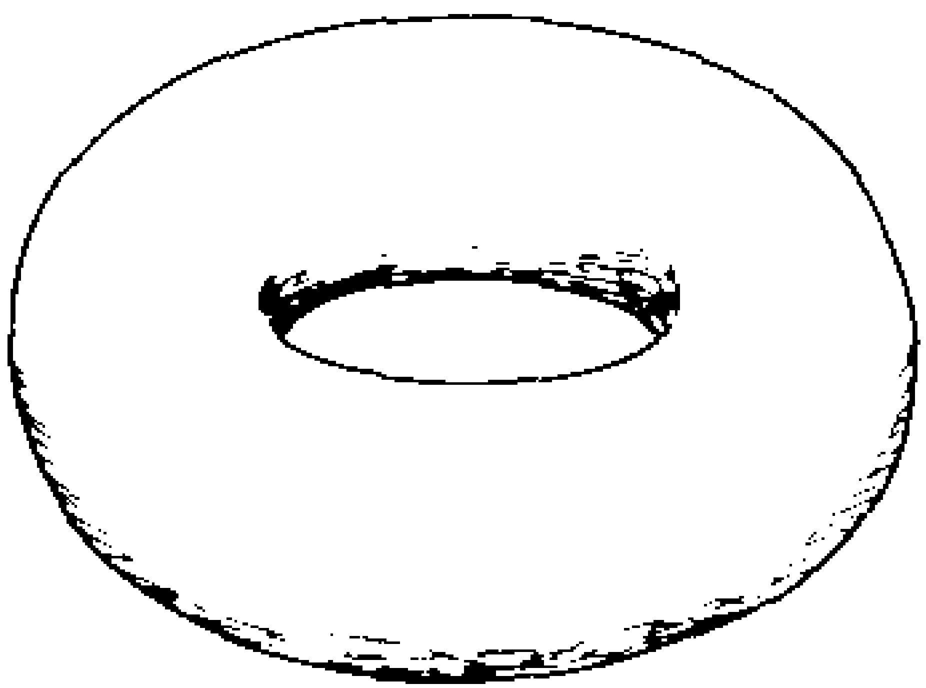
CS164: Topology II, Complexes and Homology

May 17, 2010

Topology



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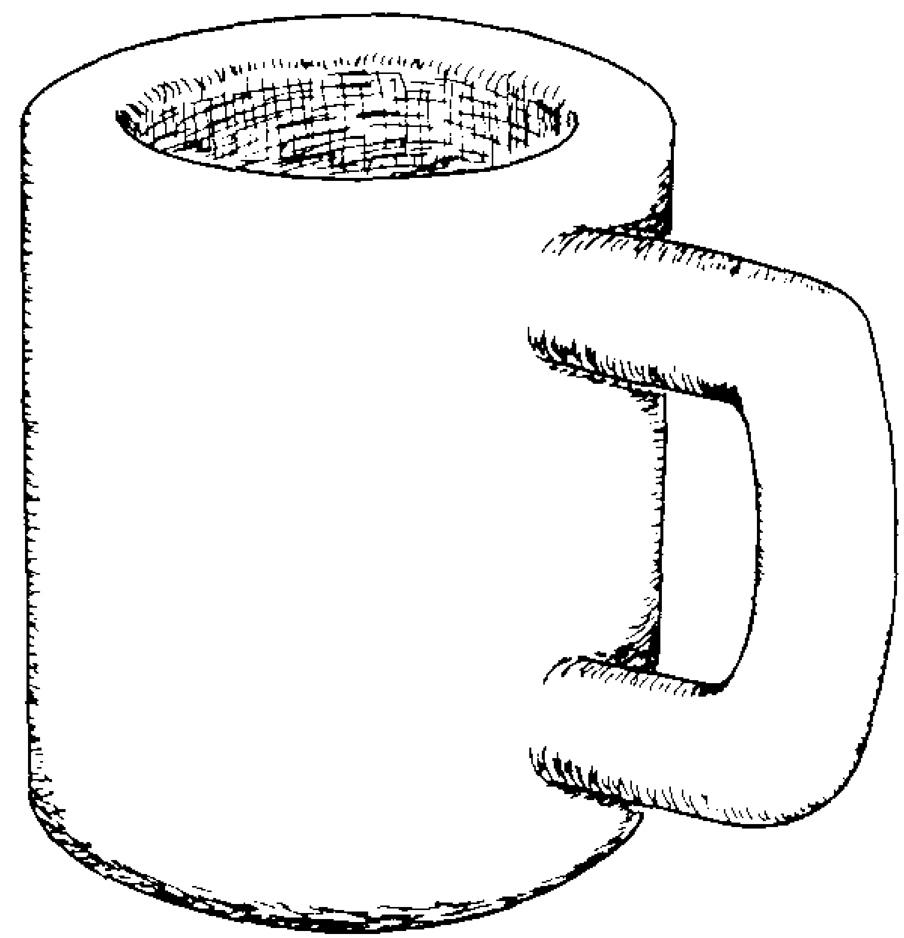
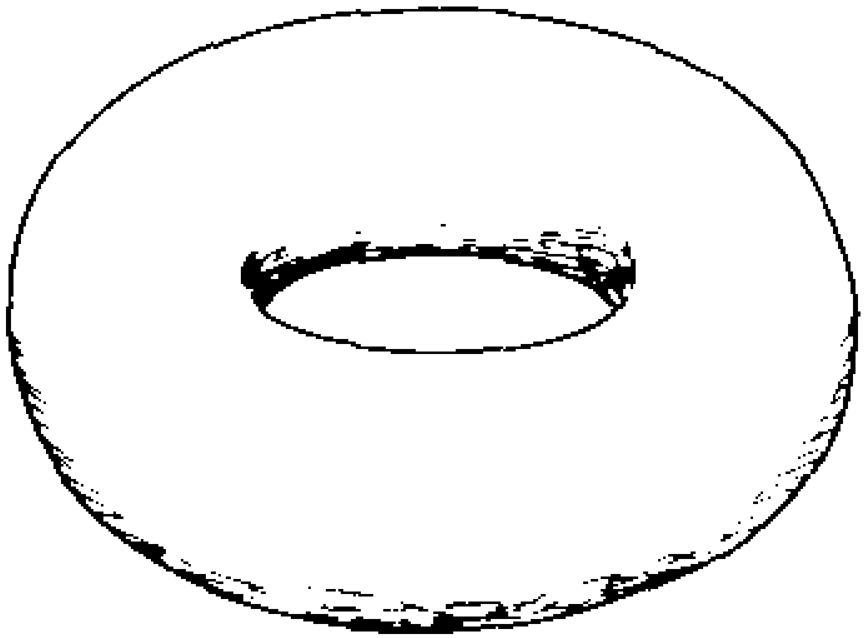
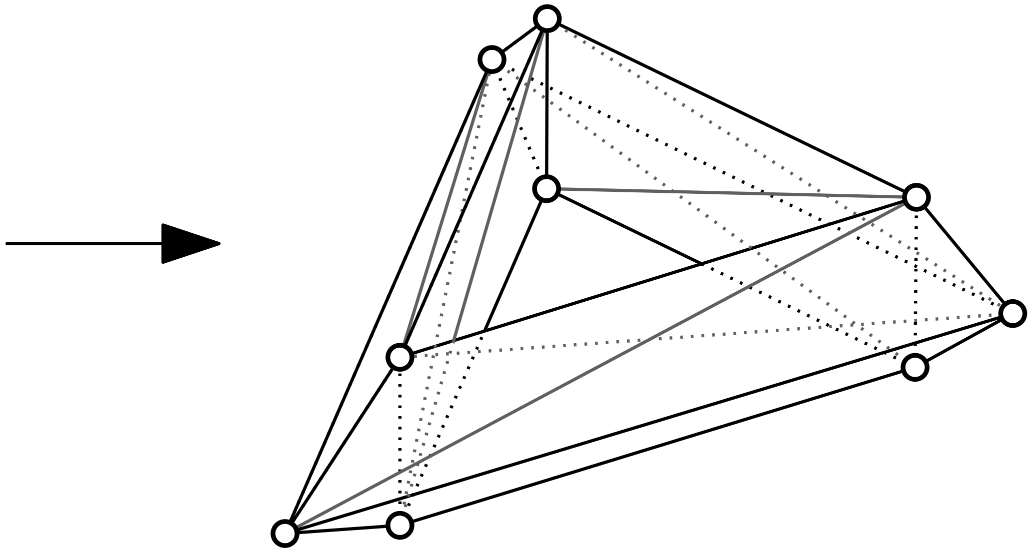


