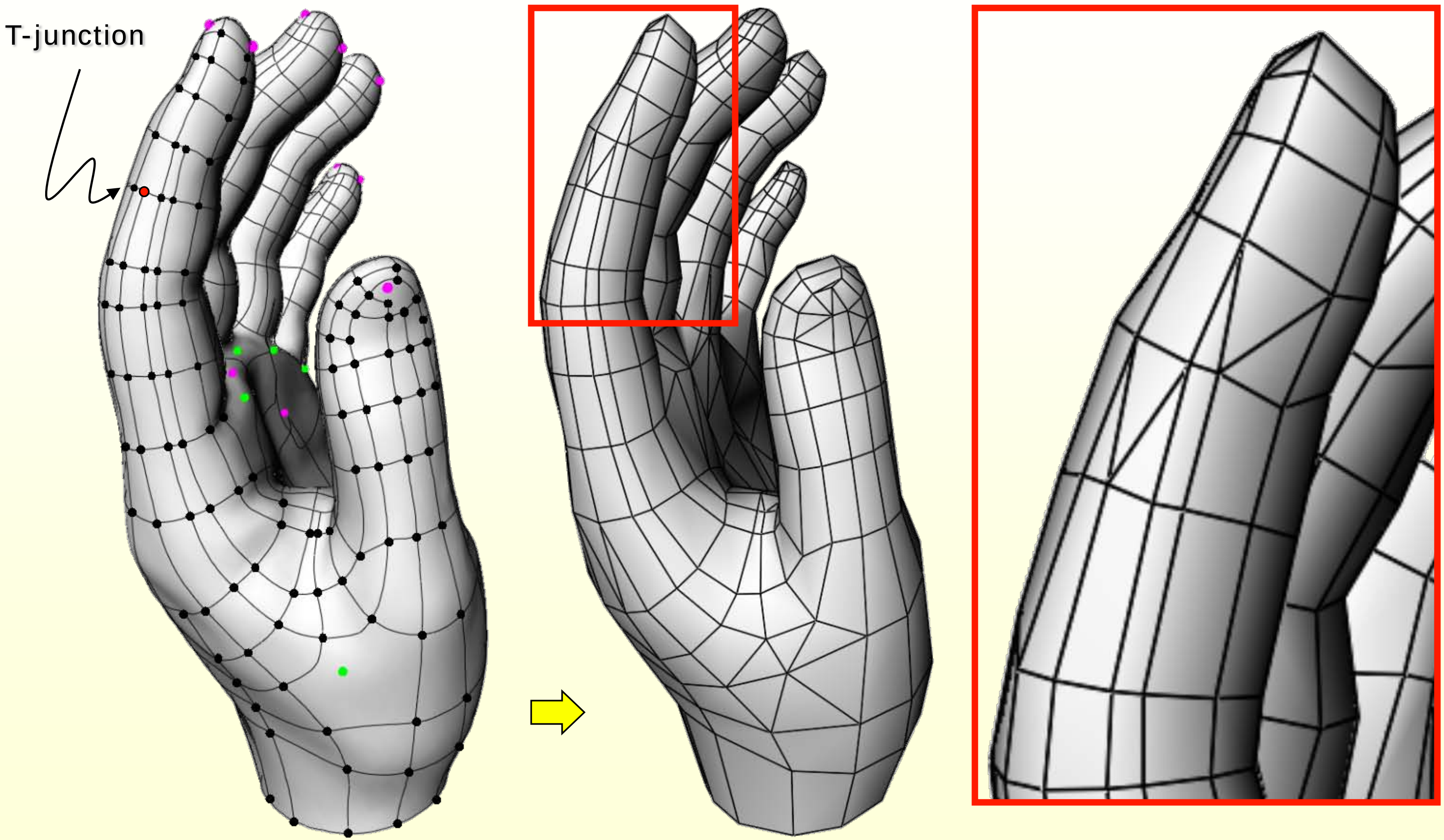
Edges



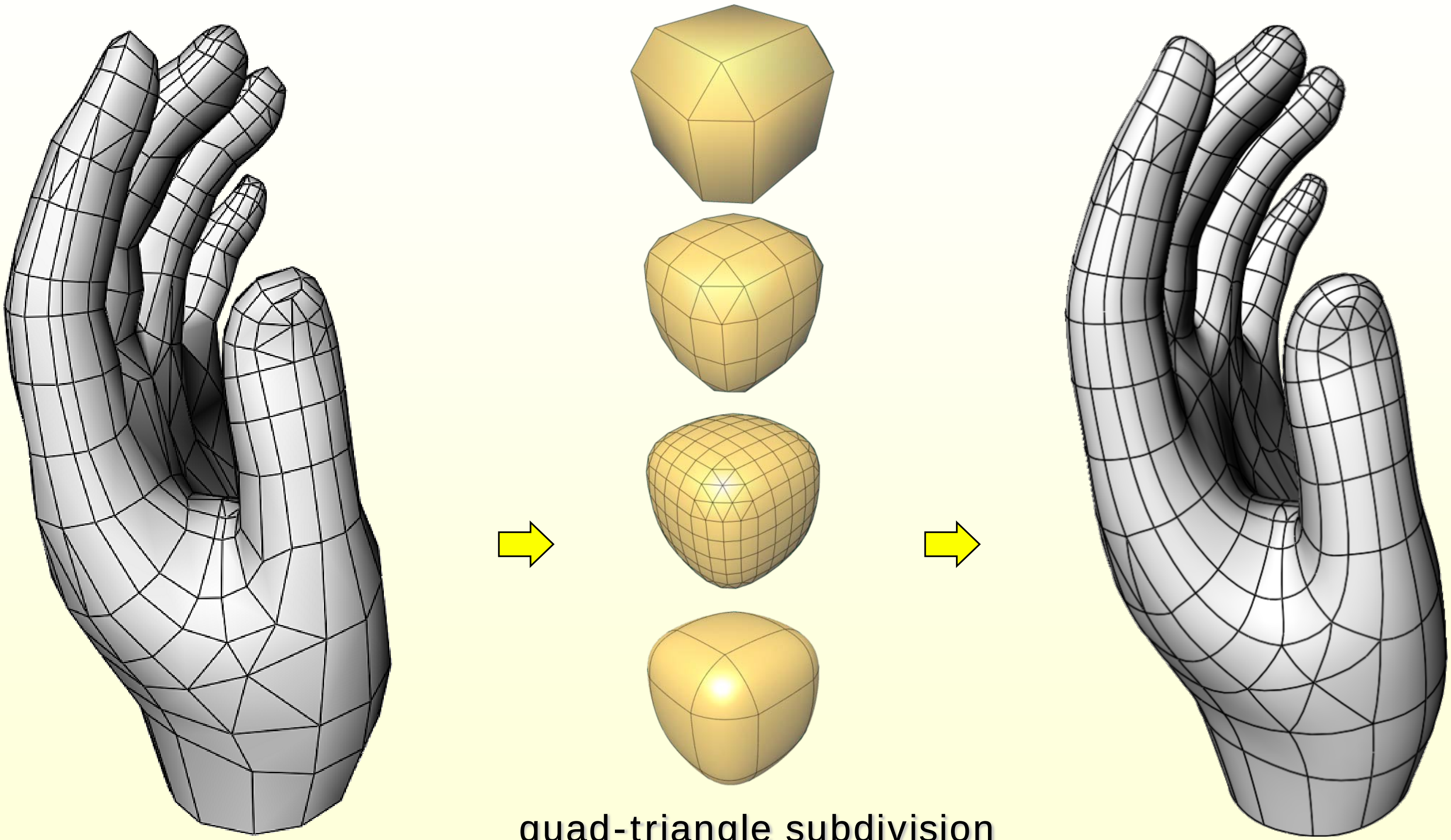
straighten lines of curvatures
+
Delaunay triangulation near umbilics



