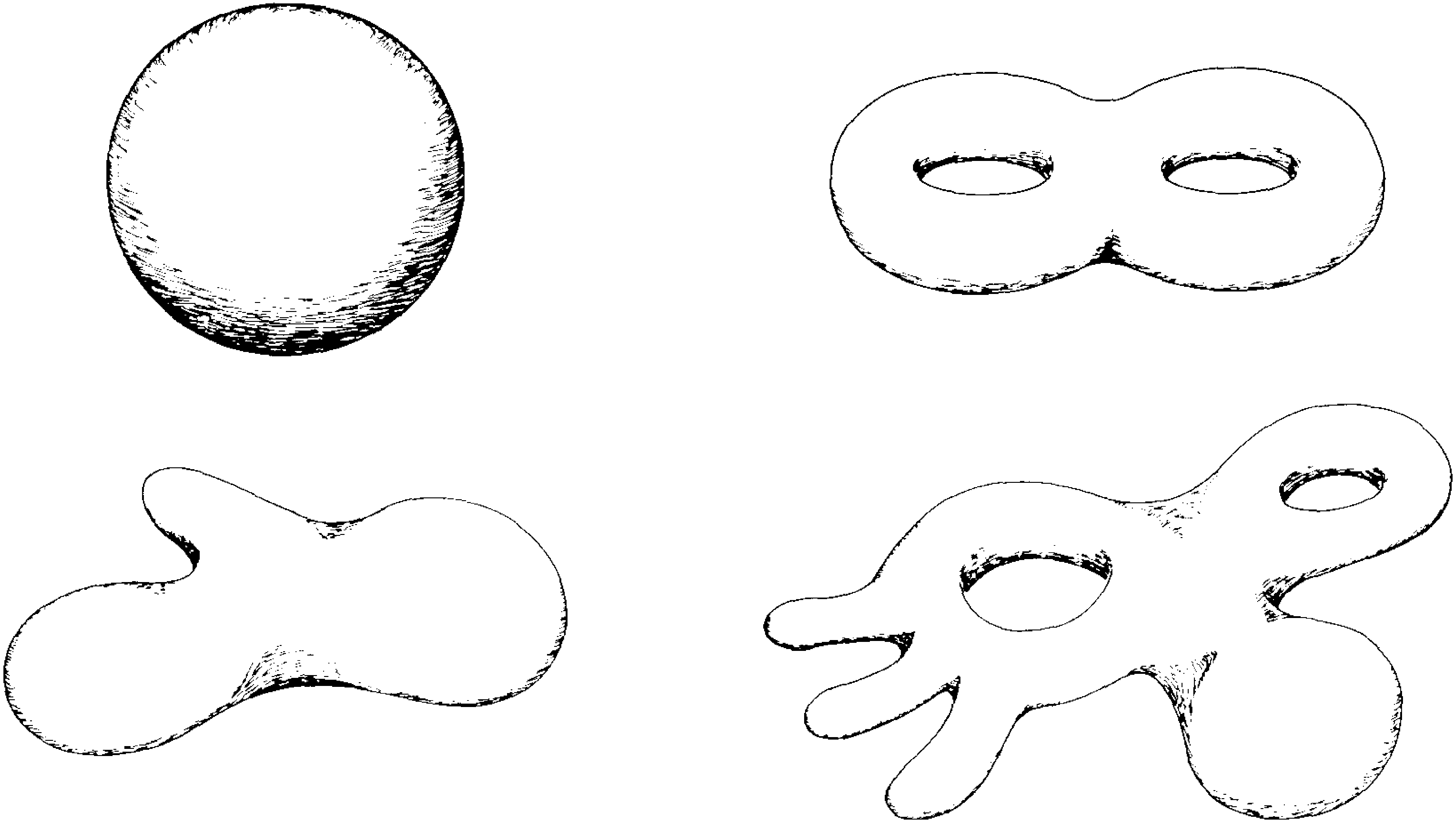
CS164: Topology I, Surfaces

May 12, 2010

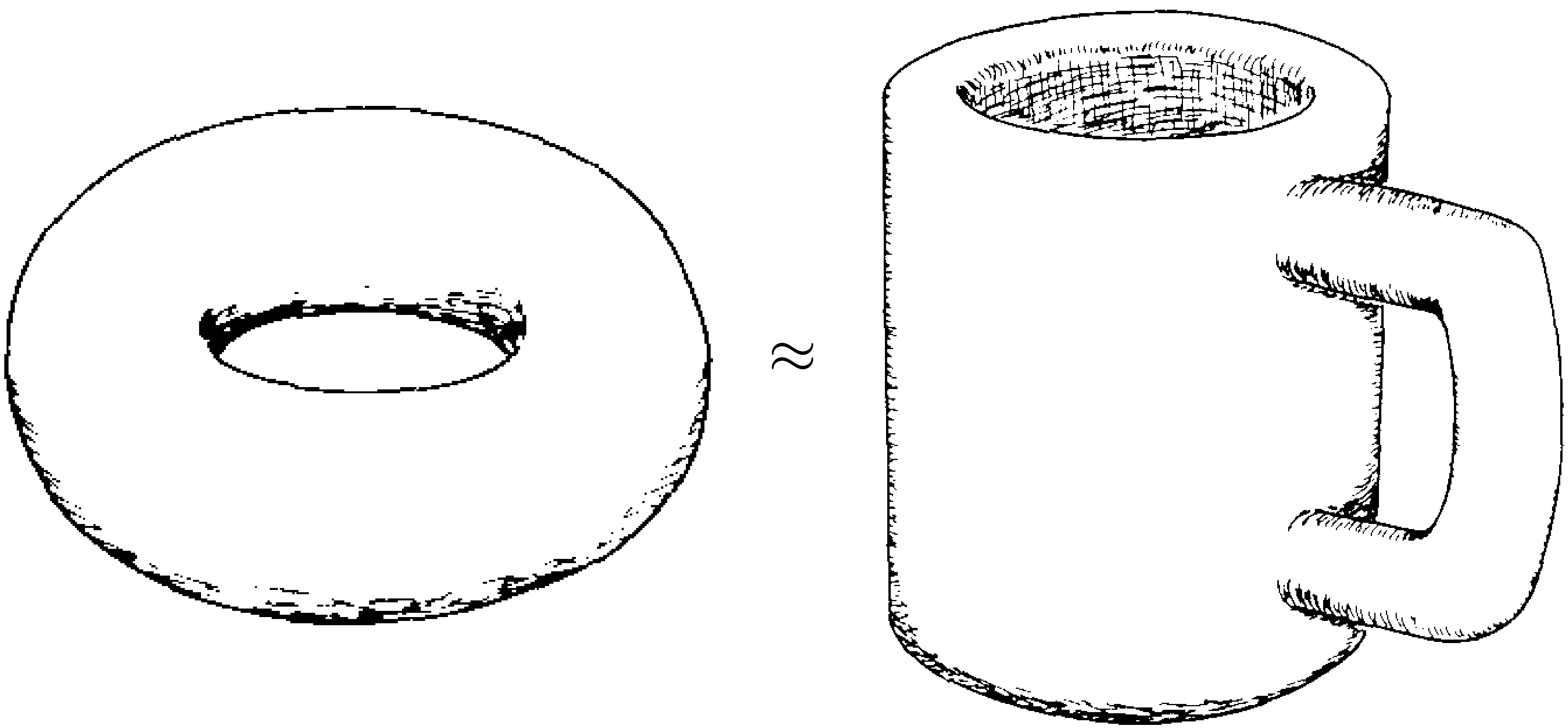
Topology vs. Geometry



Topology: properties of spaces under continuous deformations.

Topology vs. Geometry

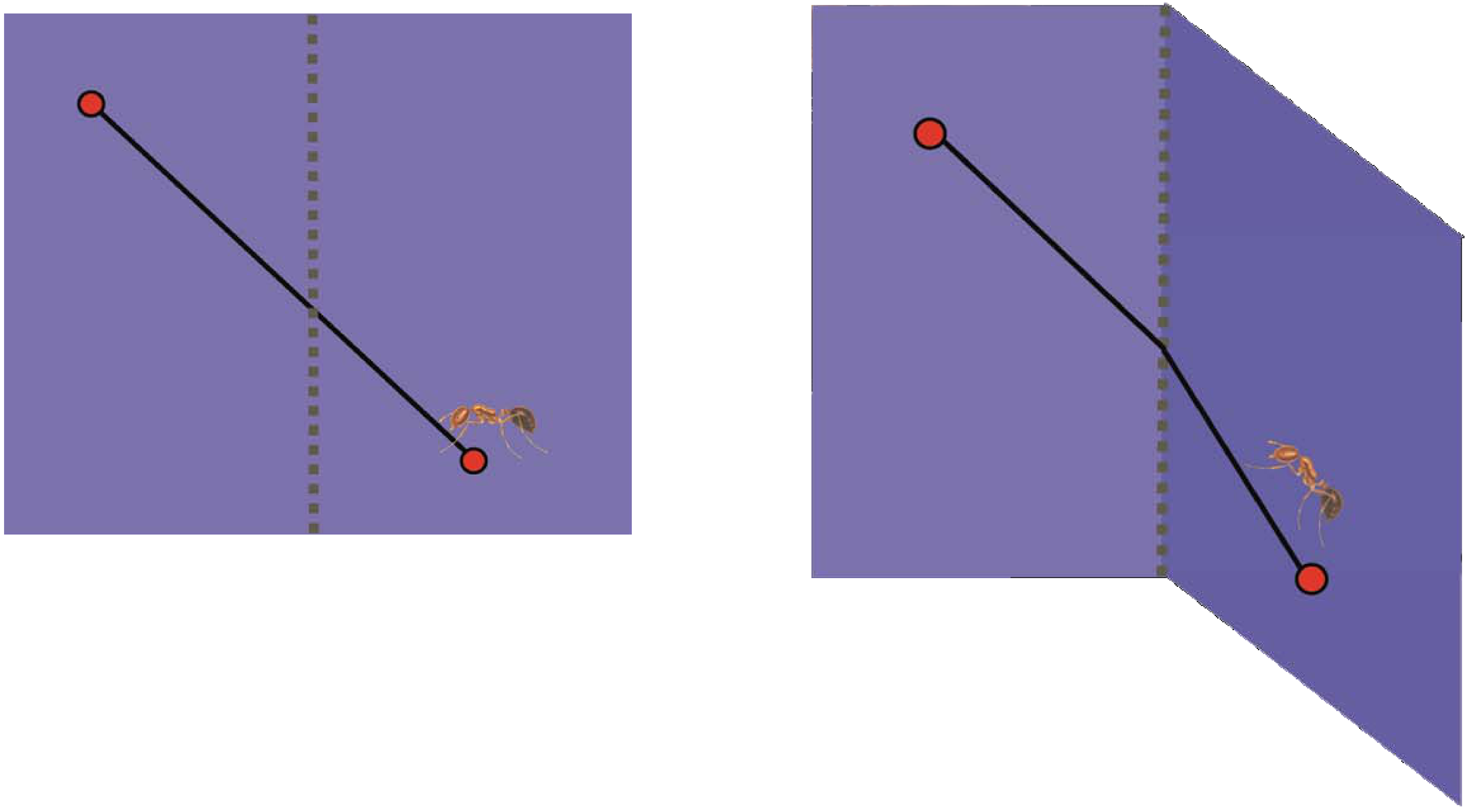
Topologist's breakfast



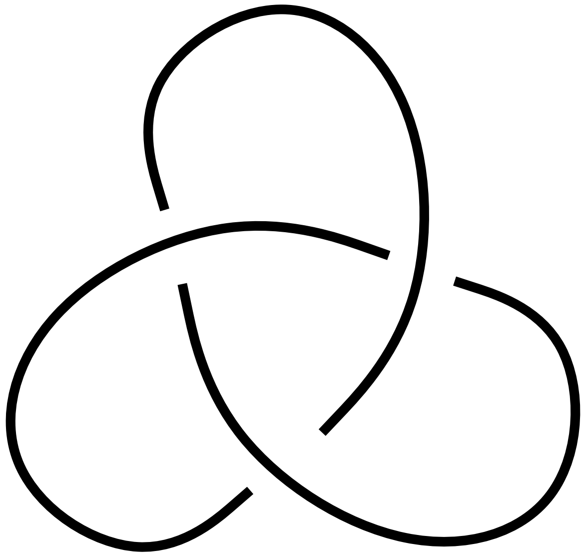
(See animation on the Wikipedia entry for "Topology")

Topology: properties of spaces under continuous deformations.