Topology Representation

We want discrete representations of continuous topological spaces





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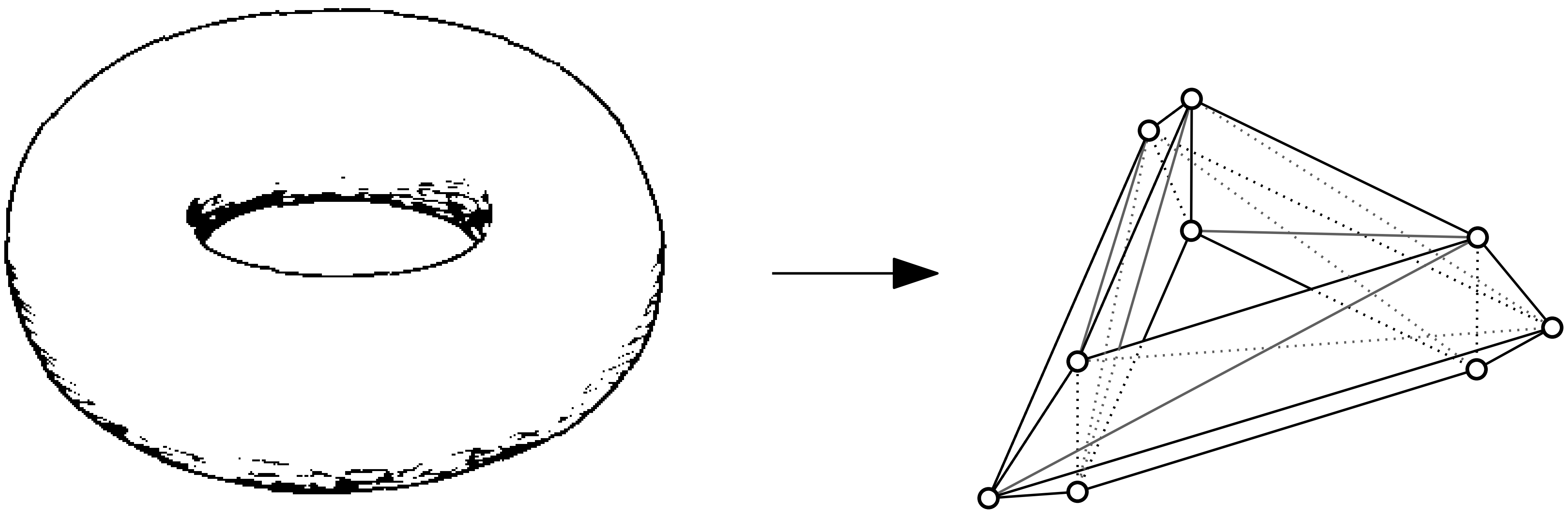
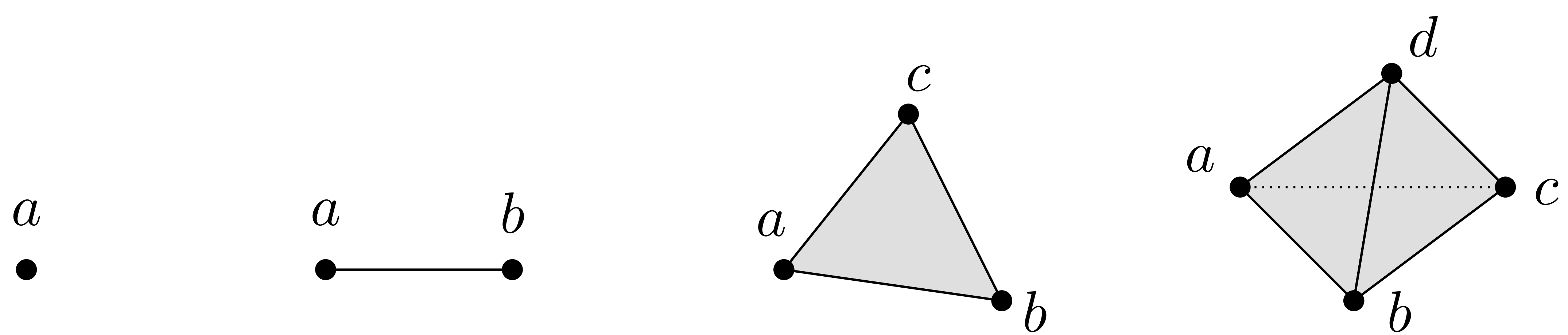


Simplicial Complexes

Simplices

- A k-simplex is the convex hull of $k+1$ affinely independent points $S = \{v_0, v_1, \ldots, v_k\}$. The points in S are the vertices of the simplex.

- A k-simplex is a k-dimensional subspace of $\mathbb{R}^d$, $\dim \sigma = k$.



vertex	edge	triangle	tetrahedron
0-simplex	1-simplex	2-simplex	3-simplex

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- σ: a k-simplex defined by $S = \{v_0, v_1, \ldots, v_k\}$
  τ defined by $T \subseteq S$ is a face of σ; $\tau \leq \sigma$
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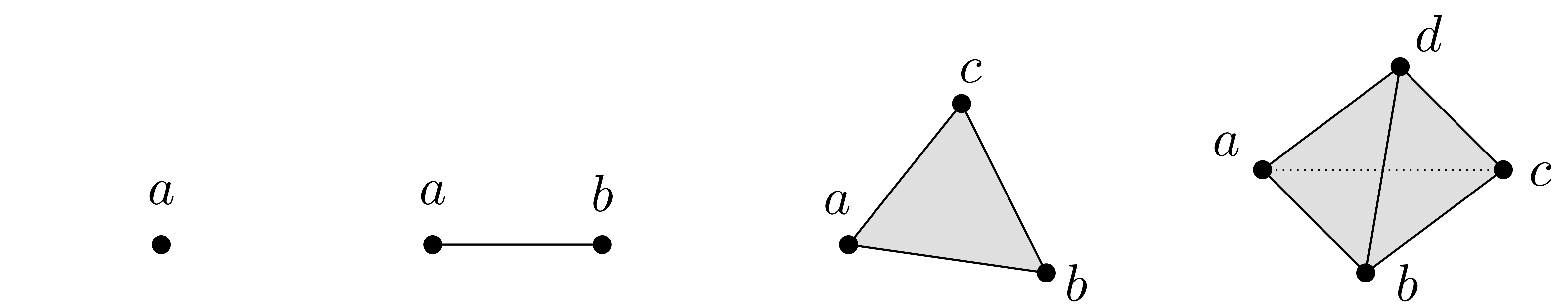

vertex 0-simplex | edge 1-simplex | triangle 2-simplex | tetrahedron 3-simplex

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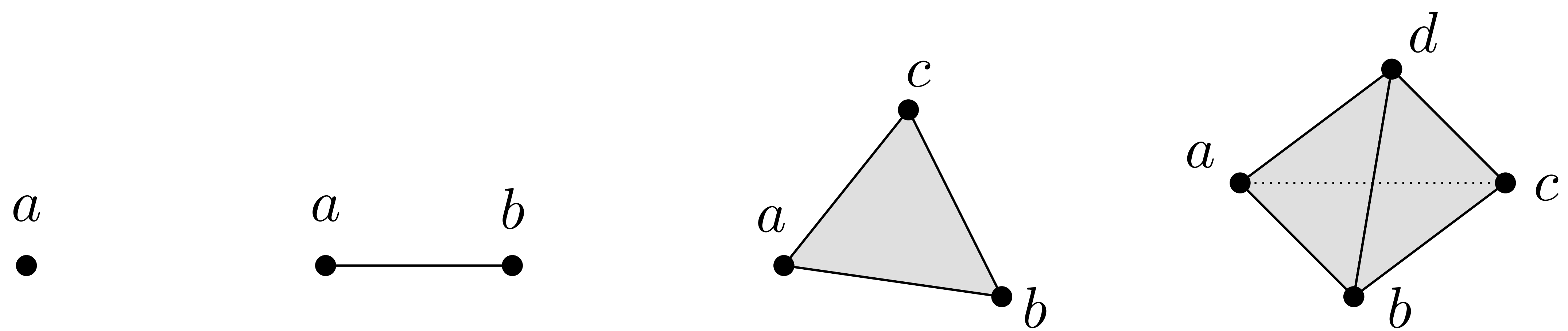
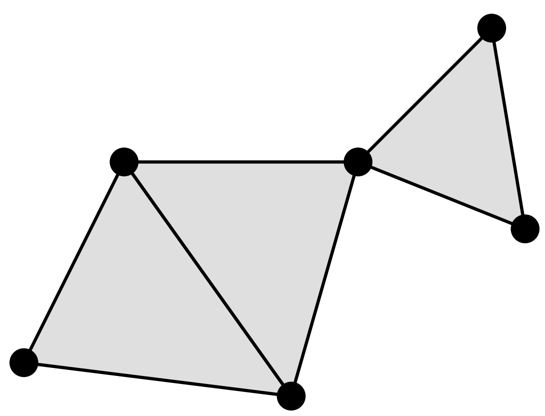
- How many faces does a k-simplex have?



vertex — 0-simplex | edge — 1-simplex | triangle — 2-simplex | tetrahedron — 3-simplex

Simplicial Complex

- A simplicial complex is a finite collection of simplices K such that
  1. $\sigma \in K$ and $\tau \leq \sigma$ implies $\tau \in K$ (closed under face relation)
  2. $\sigma, \sigma' \in K$ implies $\sigma \cap \sigma'$ is either empty or a face of both
- The dimension of K is the maximum dimension of any of its simplices; $\dim K = \max\{\dim \sigma \mid \sigma \in K\}$.