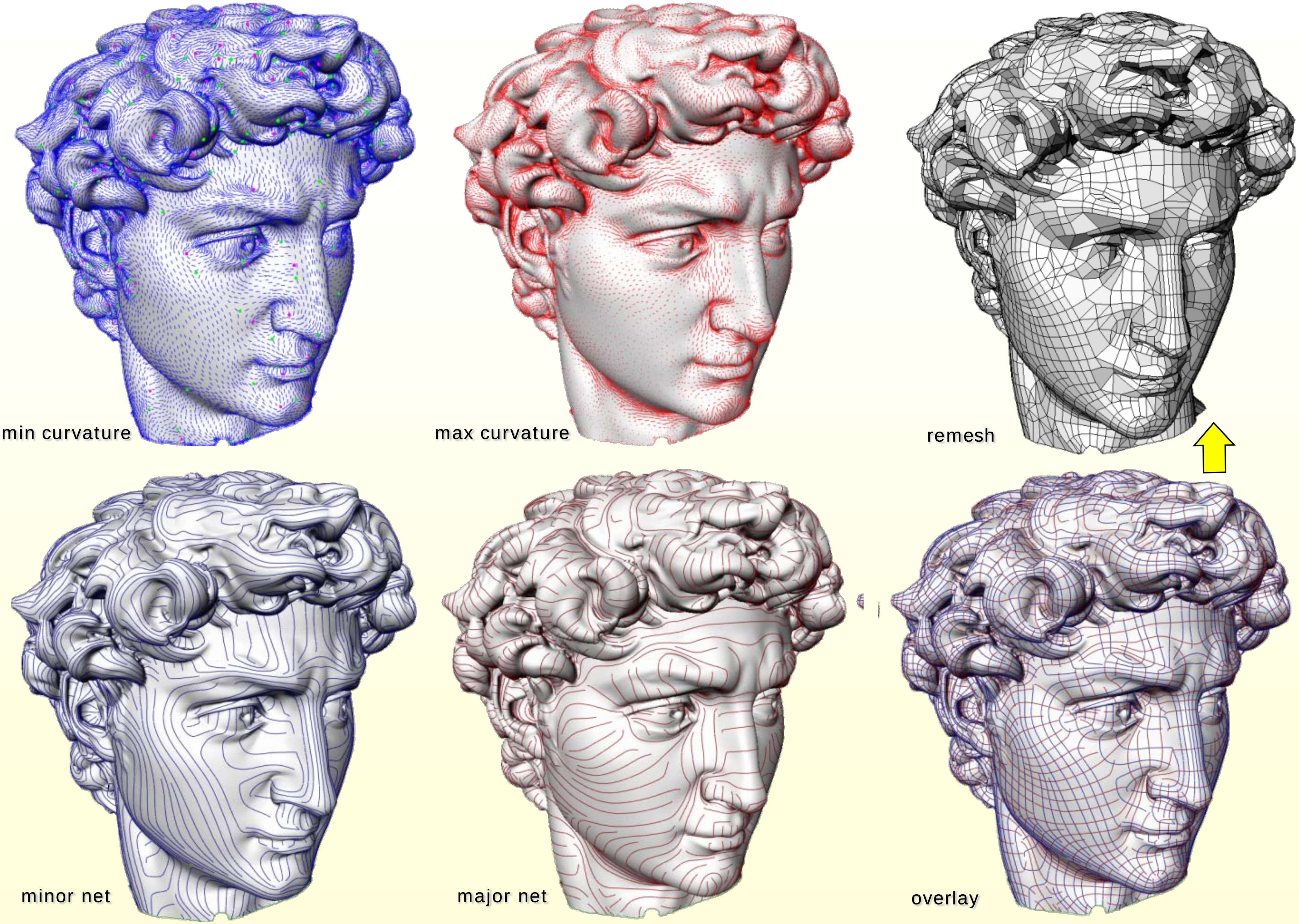
Resolve T-Junctions



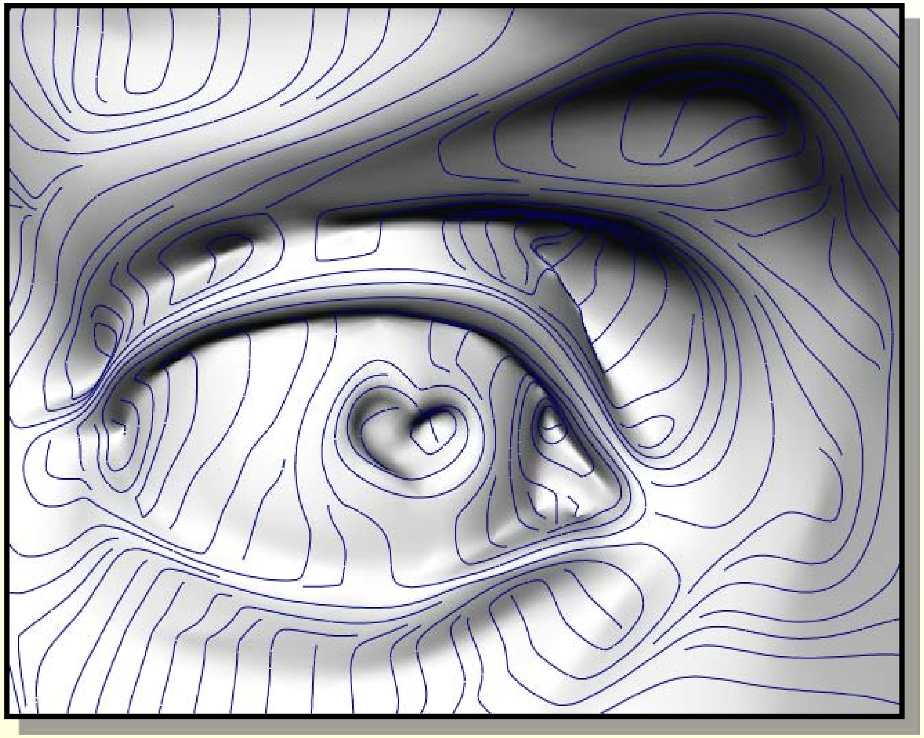
Smoothing by Subdivision



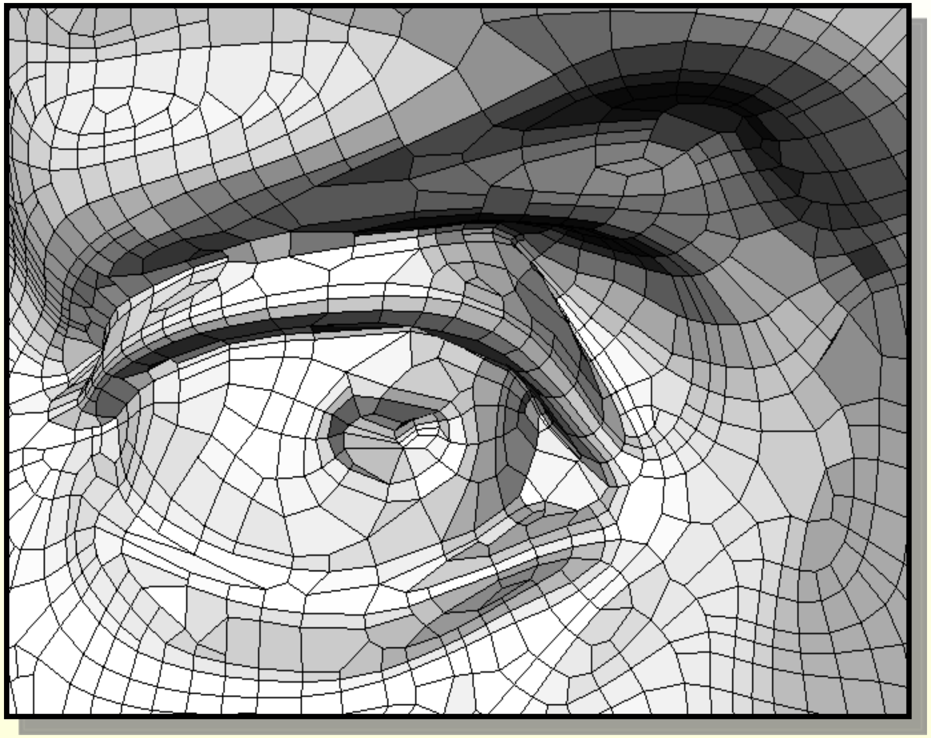
quad-triangle subdivision


min curvature
max curvature
remesh
minor net
major net
overlay

Close-Up



minor net



remesh

Limitations

- Global parameterization
  - Smart cutting required
  - Low distortion parameterization
  - Numerical issues
- Lines of curvature
  - Robust curvature tensor computation
  - Tensor smoothing required
  - Optimal placement of streamlines difficult

Two Fundamental Approaches

- Surface oriented
  - operate directly of the surface
  - treat surface as a set of points / polygons in space
  - efficient for high resolution remeshing
- Parametrization based
  - map to 2D domain / 2D problem
  - computationally more expensive (?)
  - works even for coarse resolution remeshing

Basic Operators

- Correspondence
  - Global vs. local parameterization
- Vertex density control
  - Uniform vs. adaptive
  - Isotropic vs. anisotropic
- Local alignment
  - Optimal shape approximation
- Global alignment
  - Feature sensitivity

References

- Alliez et al, *“Interactive geometry remeshing”*, SIGGRAPH 2002
- Alliez et al, *“Isotropic surface remeshing”*, SMI 2003
- Alliez et al, *“Anisotropic polygonal remeshing”*, SIGGRAPH 2003
- Vorsatz et al, *“Dynamic remeshing and applications*”, Solid Modeling 2003
- Botsch & Kobbelt, *“A remeshing approach to multiresolution modeling”*, Symp. on Geometry Processing 2004
- **Alliez et al, *“Recent advances in remeshing of surfaces”*, AIM@Shape state of the art report, 2006**