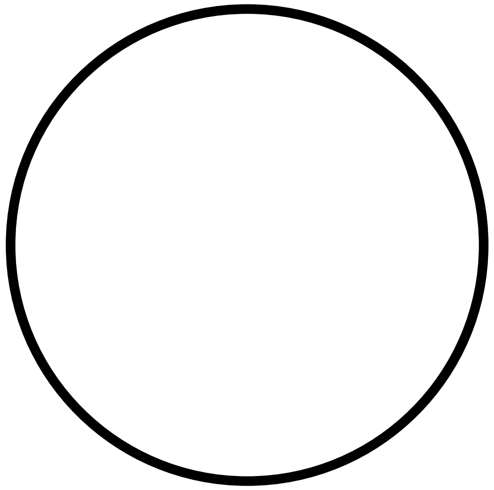
Extrinsic vs. Intrinsic



Extrinsic vs. Intrinsic

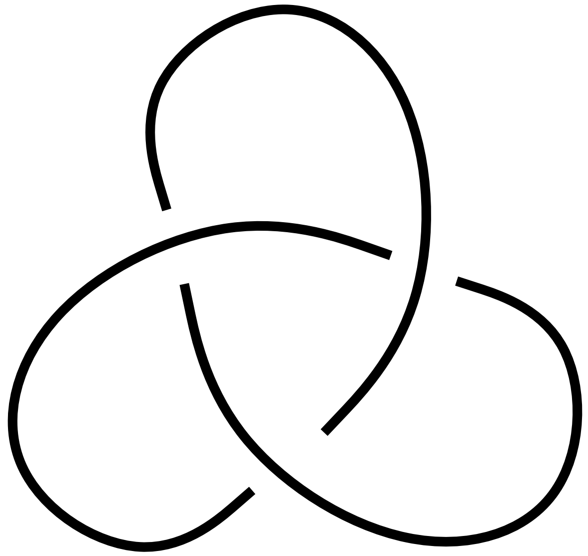


Trefoil knot

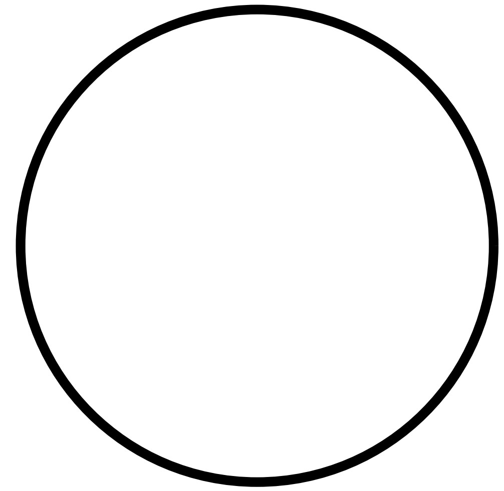


Circle

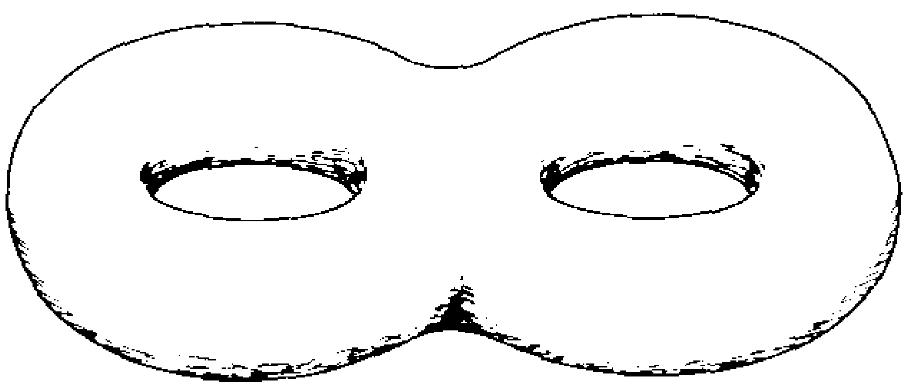
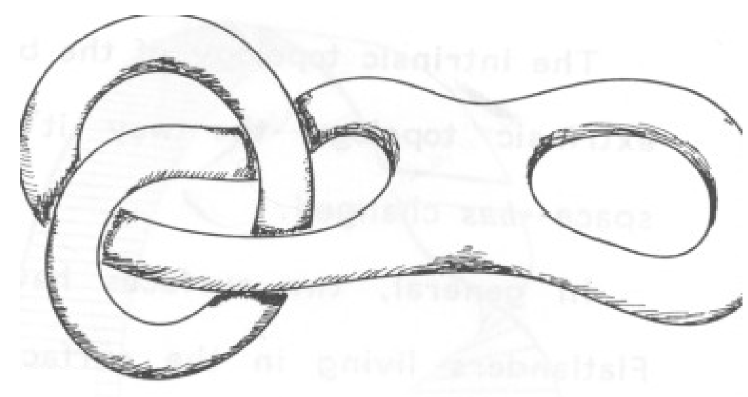
Extrinsic vs. Intrinsic



Trefoil knot

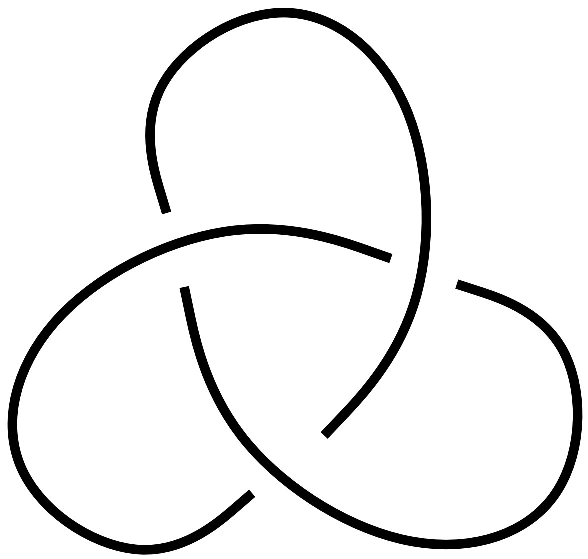


Circle

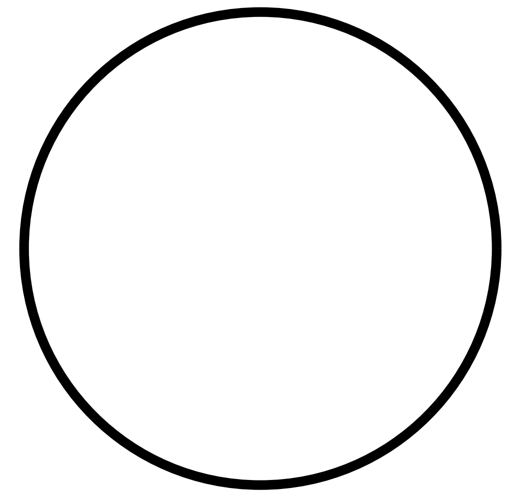
 $\approx$ 

Double torus

Extrinsic vs. Intrinsic

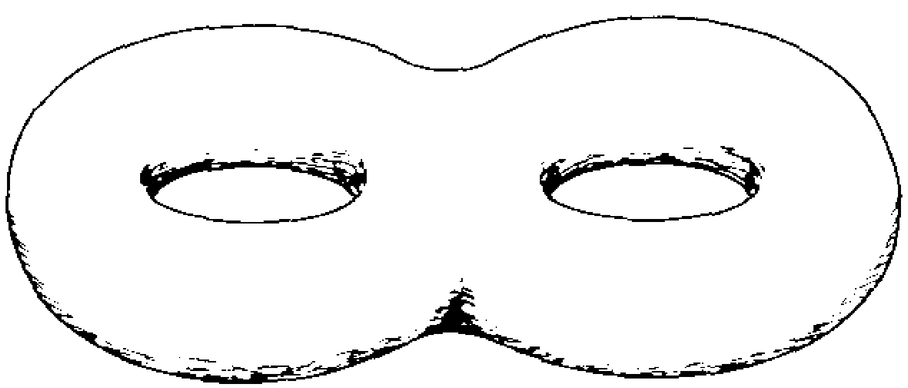


Trefoil knot

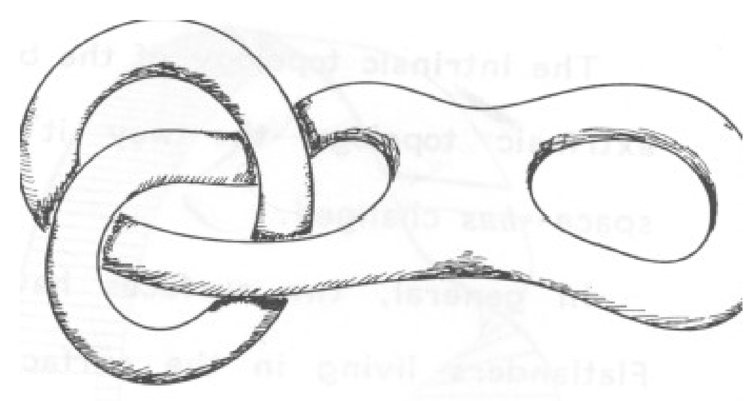


Circle

Today: classifying surfaces, only intrinsic properties matter



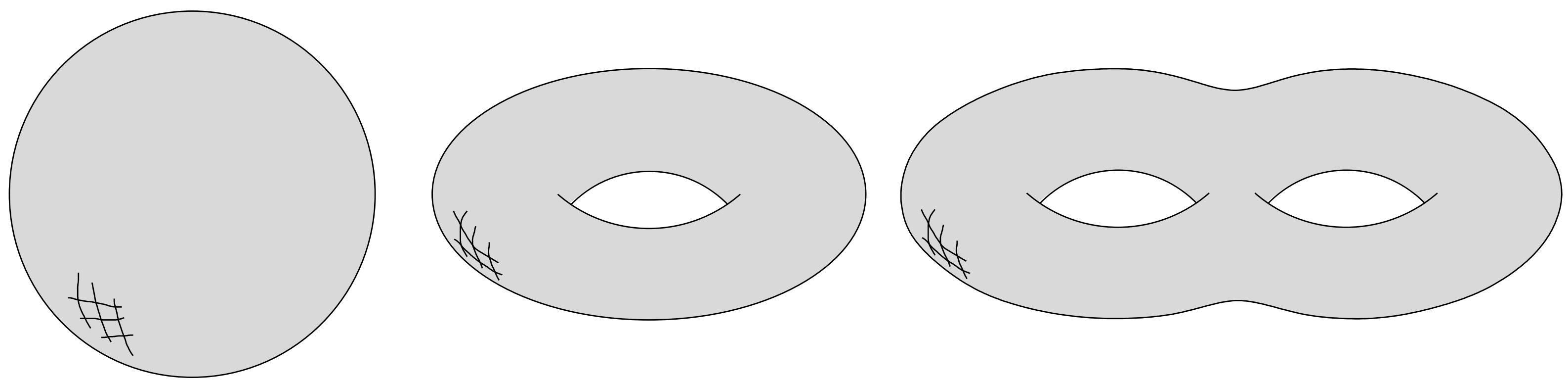
$\approx$

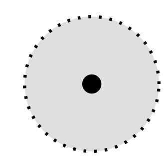


Double torus

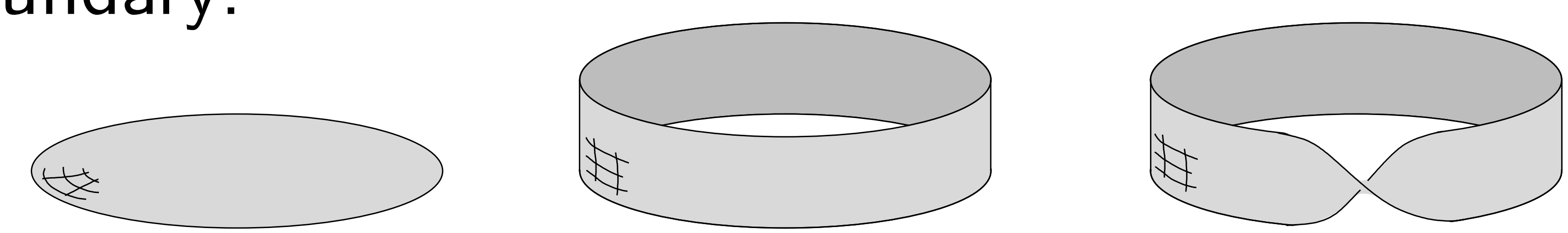
Surfaces (2-Manifolds)

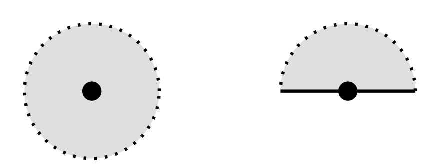
Without boundary:



locally every point: 

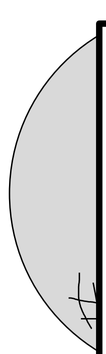
With boundary:

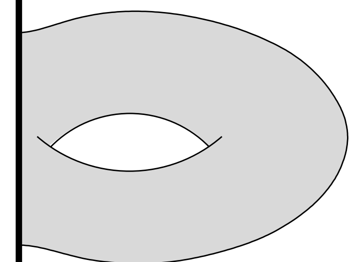


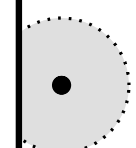
locally every point: 

Surfaces (2-Manifolds)

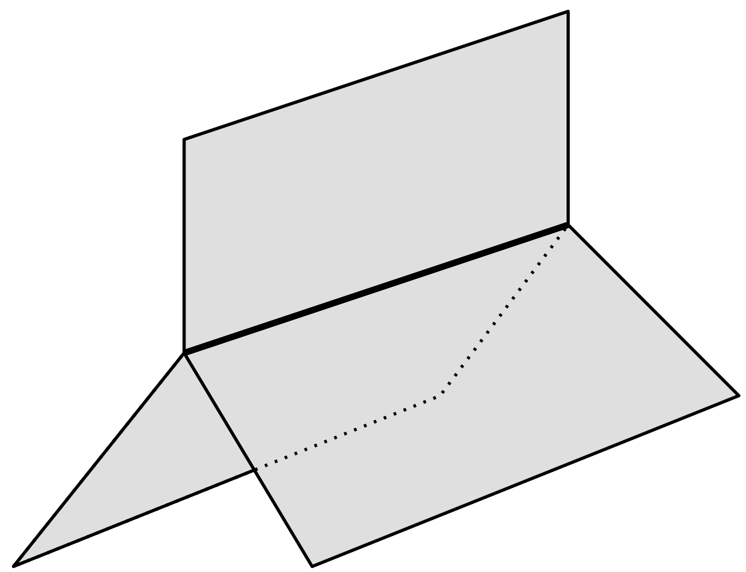
Without boundary:





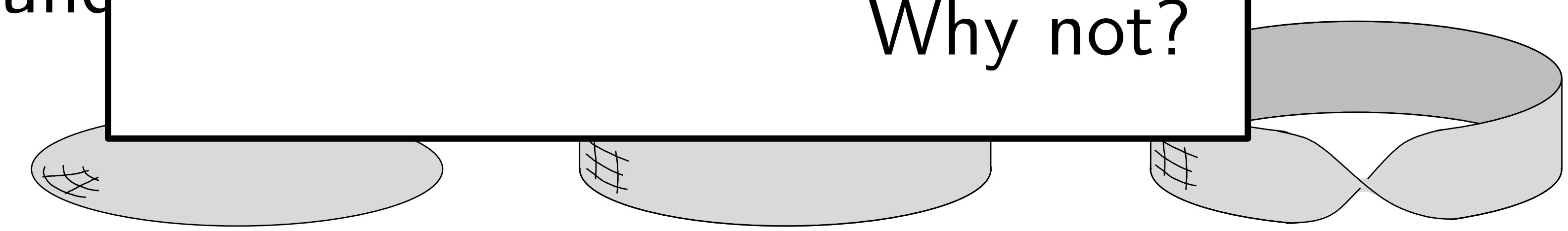


Not a manifold:

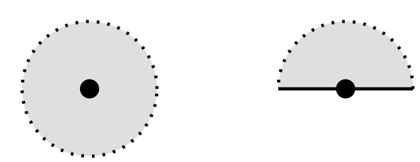


Why not?