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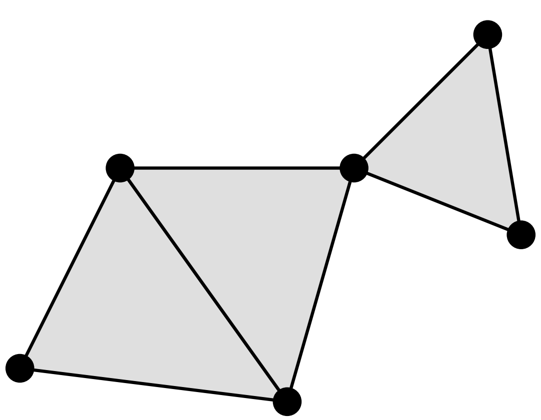
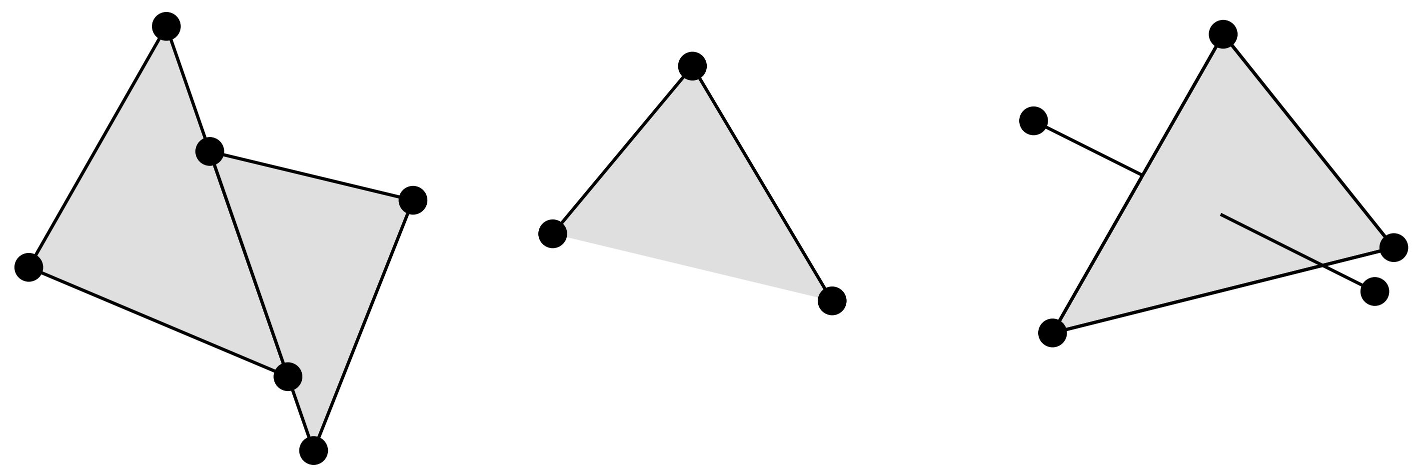


Not simplicial complexes; why not?



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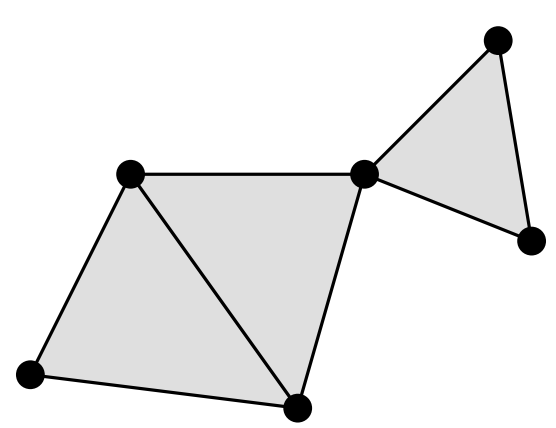
- A subcomplex is a simplicial complex $L \subseteq K$.
- A j-skeleton is the subcomplex consisting of all simplices of dimension j or less, $K^{(j)} = \{\sigma \in K \mid \dim \sigma \leq j\}$.



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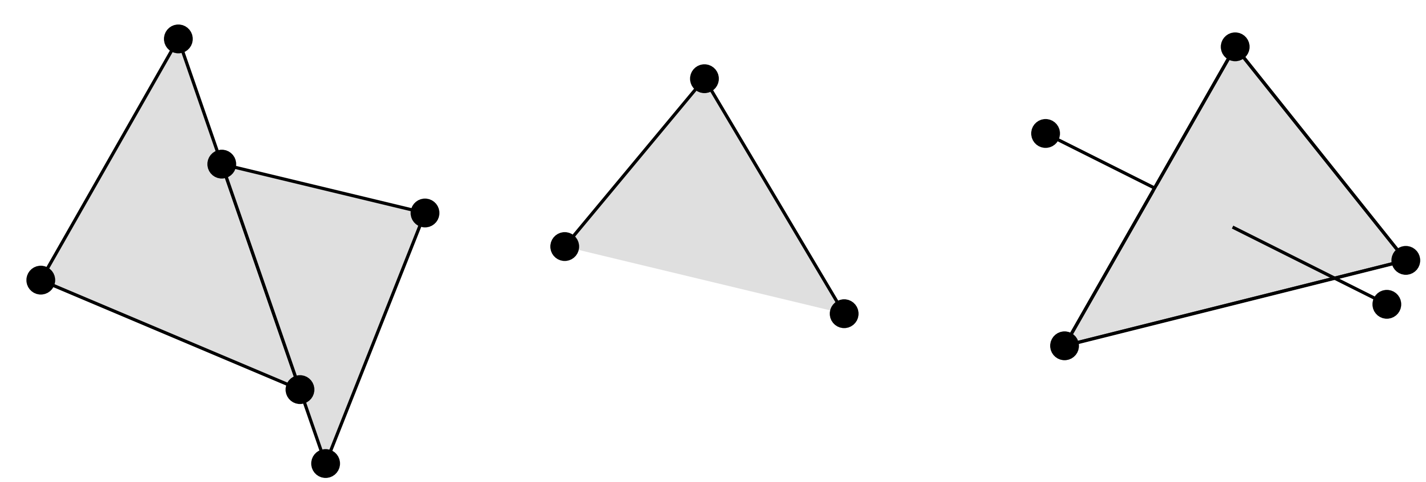
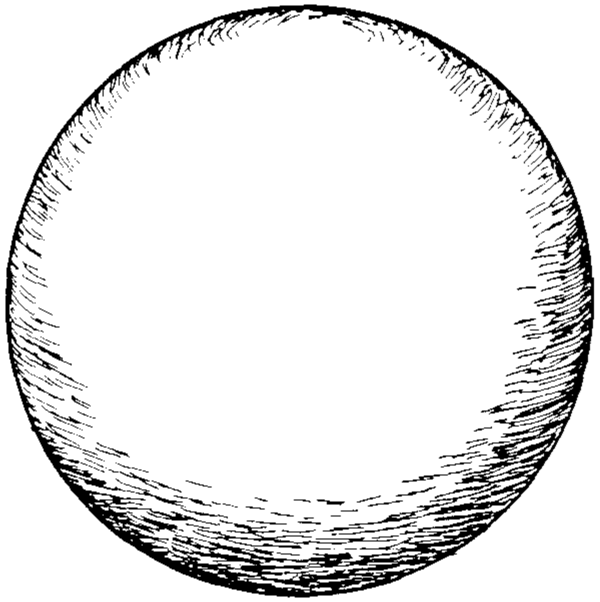
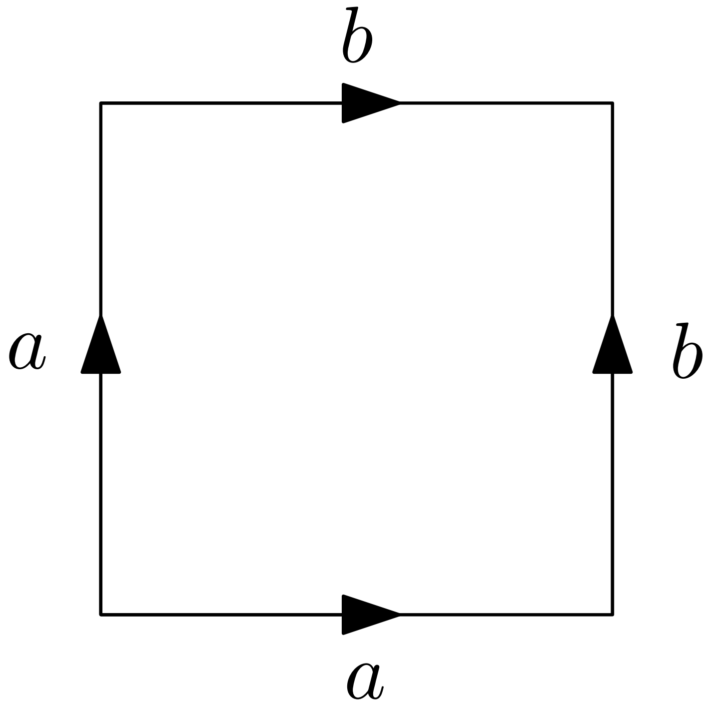


Triangulation

- The underlying space $|K|$ is the union the simplices in K, $|K| = \bigcup_{\sigma \in K} \sigma$.
- $|K|$ is a topological space (topology inherited from the ambient Euclidean space)
- A triangulation of a topological space $\mathbb{X}$ is a simplicial complex K with $|K| \approx \mathbb{X}$.

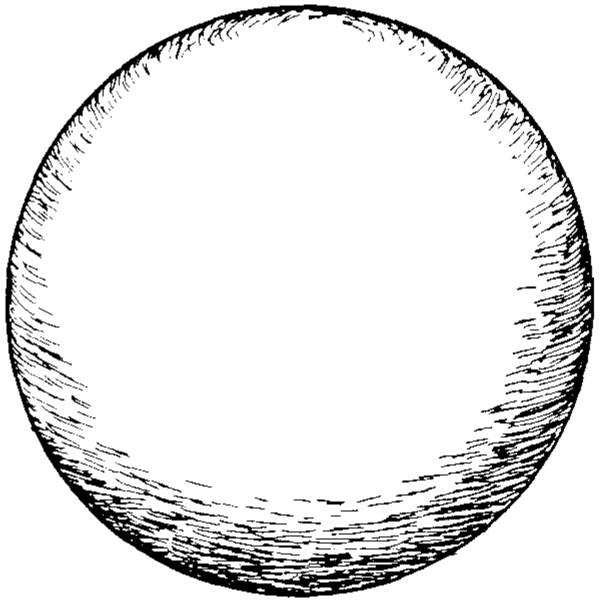
Sphere

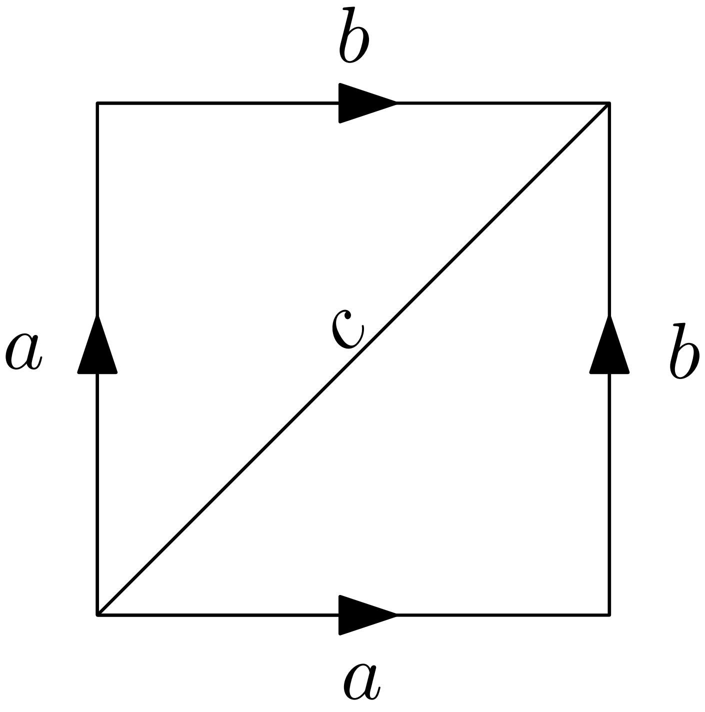




How can we triangulate a sphere?