With bound

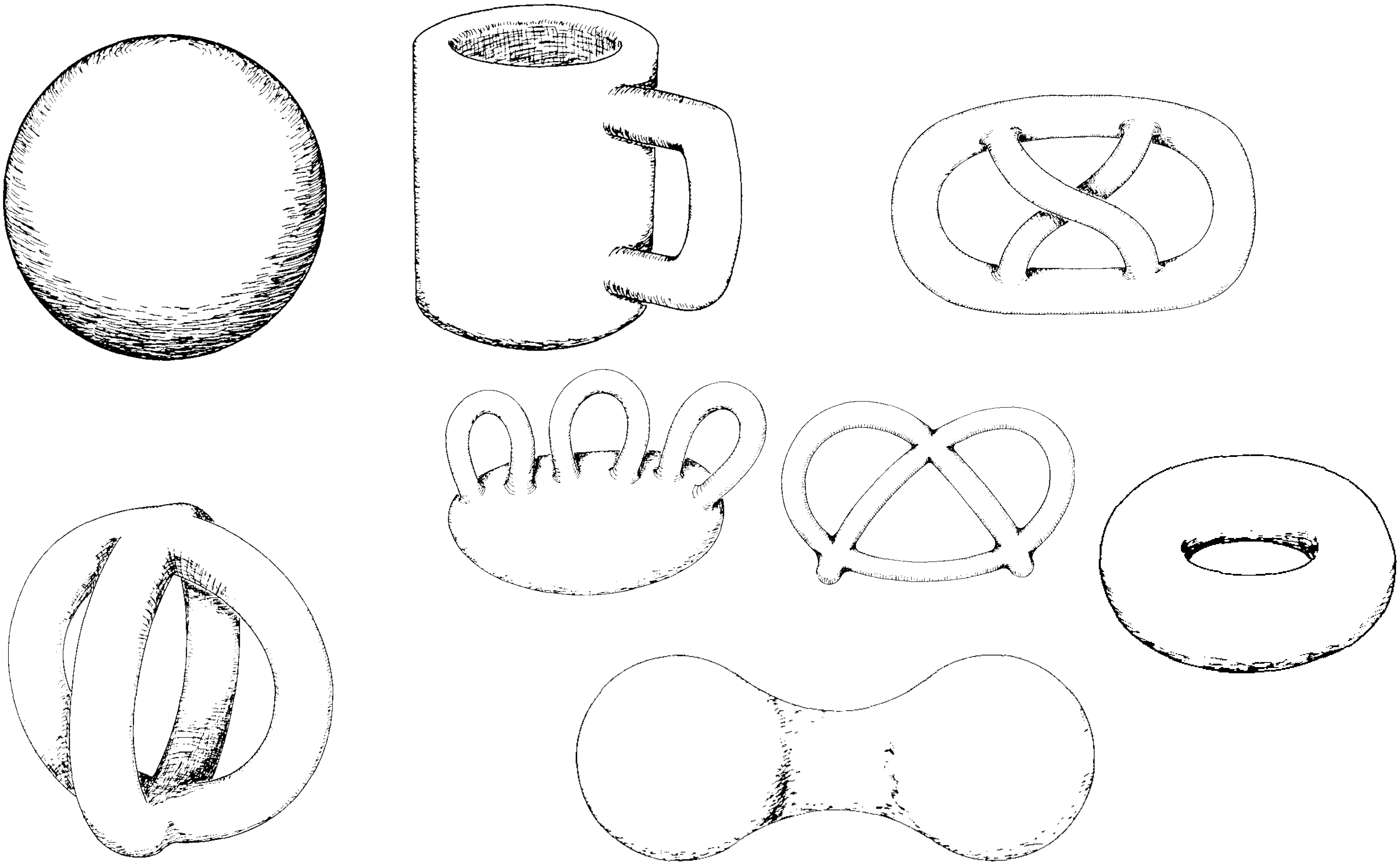


locally every point:

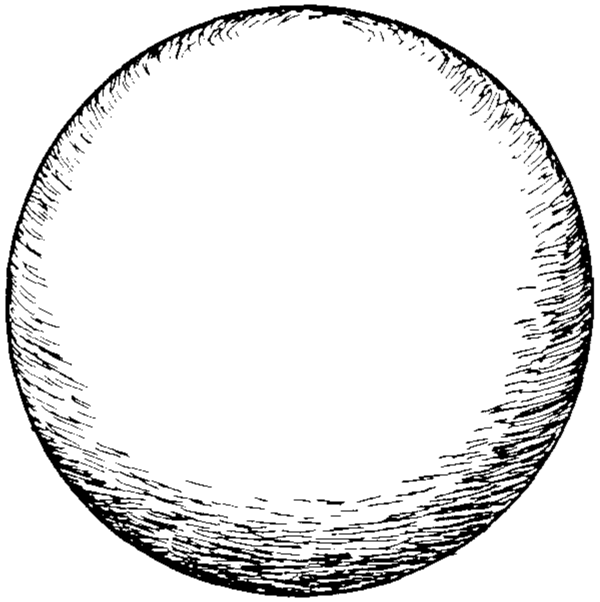


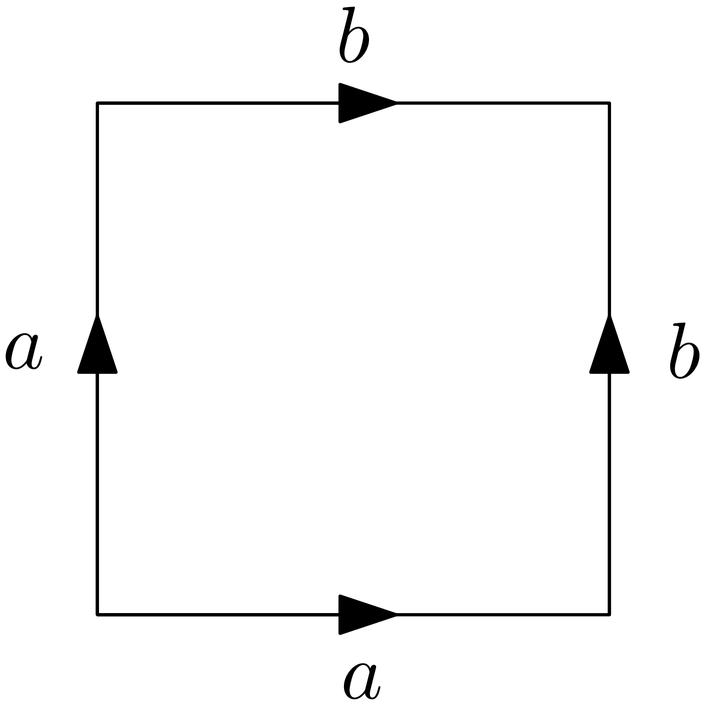
Quiz

Which surfaces are topologically the same?