Sphere





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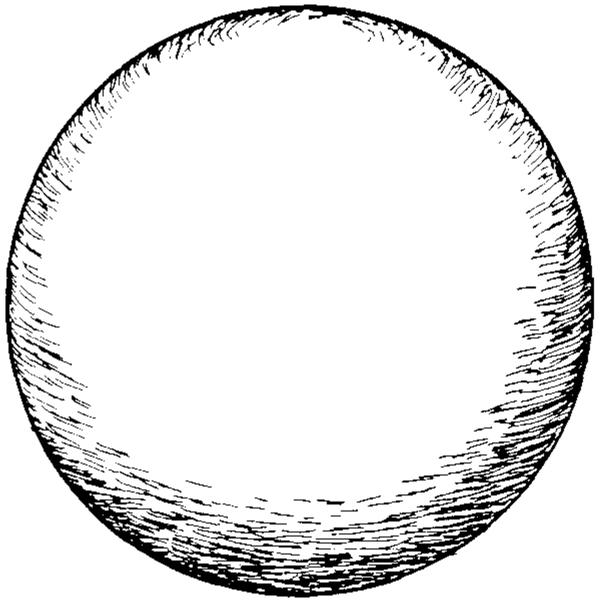

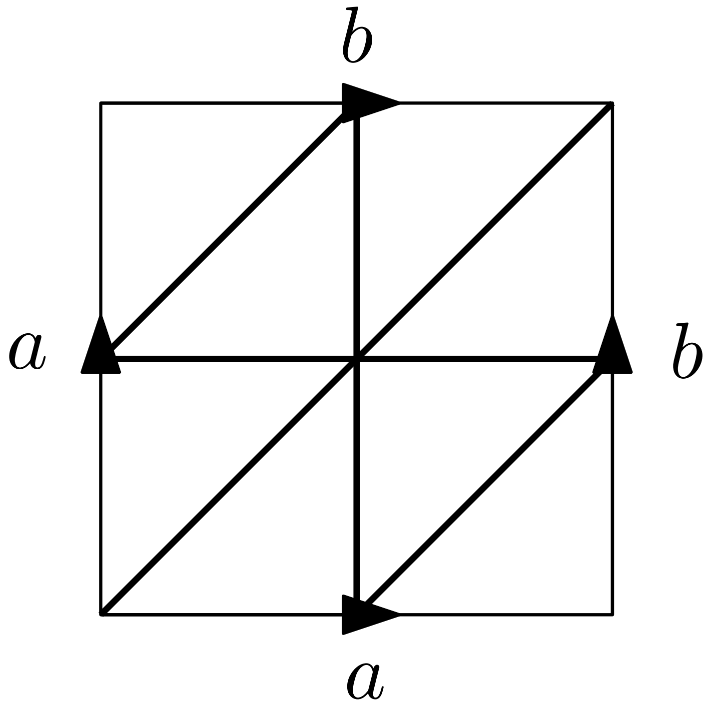

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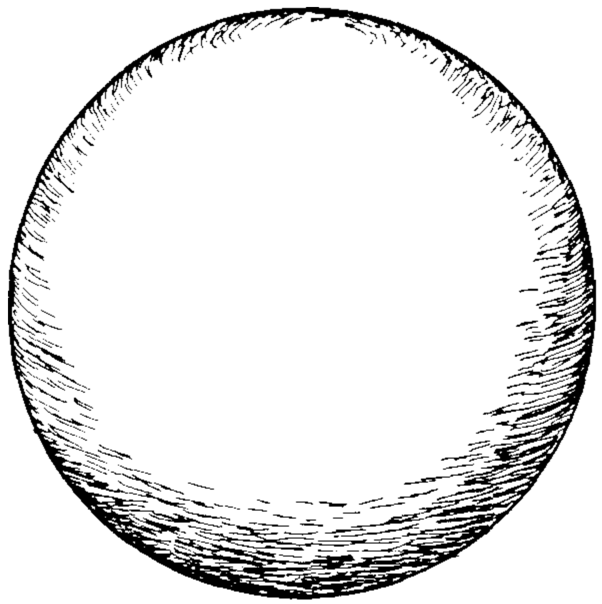

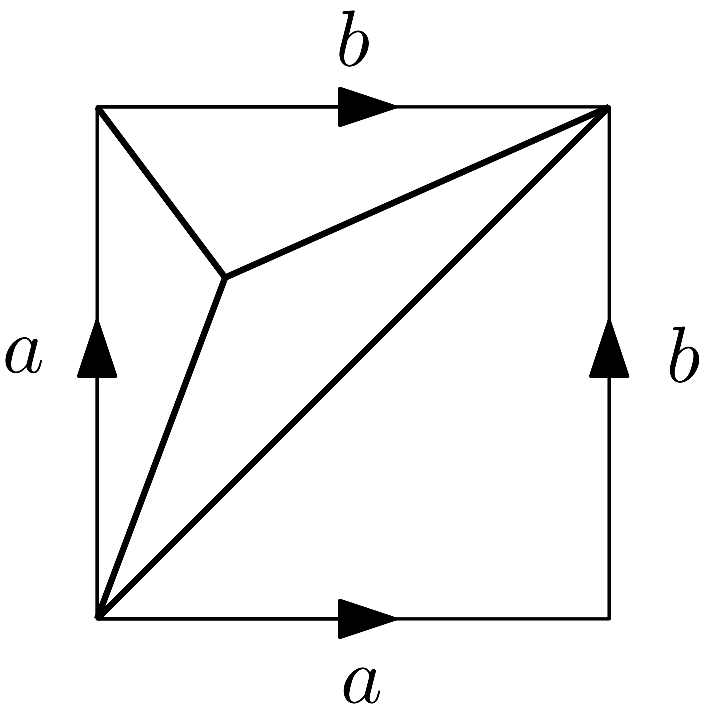

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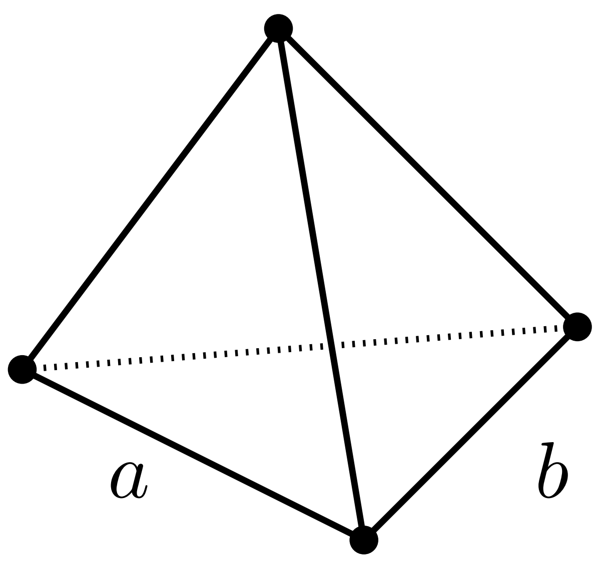