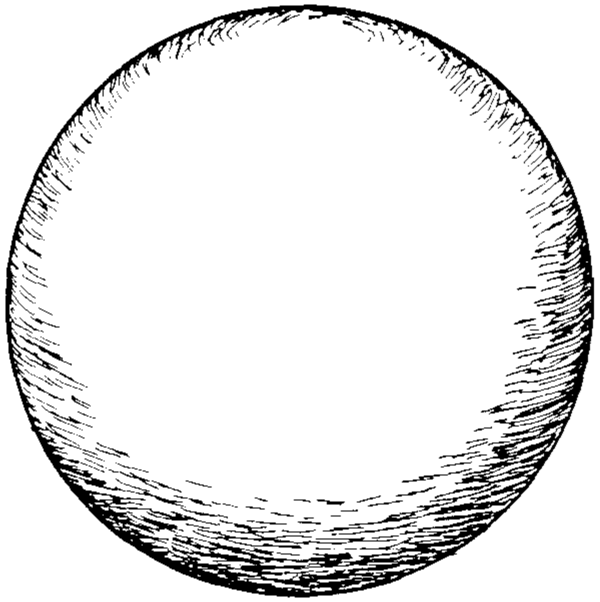


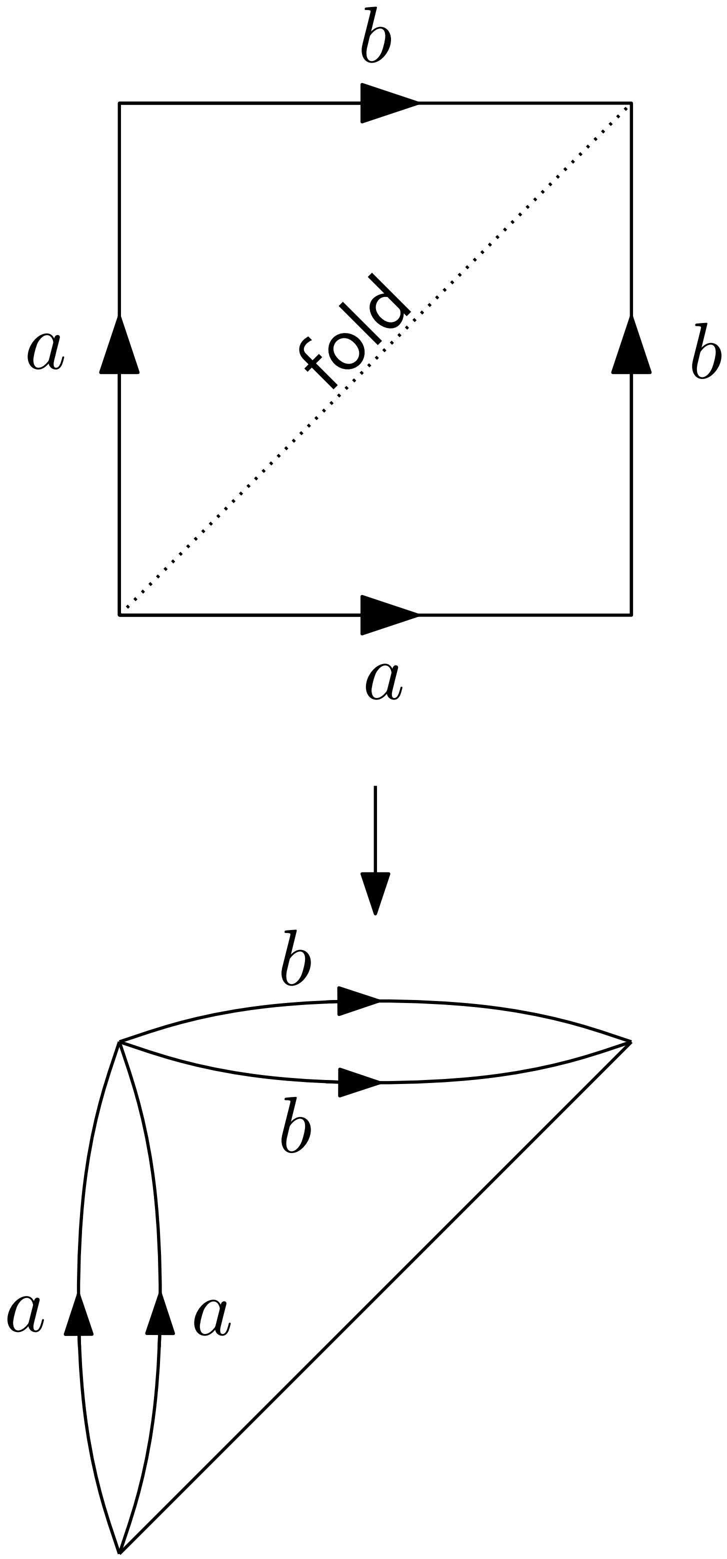
Sphere



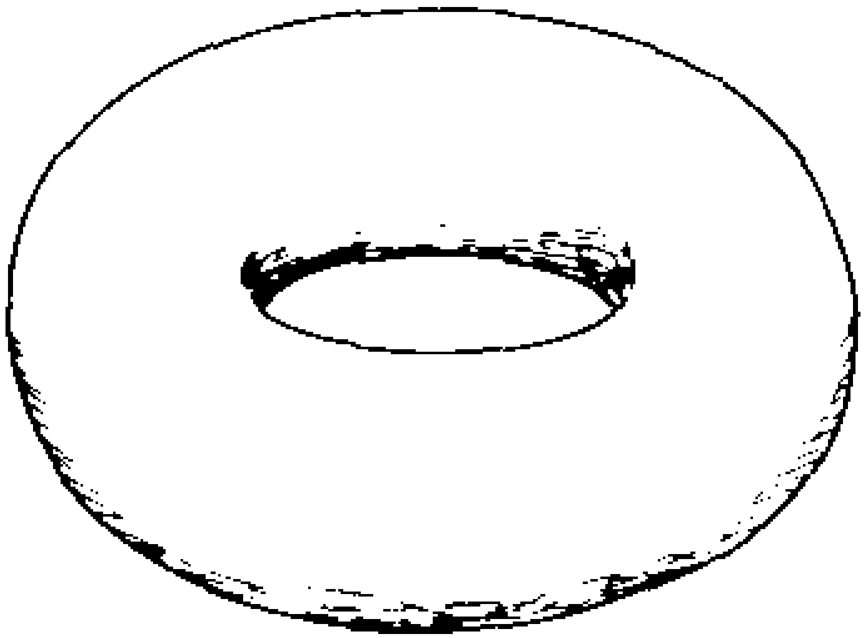


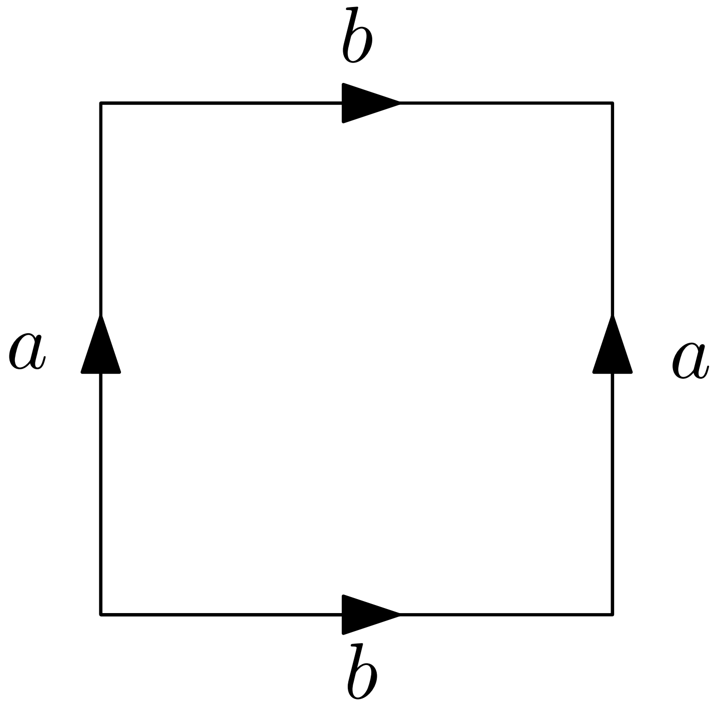
Sphere



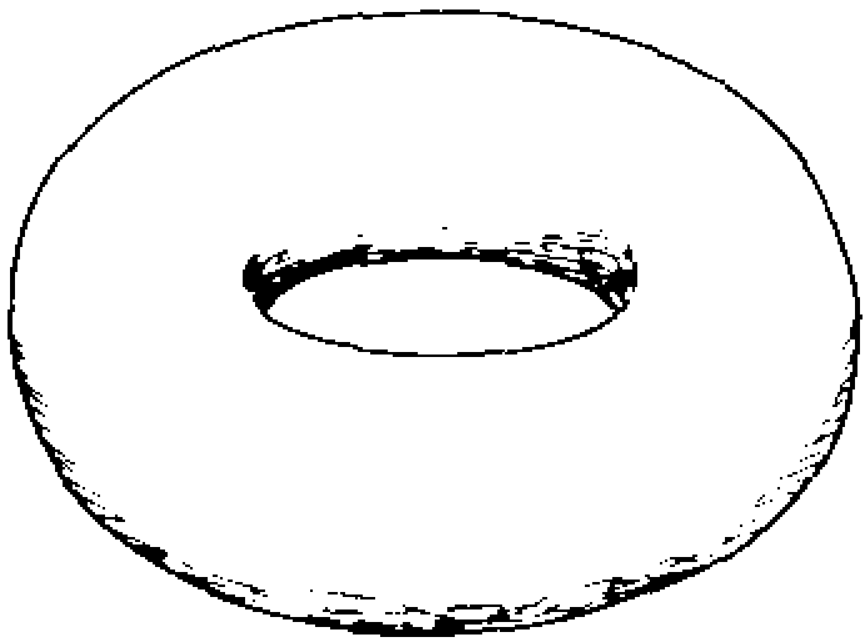


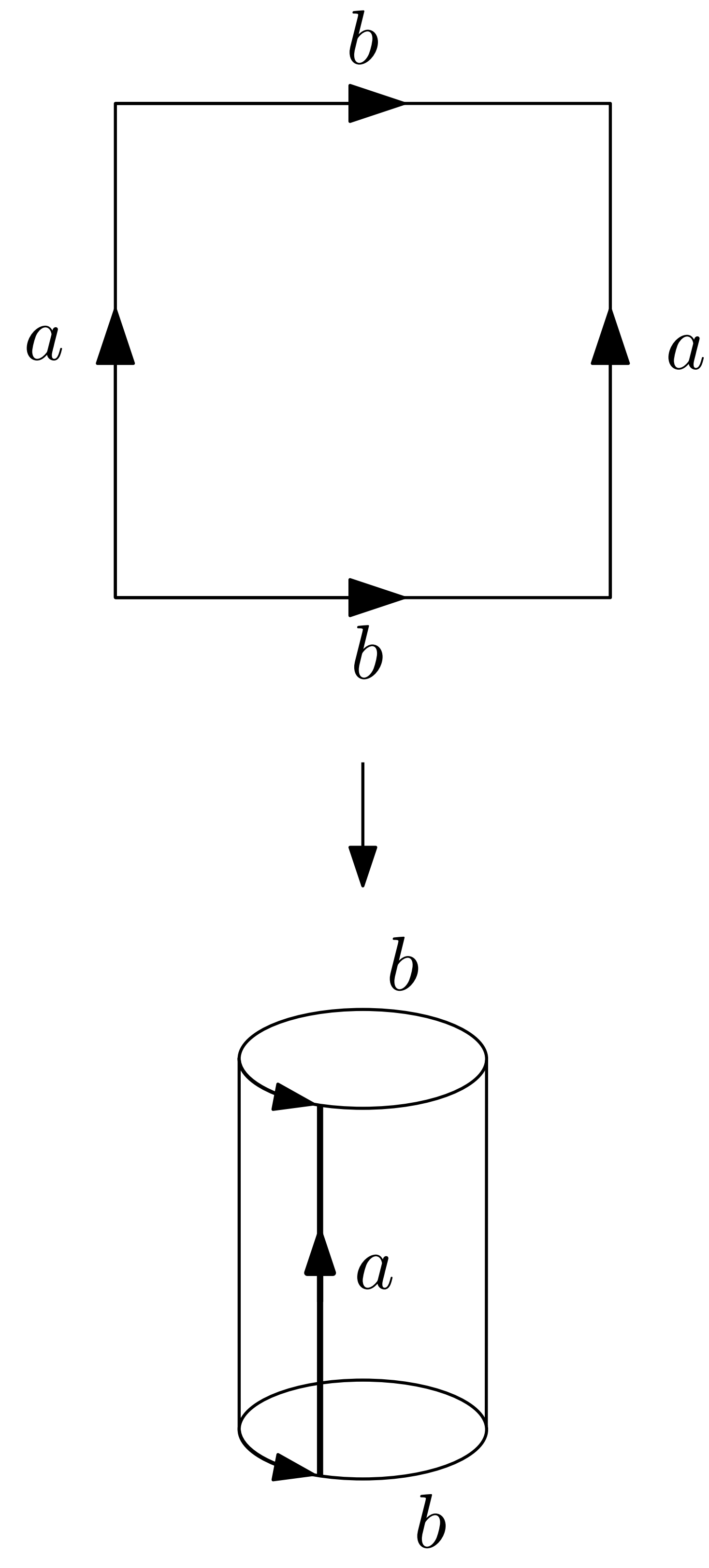
Torus



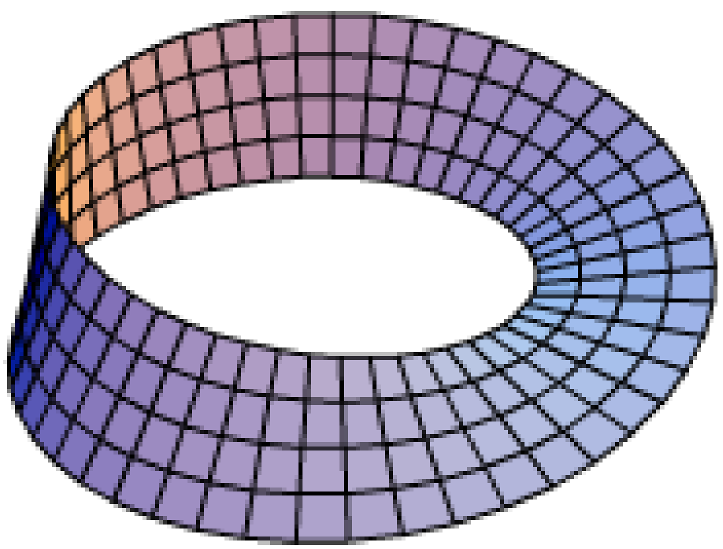


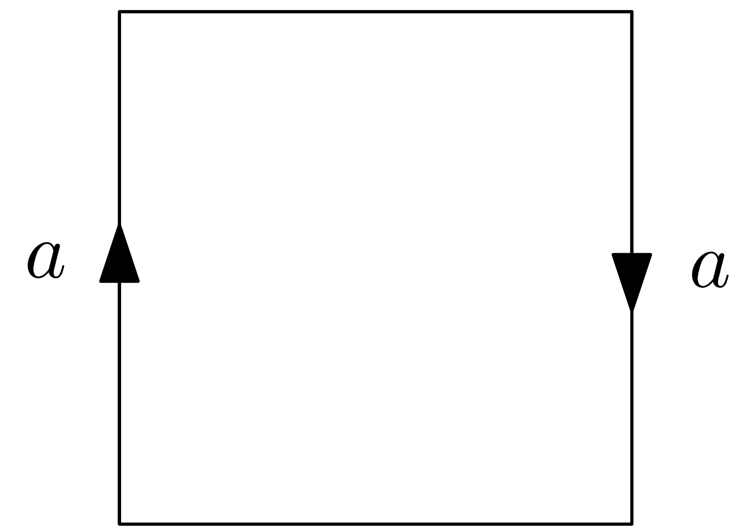
Torus





Möbius Strip

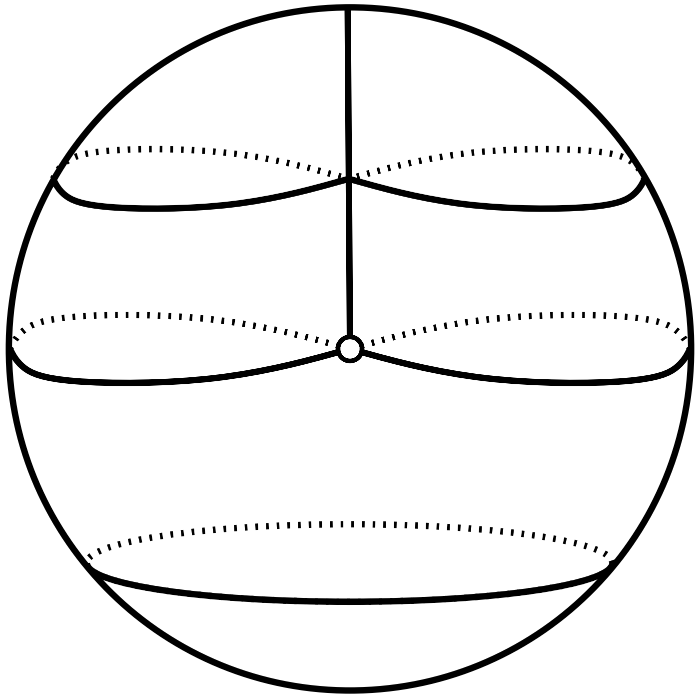


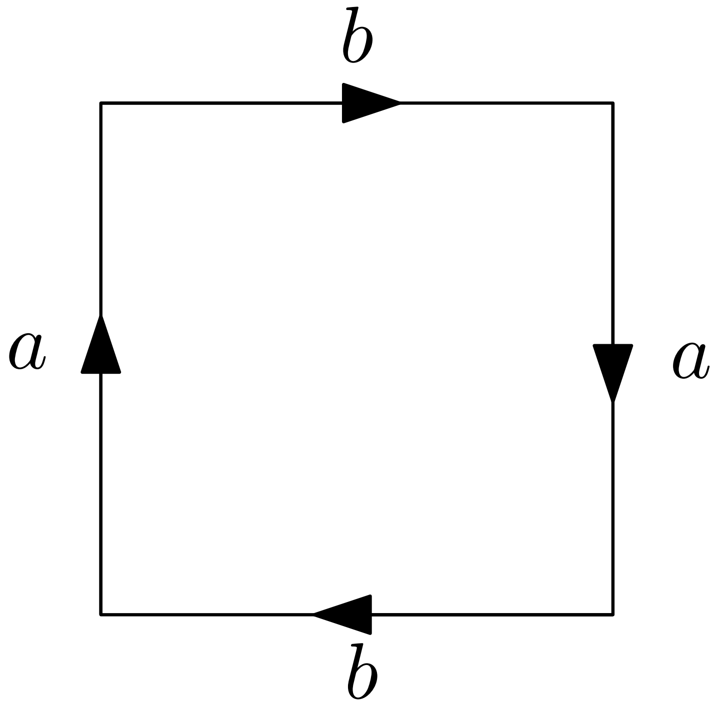


Boundary a single circle

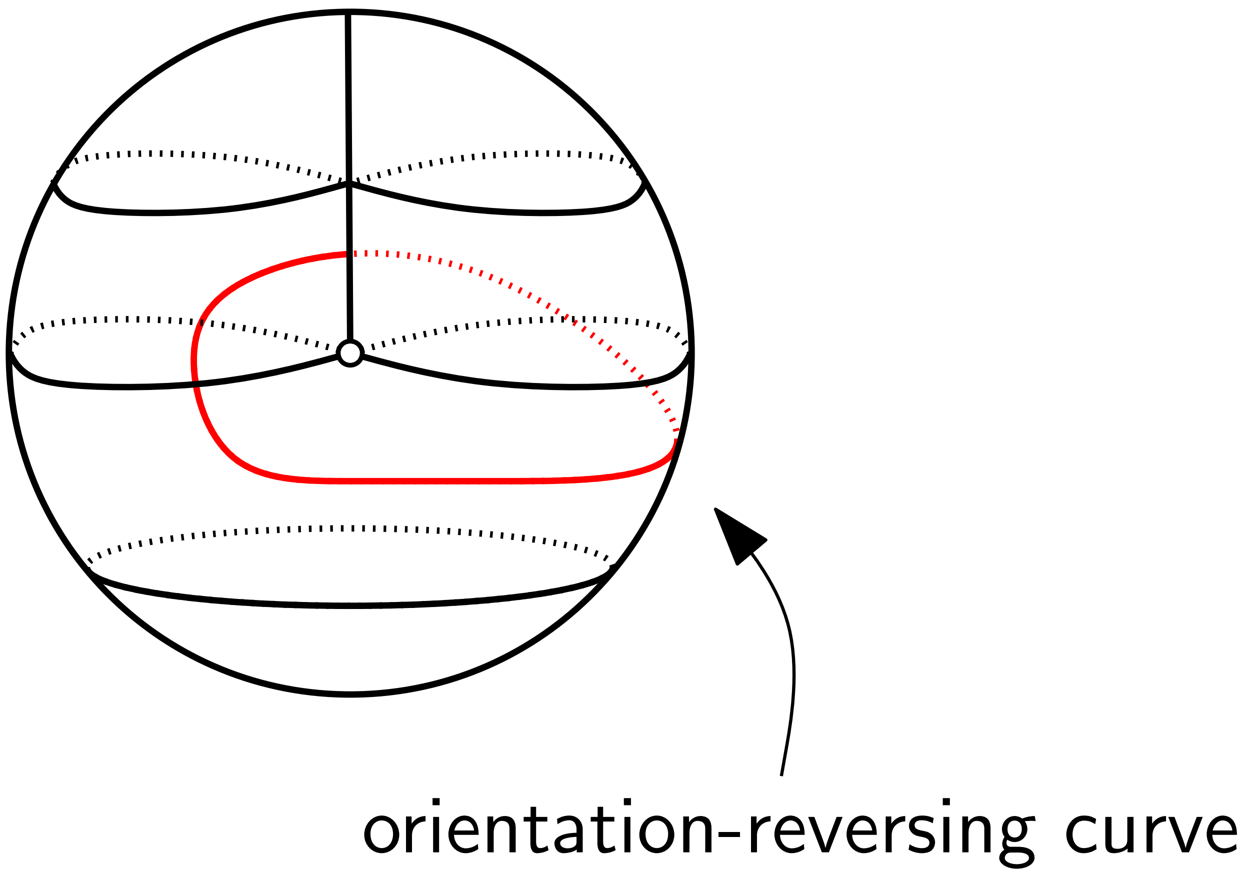
If all closed curves in a 2-manifold are orientation-preserving, then the 2-manifold is orientable; otherwise, it is non-orientable.

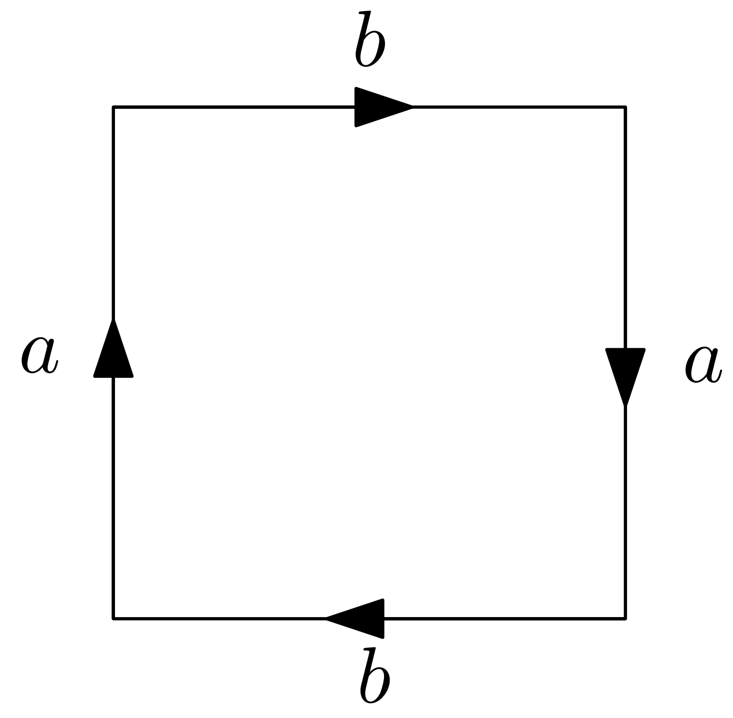
Projective Plane



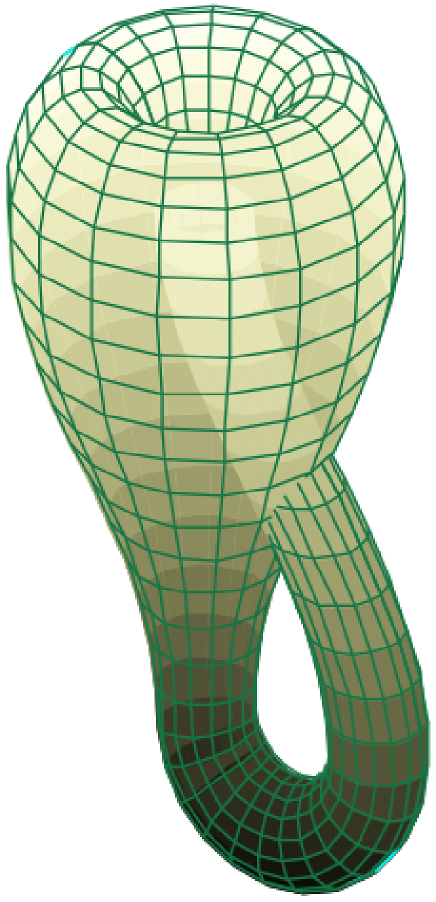


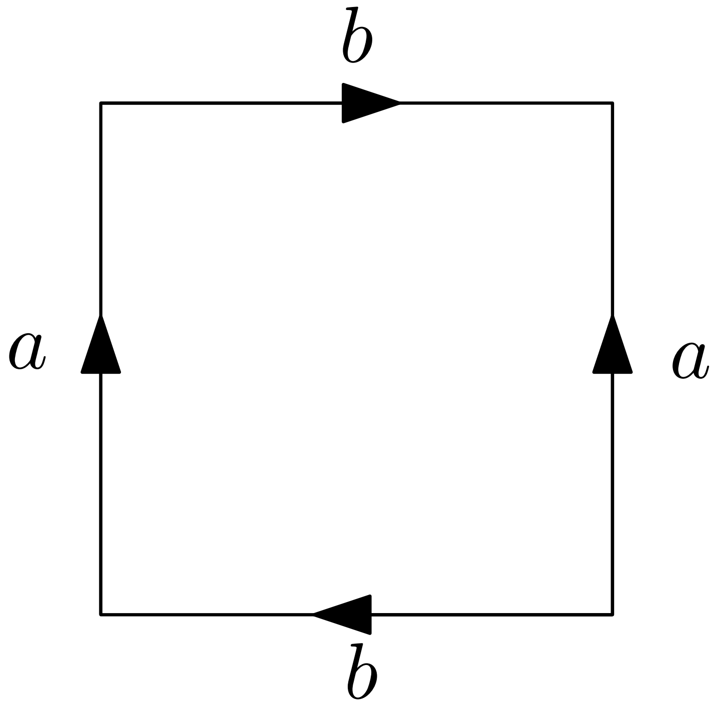
Projective Plane



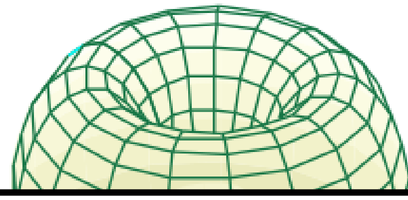


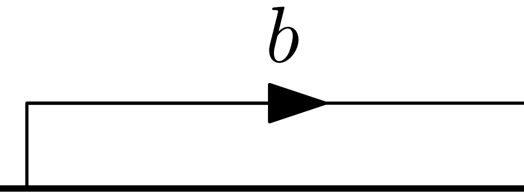
Klein Bottle

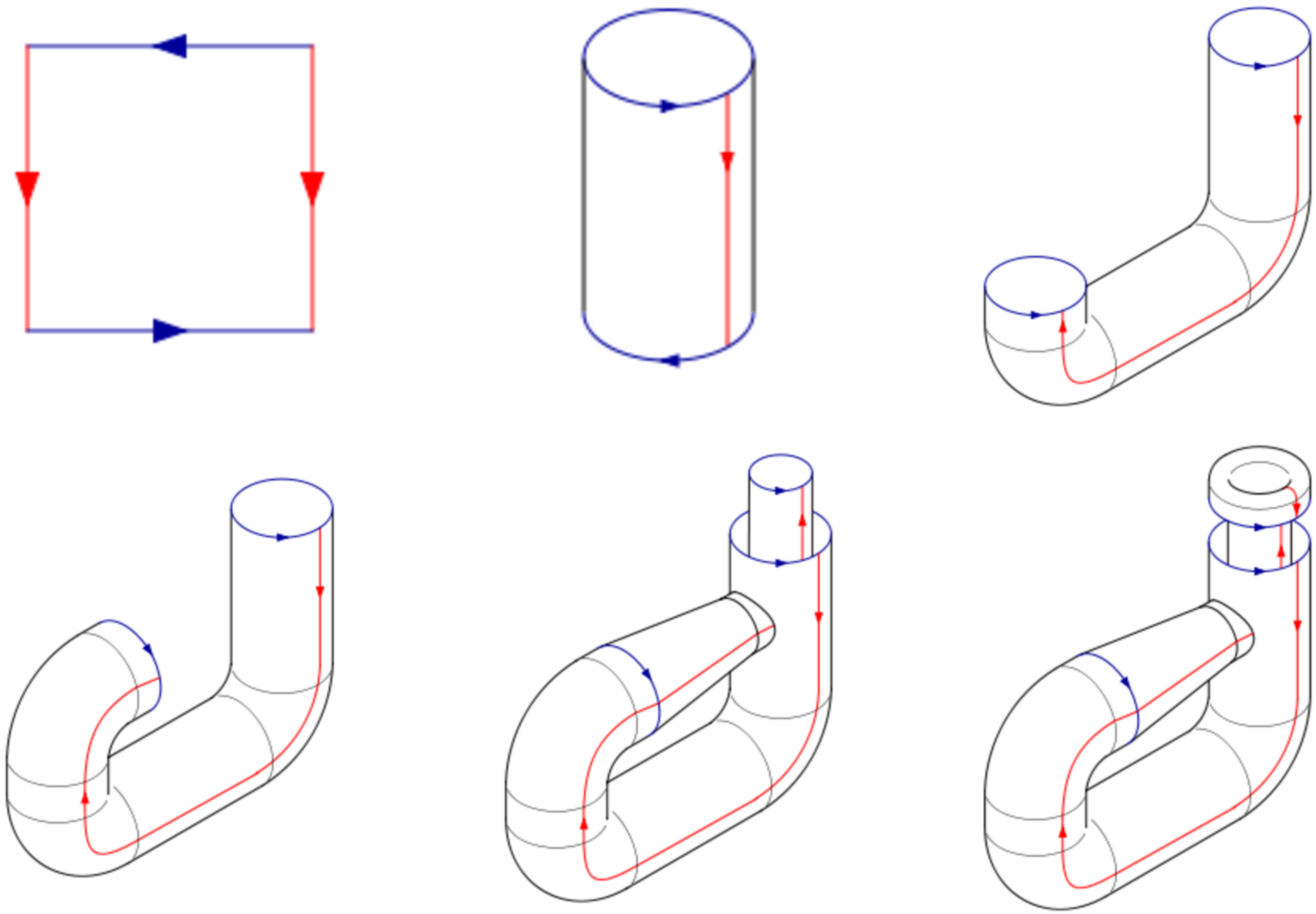




Klein Bottle

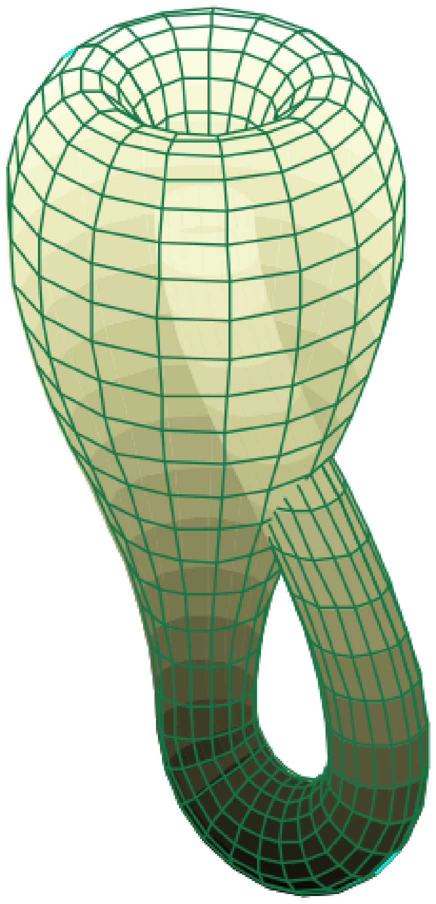


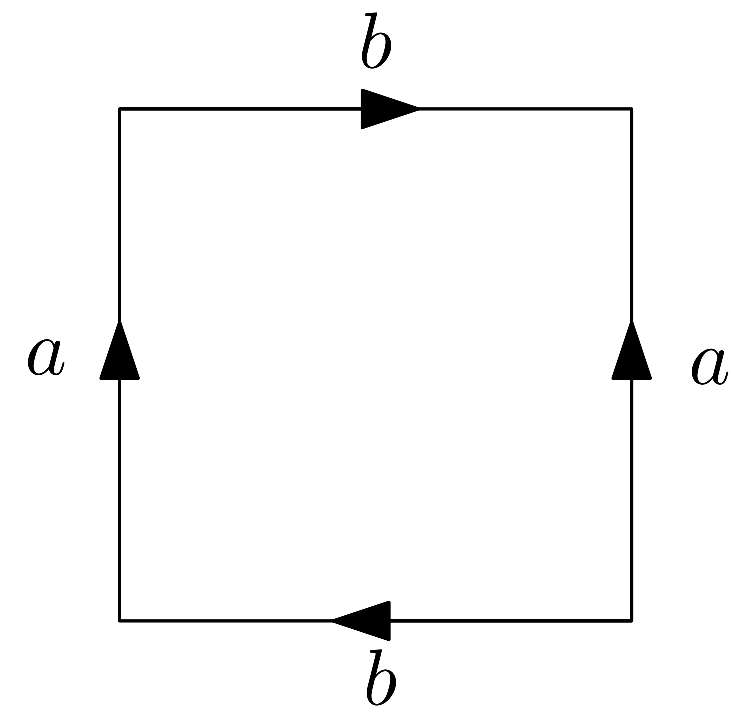




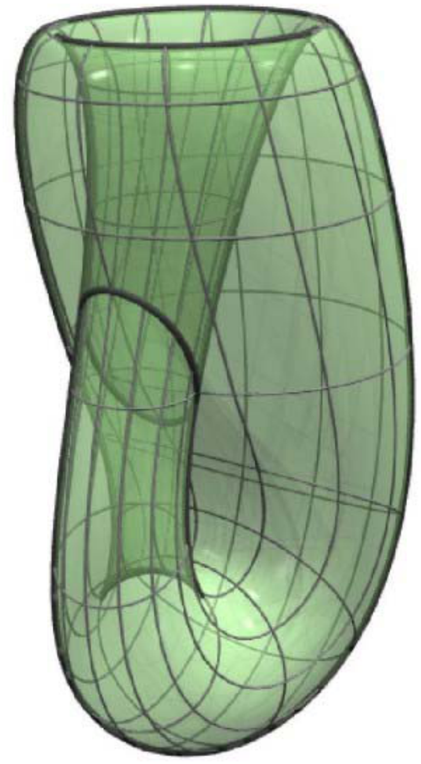
(Courtesy Wikipedia)

Klein Bottle





immersion = local embedding

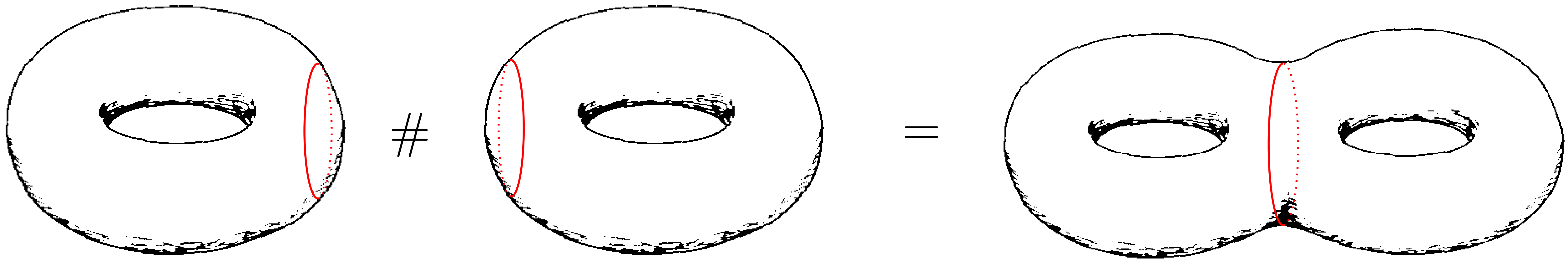


Connected sum