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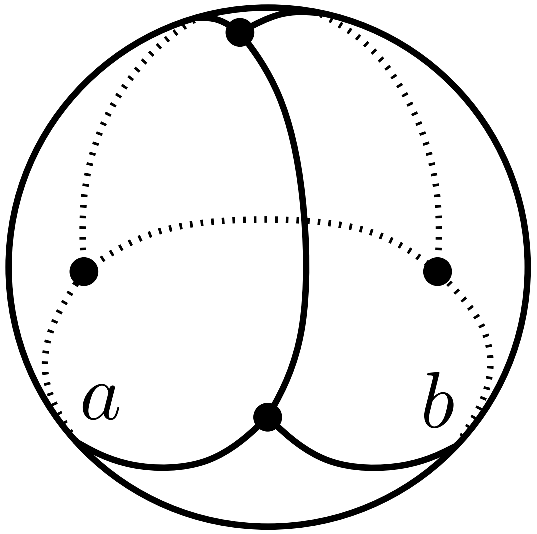
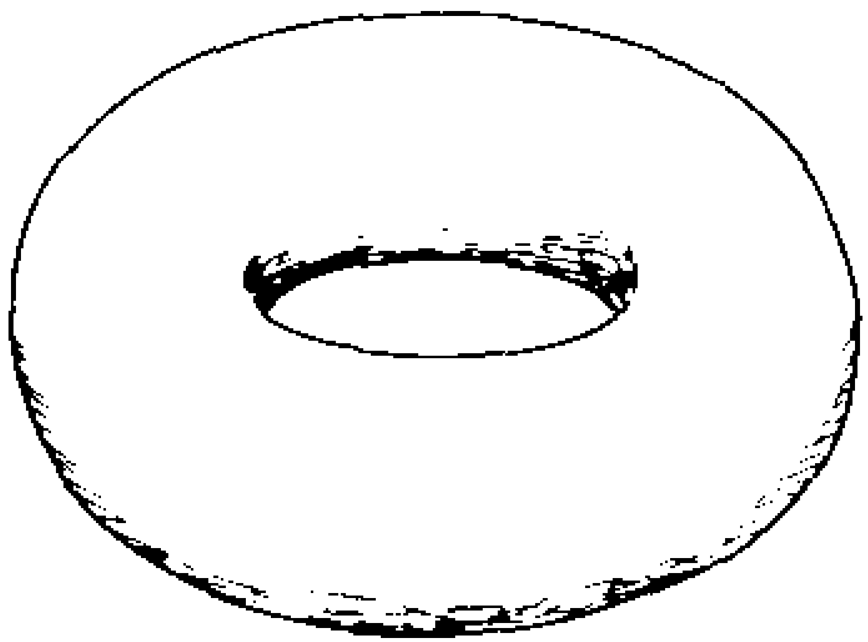
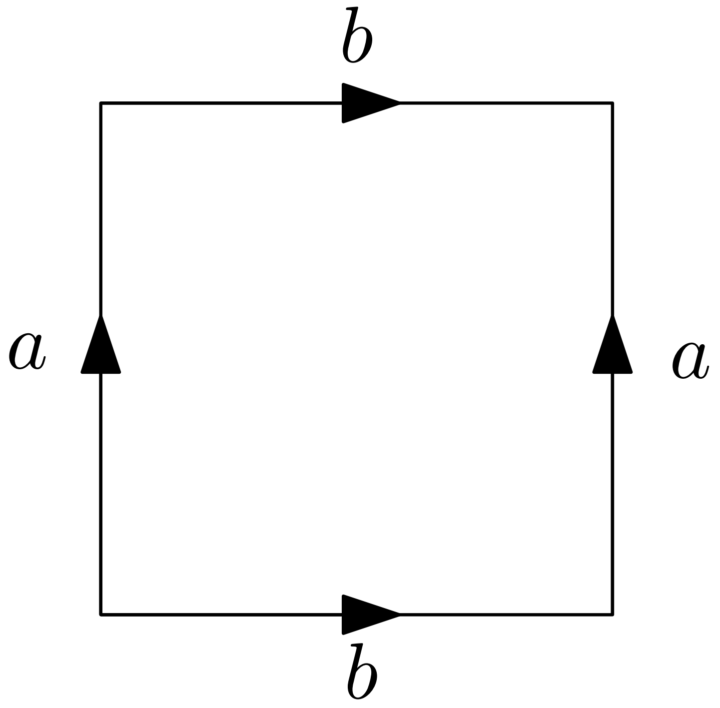


Torus





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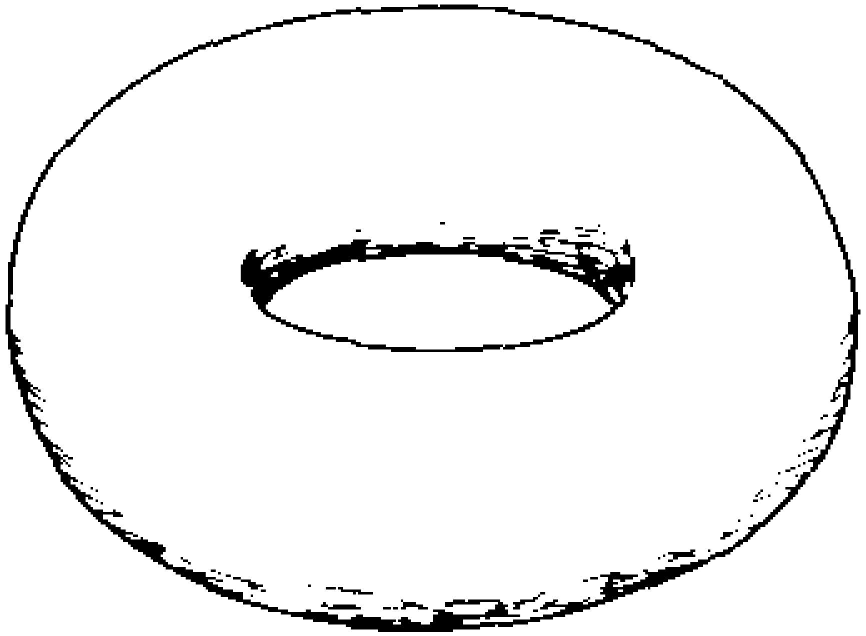

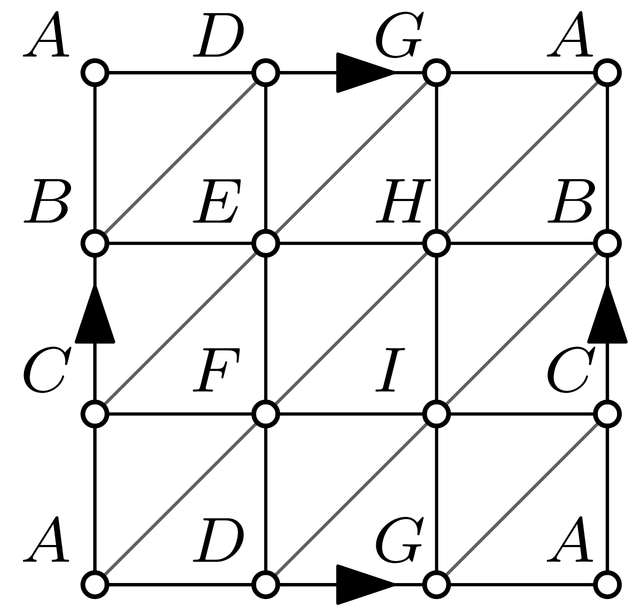

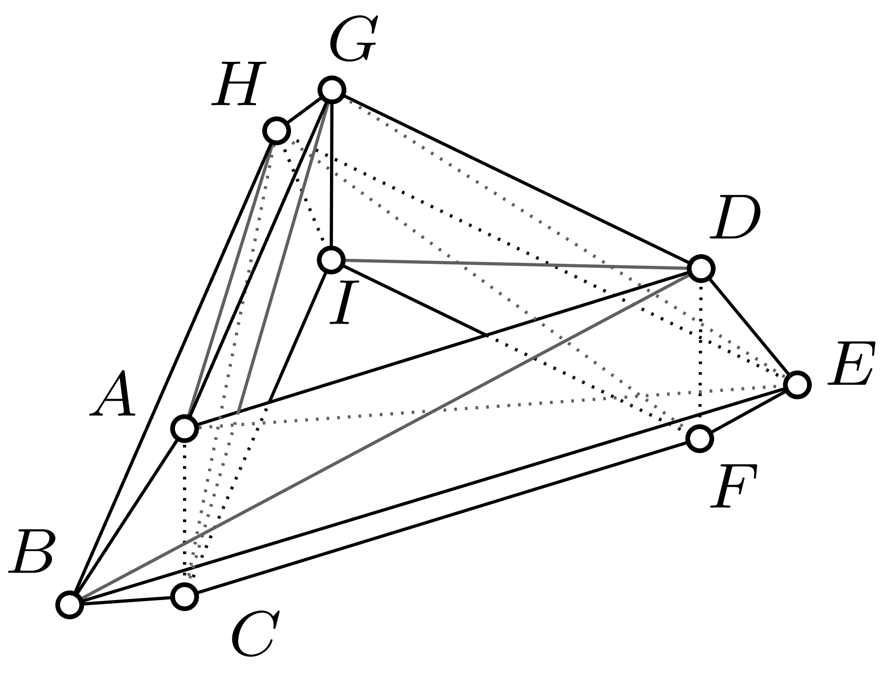

Abstract Simplicial Complex

- An abstract simplicial complex is a finite collection of sets A such that $\alpha \in A$ and $\beta \subseteq \alpha$ implies $\beta \in A$.

- The sets in A are (abstract) simplices.

- $\alpha, \beta \in A$, β is a face of α if $\beta \subseteq \alpha$; α is a coface of β.

- The dimension of a simplex is $\dim \alpha = \operatorname{card} \alpha - 1$.

- The vertex set is the union of all simplices, $\operatorname{Vert} A = \bigcup A$.

- Two abstract simplicial complexes are isomorphic if there is a bijection $b : \operatorname{Vert} A \to \operatorname{Vert} B$ such that $\alpha \in A$ iff $b(\alpha) \in B$. I.e. we can get B by renaming the vertices in A, and vice versa.

Geometric Realization

- Let K be a simplicial complex with vertices V, let A be the collection of all subsets $\{v_0, \ldots, v_k\}$ such that the vertices $v_0, \ldots, v_k$ span a simplex of K. Then A is a vertex scheme of K.

- Symmetrically, we call K a geometric realization of A.

Geometric Realization Theorem. Every abstract simplicial complex of dimension d has a geometric realization in $\mathbb{R}^{2d+1}$.

Star and Link

- A star of τ consists of all cofaces of τ, $\operatorname{St} \tau = \{\sigma \in K \mid \tau \leq \sigma\}$.
- A closed star $\bar{\operatorname{St}} \tau$ is the smallest subcomplex that contains the star.
- A link consists of all simplices in the closed star that are disjoint from τ, $\operatorname{Lk} \tau = \{\nu \in \bar{\operatorname{St}} \tau \mid \nu \cap \tau = \emptyset\}$.