The connected sum of two surfaces $\mathbb{M}_1$, $\mathbb{M}_2$ is

$$\mathbb{M}_1 \# \mathbb{M}_2 = (\mathbb{M}_1 - D_1) \bigcup_{\partial D_1 = \partial D_2} (\mathbb{M}_2 - D_2)$$

where D_1, D_2 are closed disks in $\mathbb{M}_1, \mathbb{M}_2$, respectively.



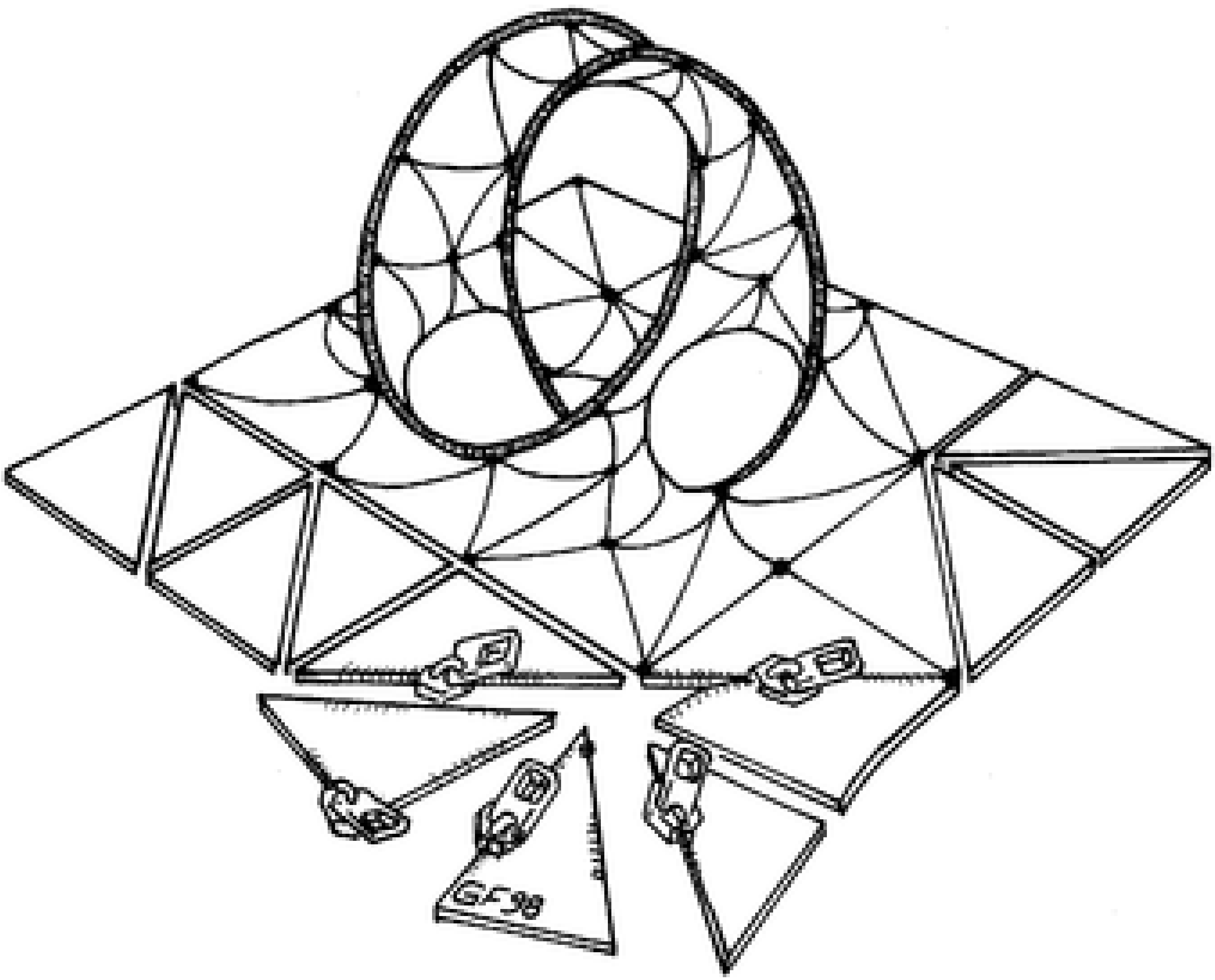
Classification Theorem

Theorem. Every closed compact surface is homeomorphic to a sphere, a connected sum of tori, or connected sum of projective planes.

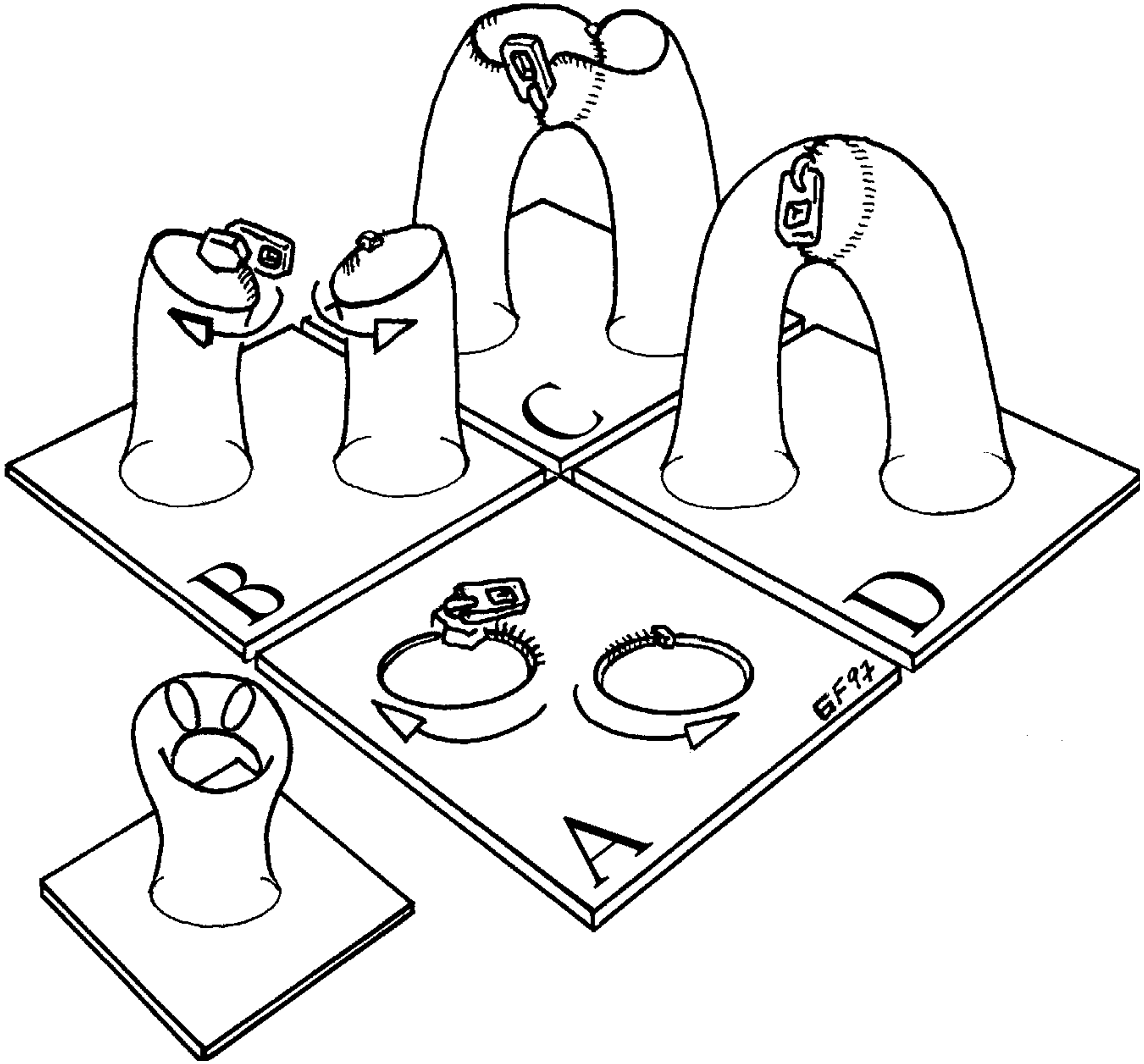
- Known since 1860s
- Today: Conway's Zero Irrelevancy Proof or ZIP (1992), written down by Francis and Weeks (1999)

Zip Pairs

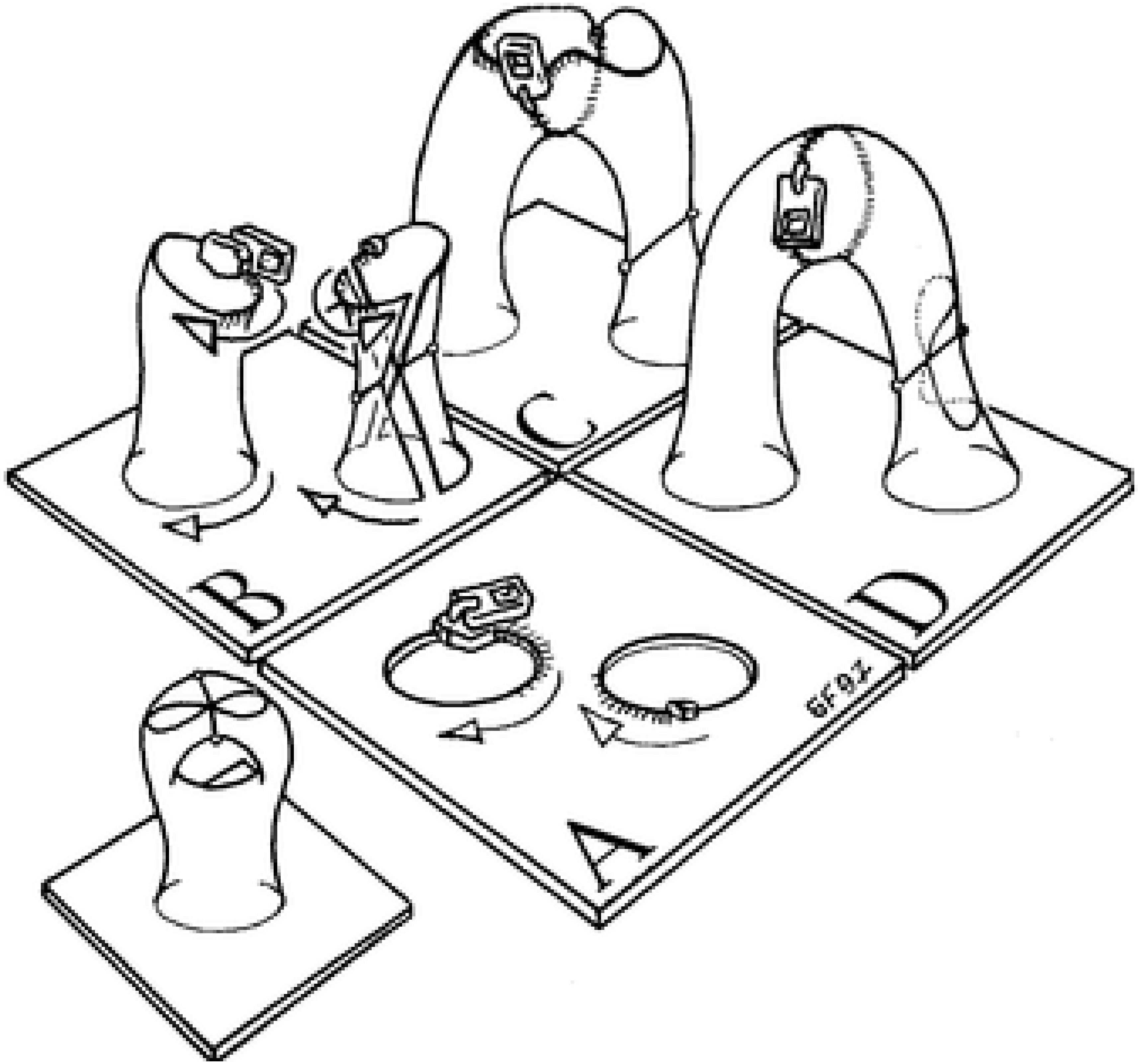
Triangulate the surface, and "unzip" it



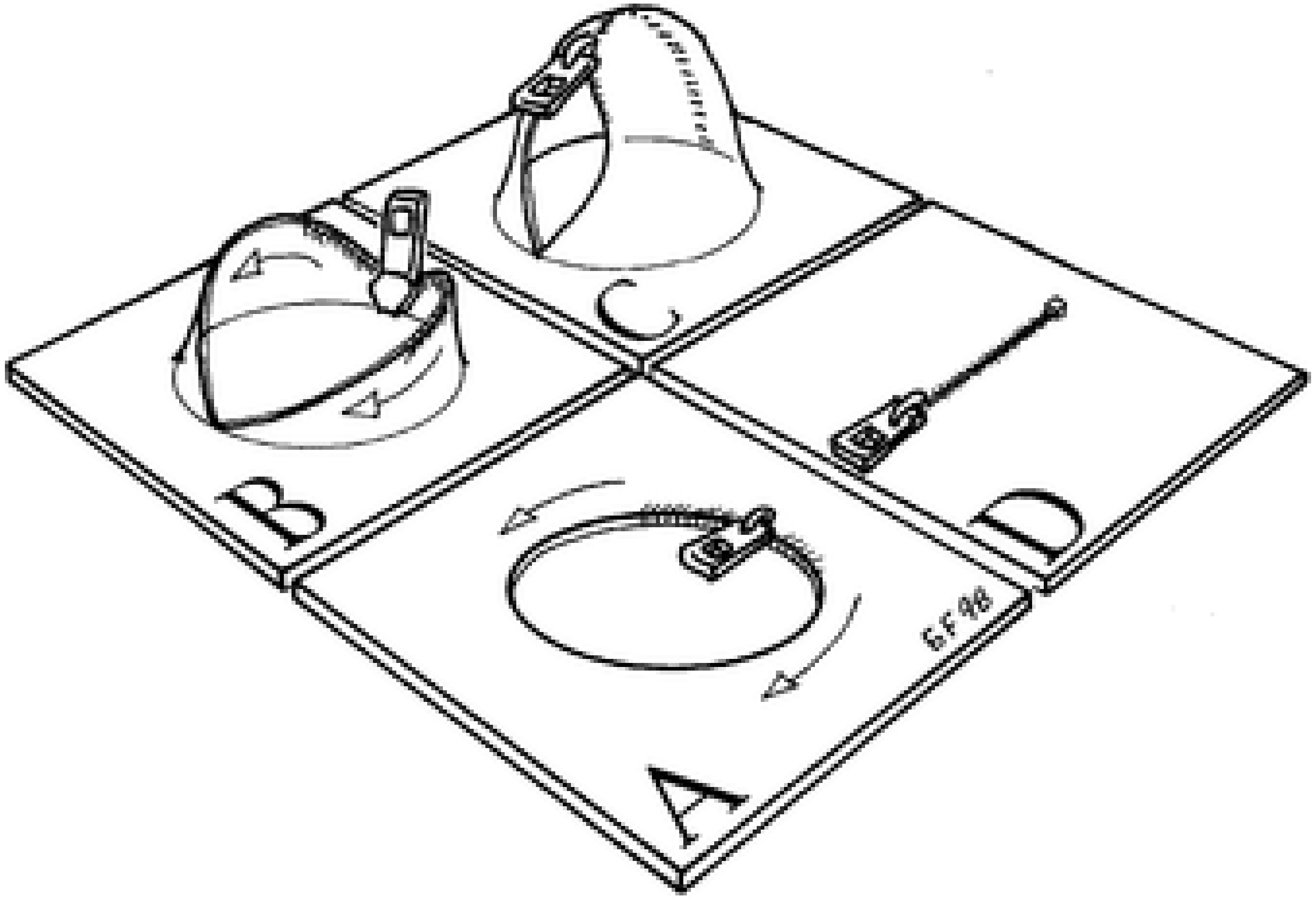
Handle



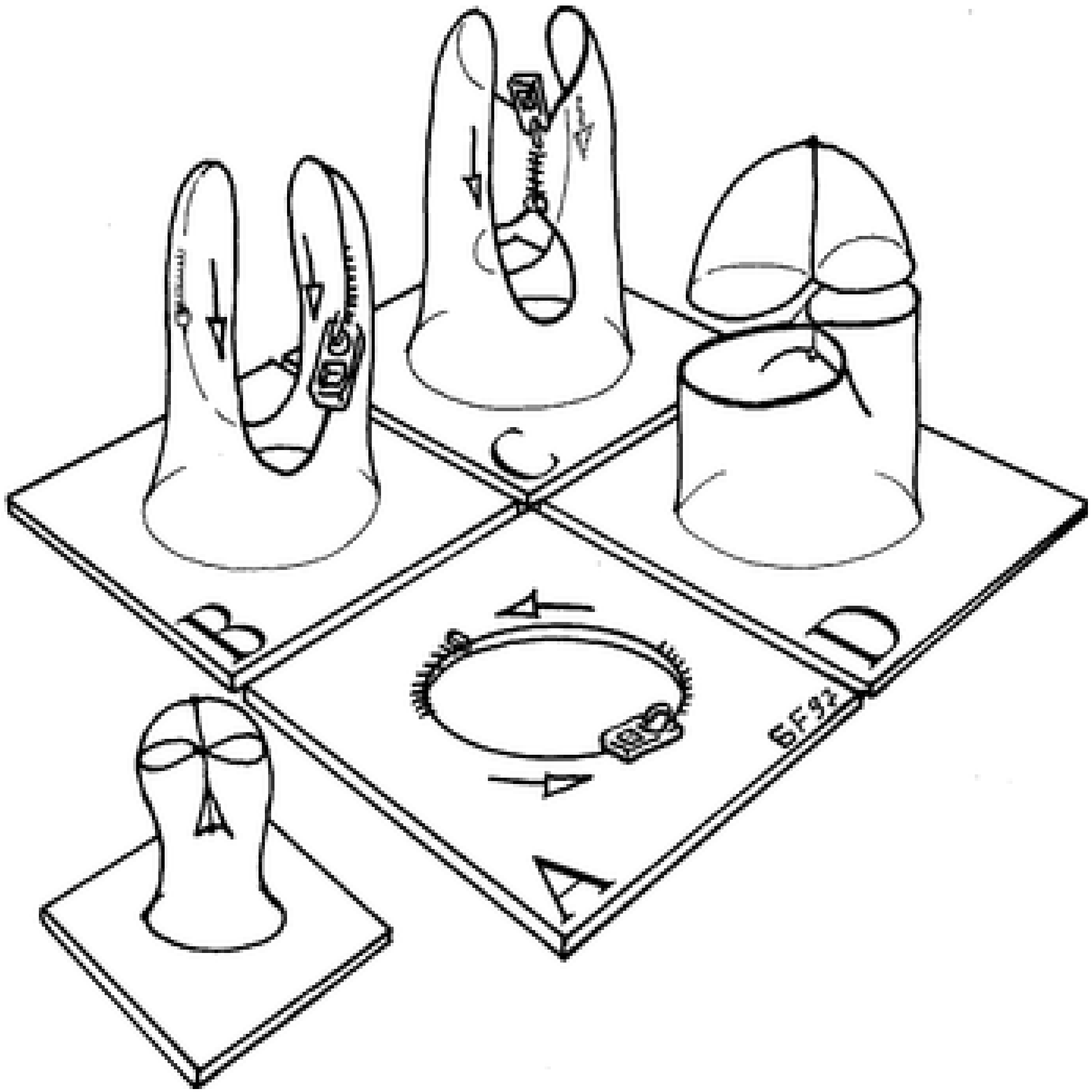
Cross handle



Cap



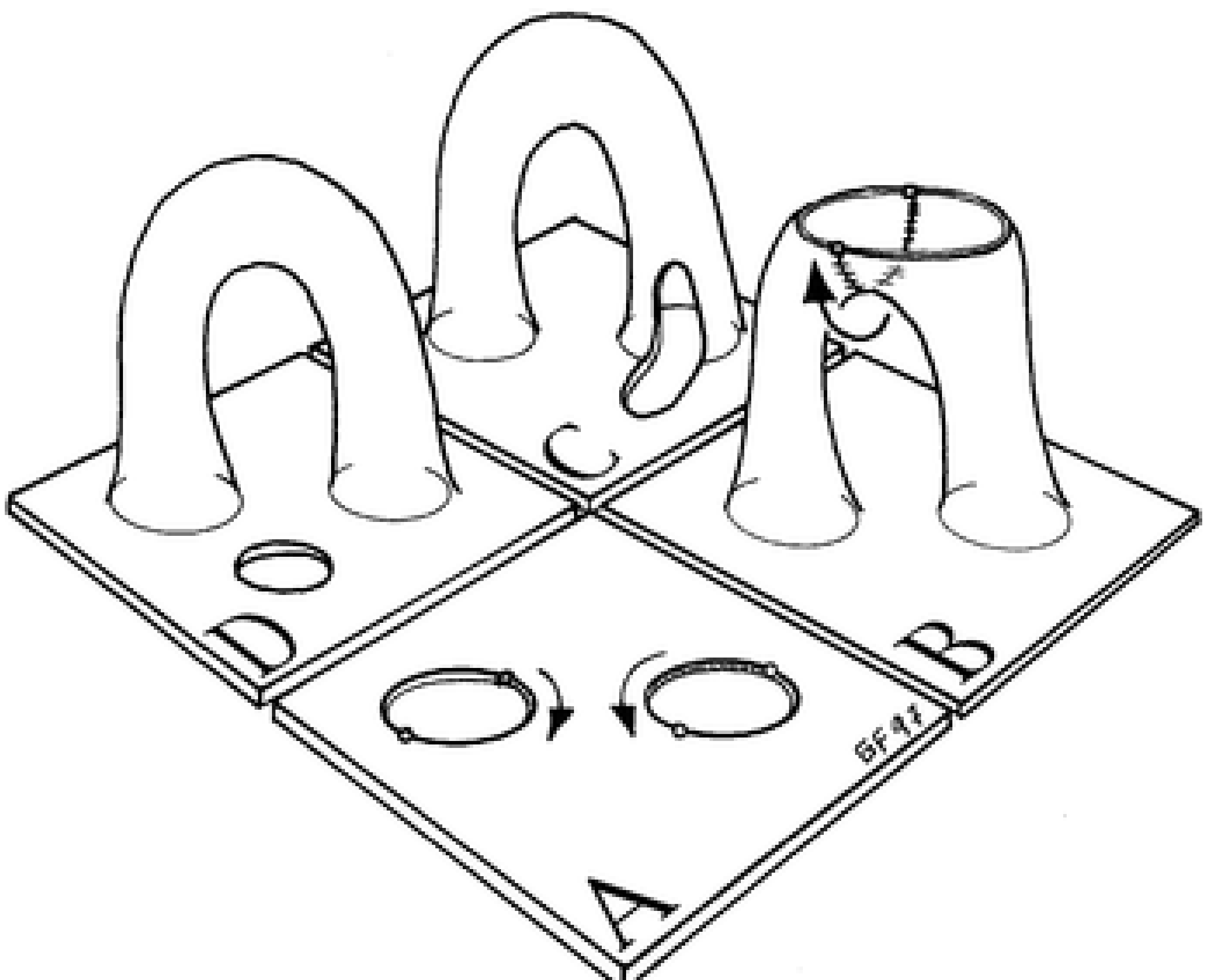
Cross cap



Ordinary surfaces

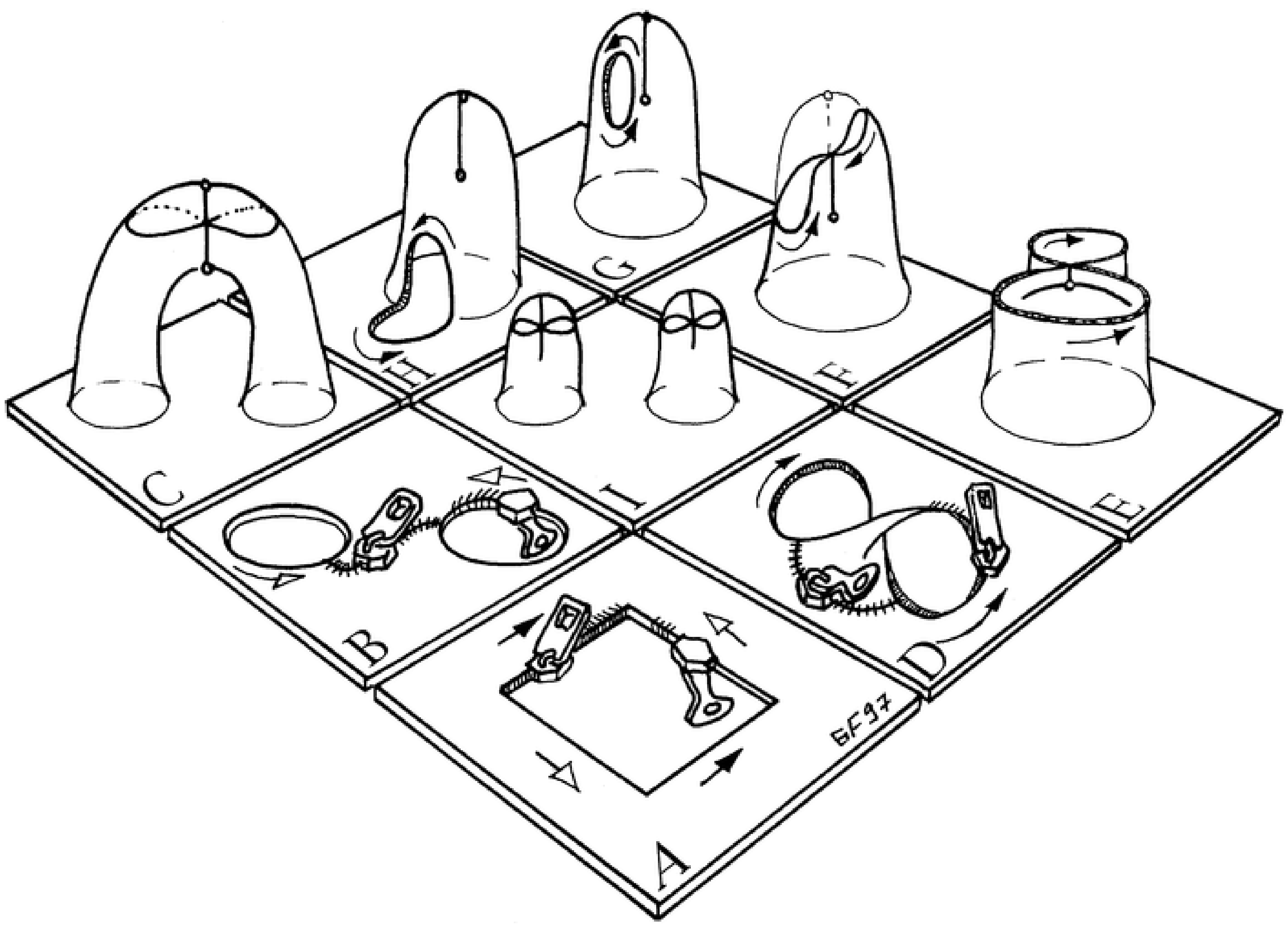
ordinary surface = finite collection of spheres, each with a finite number of handles, crosshandles, crosscaps, and perforations

Lemma 1. If the surface is ordinary before two zips are zipped together, it is ordinary afterwards as well.



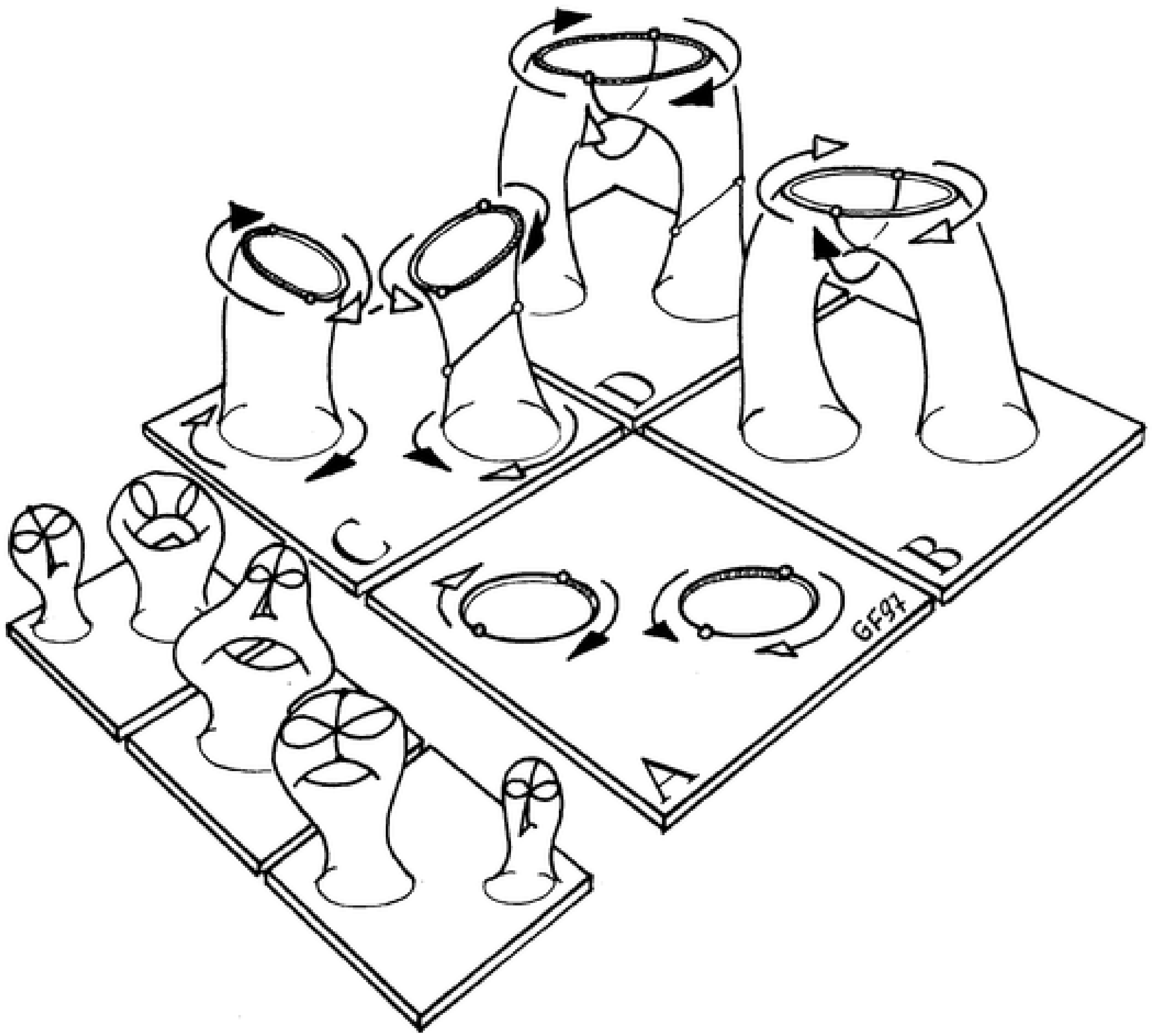
Order of operations

Lemma 2. XHandle = 2 XCaps



Order of operations

Lemma 3. XHandle + XCap = Handle + XCap



Proof summary

- triangles zipped together $\Rightarrow$ ordinary surfaces (spheres with handles and crosscaps)
- Lemma 2: crosshandle = 2 crosscaps
- Lemma 3: crosshandle + crosscap = handle + crosscap
- Ordinary surfaces = sphere, sphere with handles, or sphere with crosscaps.

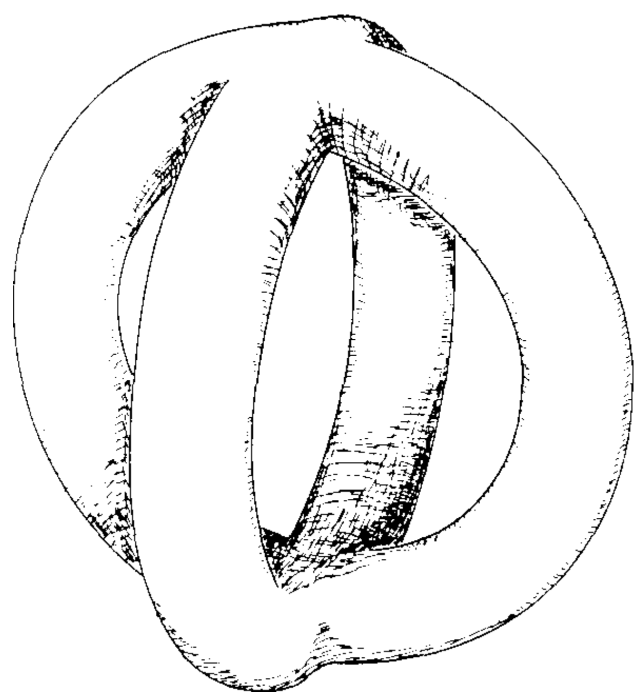
Manifolds with boundary = above + perforations

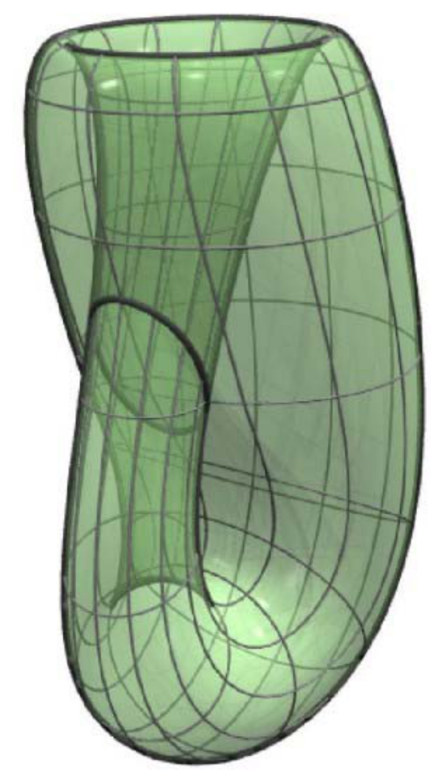
Proof summary

- tria … dles and…