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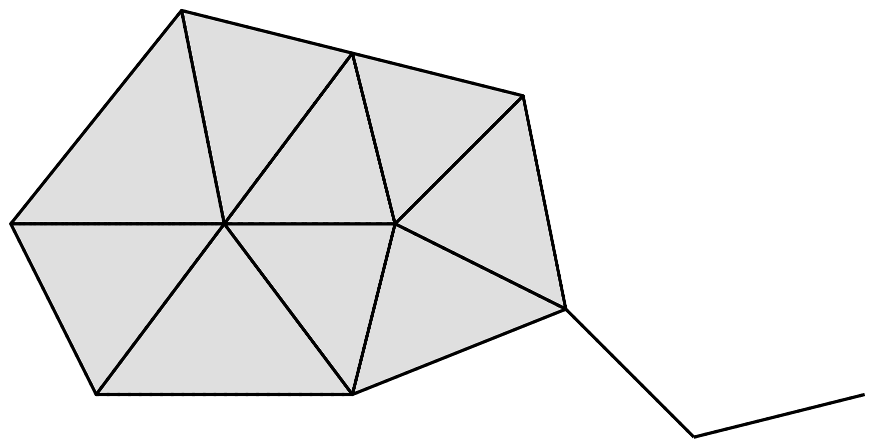

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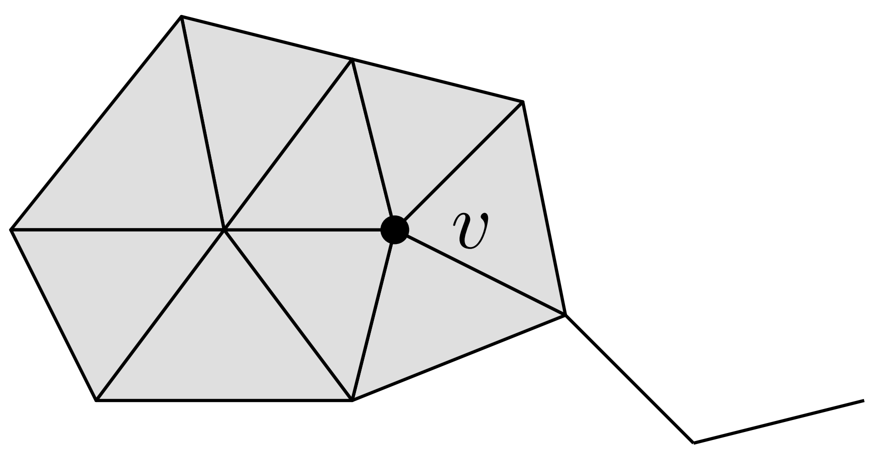

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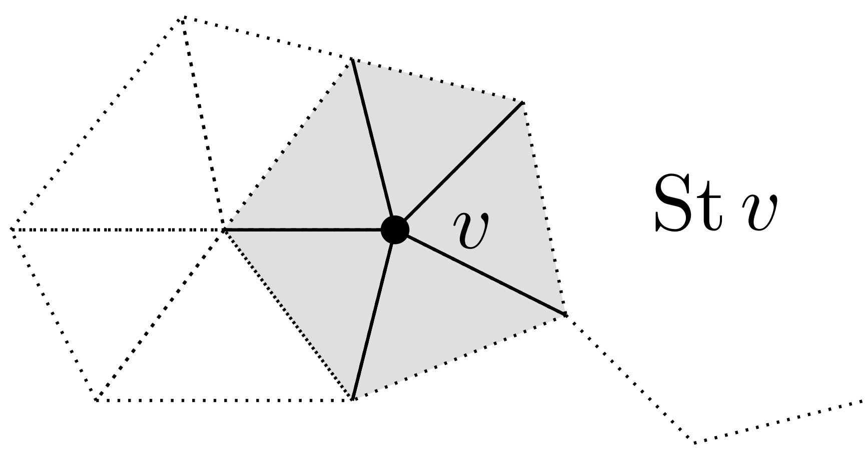

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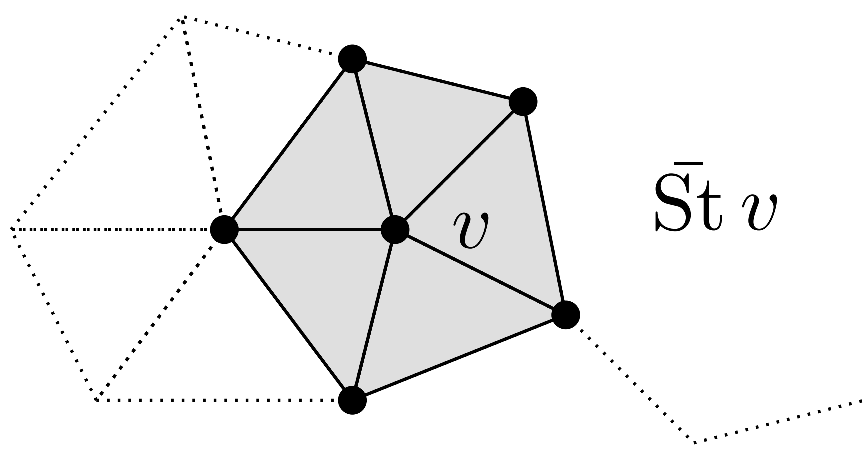

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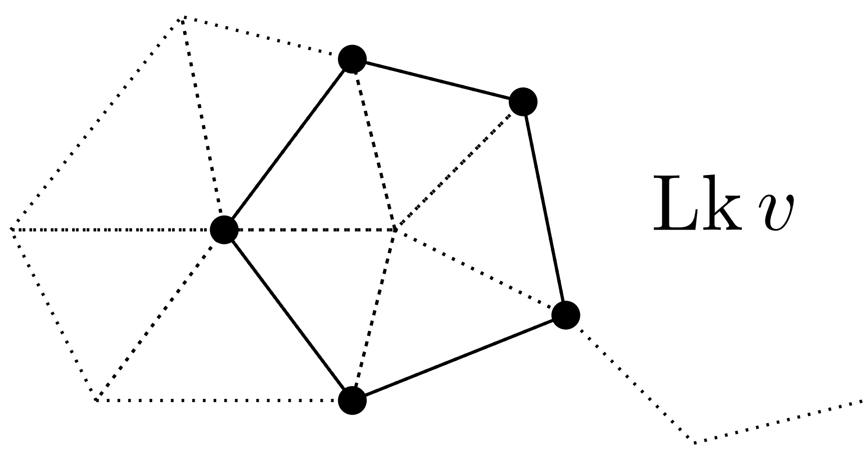

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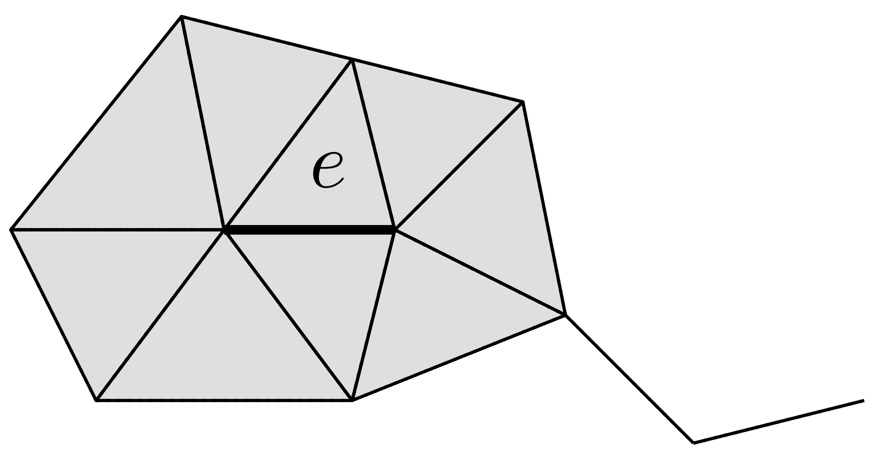

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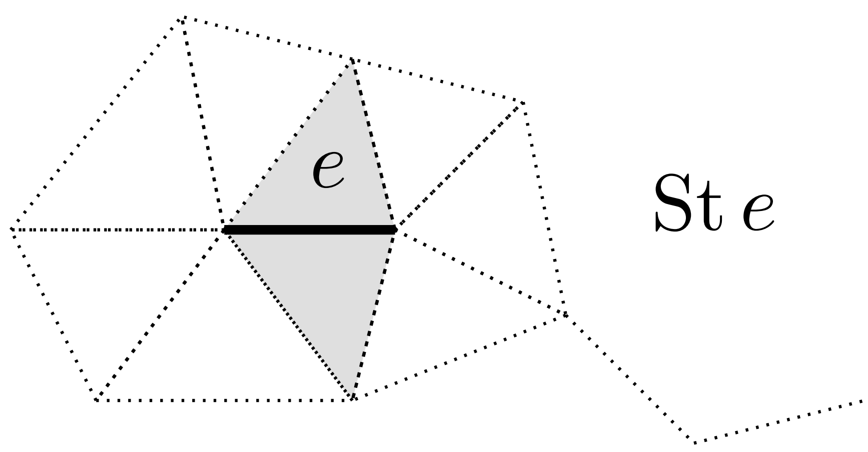

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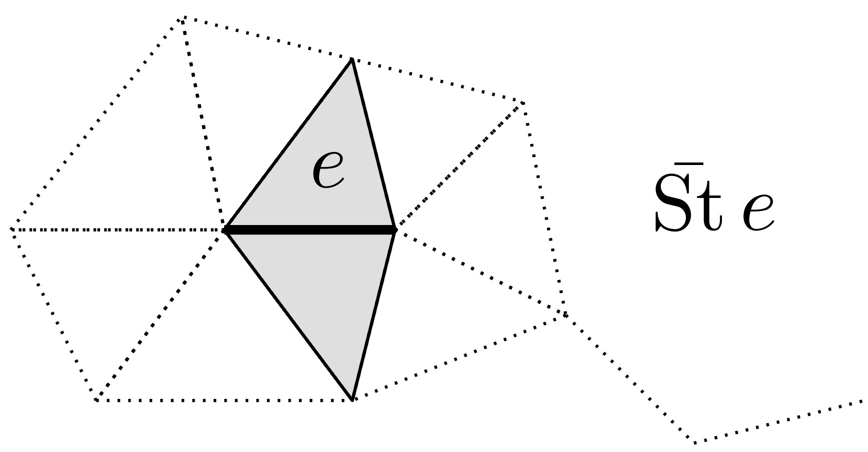

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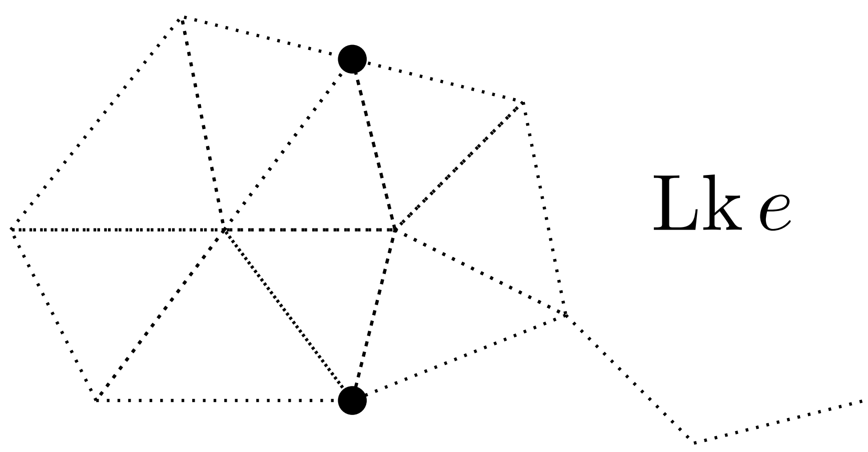


$\mathrm{Lk}\,e$

Orientability

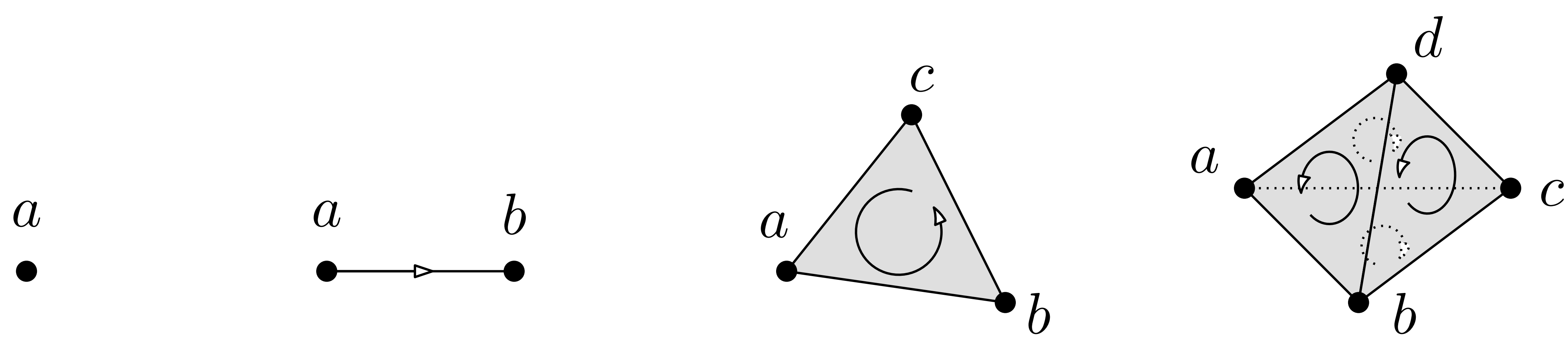
- An orientation of a k-simplex $\sigma \in K$, $\sigma = \{v_0, \ldots, v_k\}$, is an equivalence class of orderings of the vertices of σ, where

$$(v_0, \ldots, v_k) \sim (v_{\tau(0)}, \ldots, v_{\tau(k)})$$

are equivalent orderings if the parity of the permutation τ is even.

parity of the number of inversions

- We denote an oriented simplex by $[\sigma]$.



vertex $[a]$ | edge $[a, b]$ | triangle $[a, b, c]$ | tetrahedron $[a, b, c, d]$

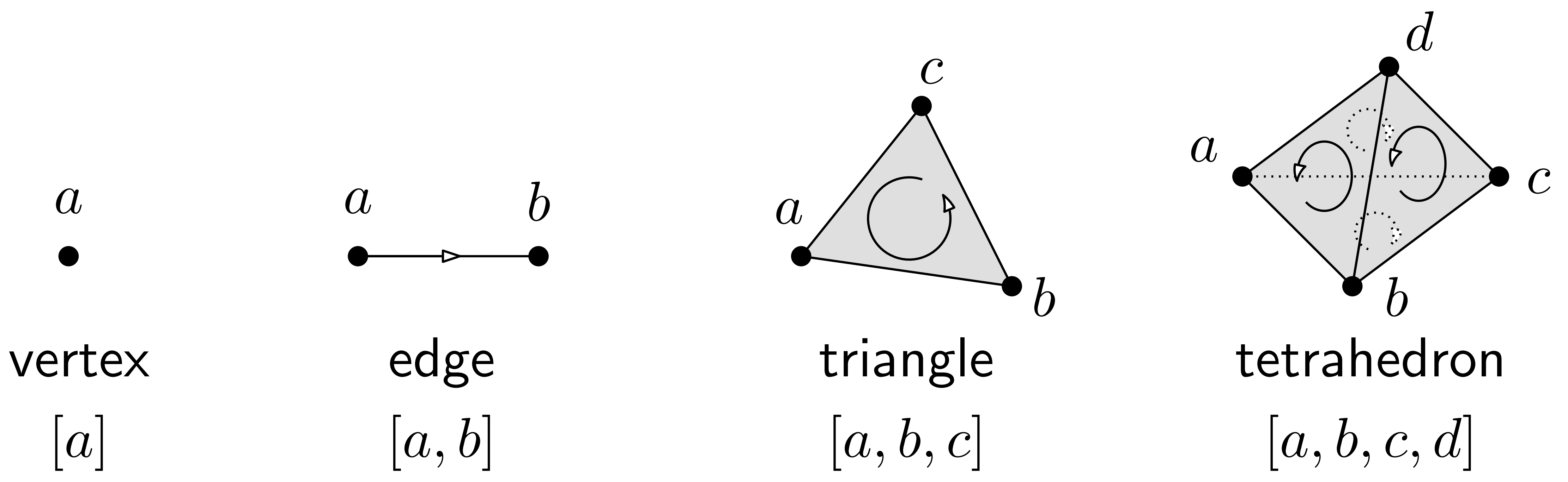
Orientability

- Two k-simplices sharing a $(k-1)$-face σ are consistently oriented if they induce different orientations on σ.

  Let $\sigma = \{v_0, v_1, \ldots, v_k\}$, $\tau_0 = \{v_1, \ldots, v_k\}$, $\tau_1 = \{v_0, v_2, \ldots, v_k\}$ two of its faces. $[\sigma]$ induces $[\tau_0]$ on τ_0. Consequently, it induces $-[\tau_1]$ on τ_1. (Why?)

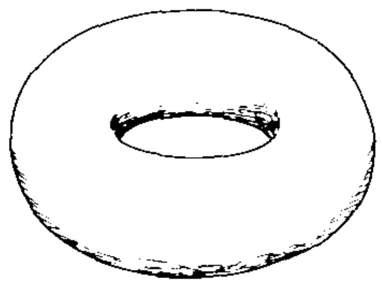
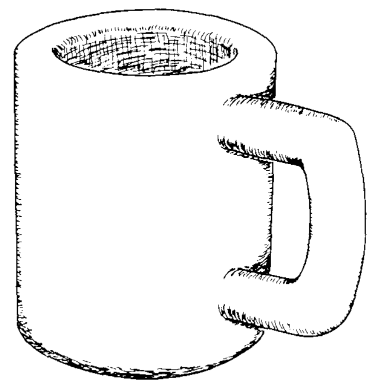
- A triangulable d-manifold is orientable if all d-simplices can be oriented consistently.

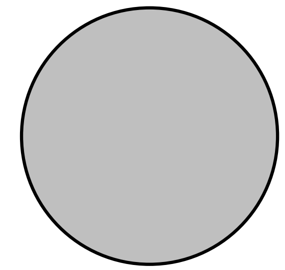
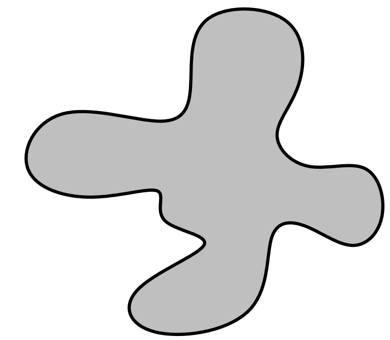
- Otherwise, the d-manifold is non-orientable.