- Ler…
- Ler…
- Or… oss-cap…

Ma…

Spheres with handles and cross caps is all there is!





Euler Characteristic

Euler characterstic: $\chi = v - e + f$ (independent of triangulation)

v = number of vertices,
e = number of edges,
f = number of triangles

Euler Characteristic for Compact 2-Manifolds. A sphere with g tubes has $\chi = 2 - 2g$ and a sphere with g cross-caps has $\chi = 2 - g$.

g = genus of $\mathbb{M}$; it is the maximum number of disjoint closed curves along which we can cut without disconnecting $\mathbb{M}$.

$\chi(\mathbb{N})$	$g(\mathbb{N})$	$\mathbb{N}$	$\mathbb{M}$	$g(\mathbb{M})$	$\chi(\mathbb{M})$
1	1	$\mathbb{P}^2$	$\mathbb{S}^2$	0	2
0	2	$\mathbb{P}^2 \# \mathbb{P}^2$	$\mathbb{T}^2$	1	0
-1	3	$\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$	$\mathbb{T}^2 \# \mathbb{T}^2$	2	-2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Thank you for your time and attention!

Acknowledgements

Most figures from:

- "Topology vs. Geometry" by Herb Ling
  `http://www.austincc.edu/herbling/shape-of-space.pdf`
- "Conway's ZIP Proof" by Francis and Weeks
  `http://new.math.uiuc.edu/zipproof/zipproof.pdf`
- Wikipedia