Homeomorphism and Homotopy

"Topologically equivalent" $=$ homeomorphic

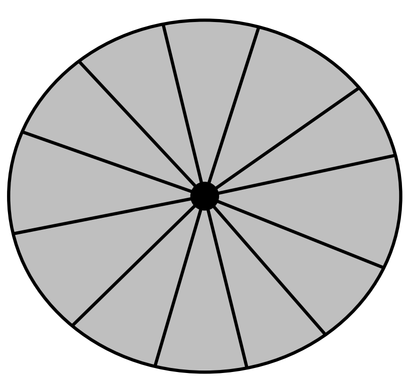
 $\approx$ 

 $\approx$ 

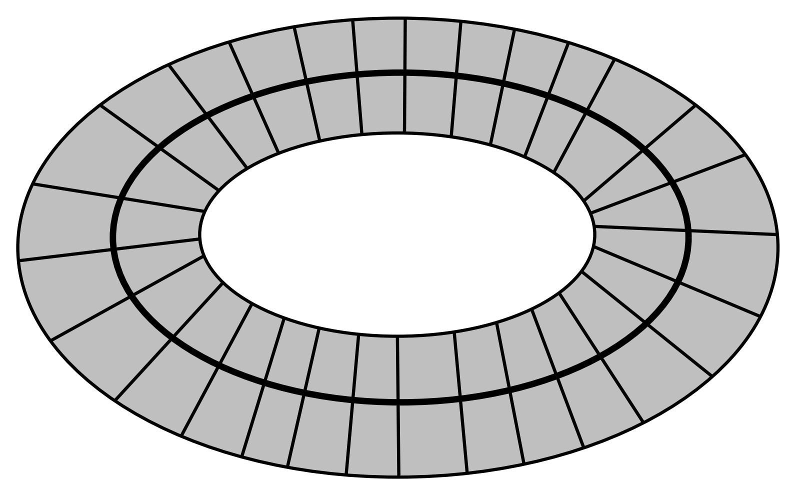
(not quite, but close enough for today)

Homotopy equivalent $\approx$ can continuously retract one space to the other; in particular, we don't care if the dimension ("thickness") changes

Two homotopy equivalent spaces are said to have the same homotopy type.



disk $\simeq$ point

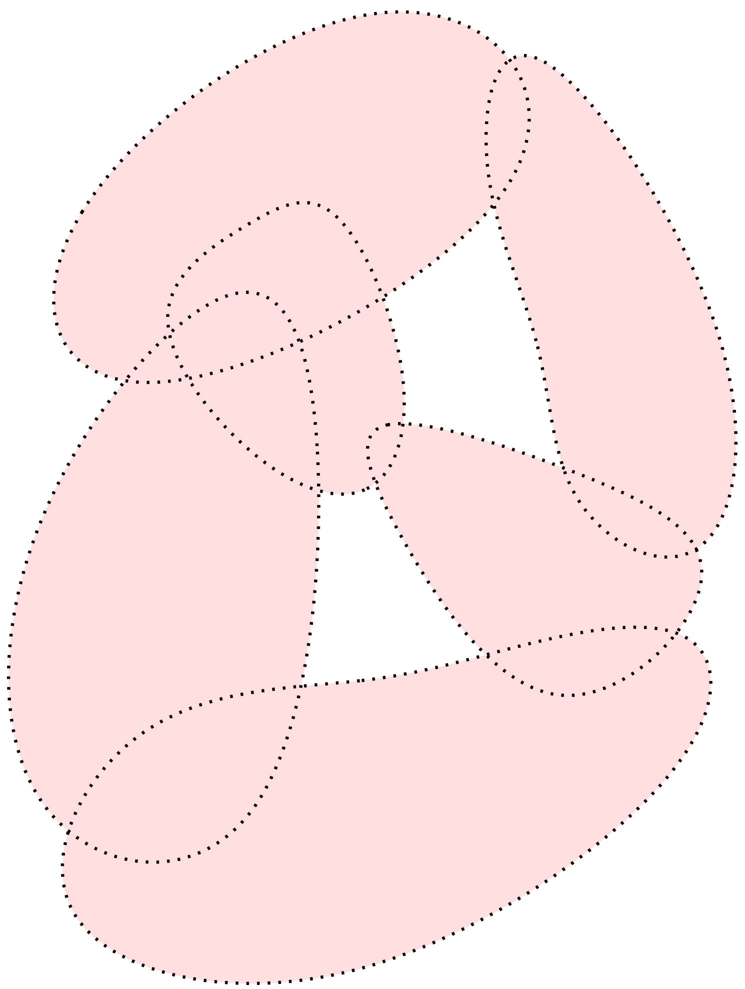


annulus $\simeq$ circle

A topological space is contractible if it has the homotopy type of a point.

Nerve

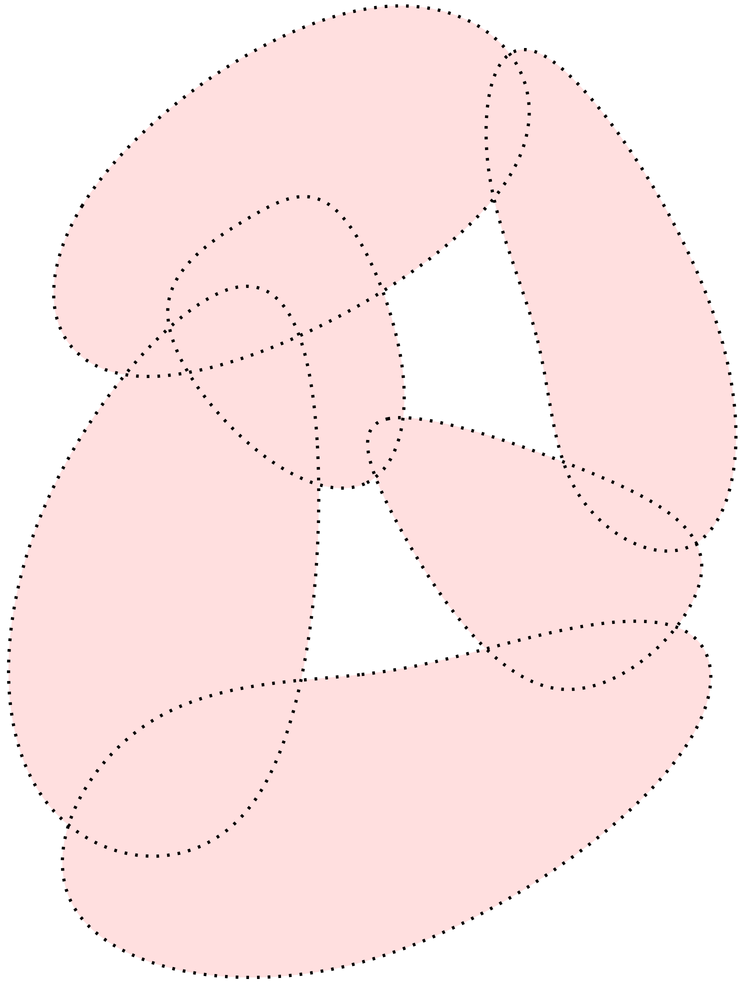
- Given collection of sets $\mathcal{U} = \{U_i\}$, what is the topology of $\mathcal{U}$? (e.g., a cover of a topological space)



Nerve

- Given collection of sets $\mathcal{U} = \{U_i\}$, what is the topology of $\mathcal{U}$? (e.g., a cover of a topological space)
- The nerve of the collection consists of all non-empty subcollections whose sets have a non-empty intersection,

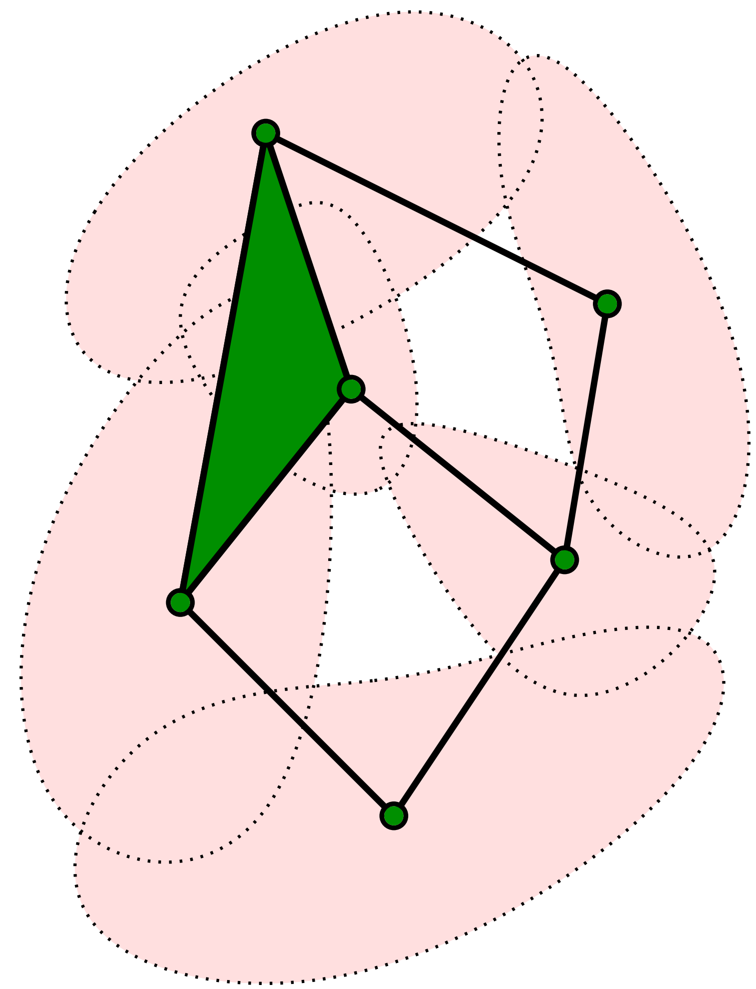
$$\operatorname{Nrv} \mathcal{U} = \{X \subseteq \mathcal{U} \mid \bigcap X \neq \emptyset\}.$$



Nerve

- Given collection of sets $\mathcal{U} = \{U_i\}$, what is the topology of $\mathcal{U}$? (e.g., a cover of a topological space)
- The nerve of the collection consists of all non-empty subcollections whose sets have a non-empty intersection,

$$\mathrm{Nrv}\,\mathcal{U} = \{X \subseteq \mathcal{U} \mid \bigcap X \neq \emptyset\}.$$



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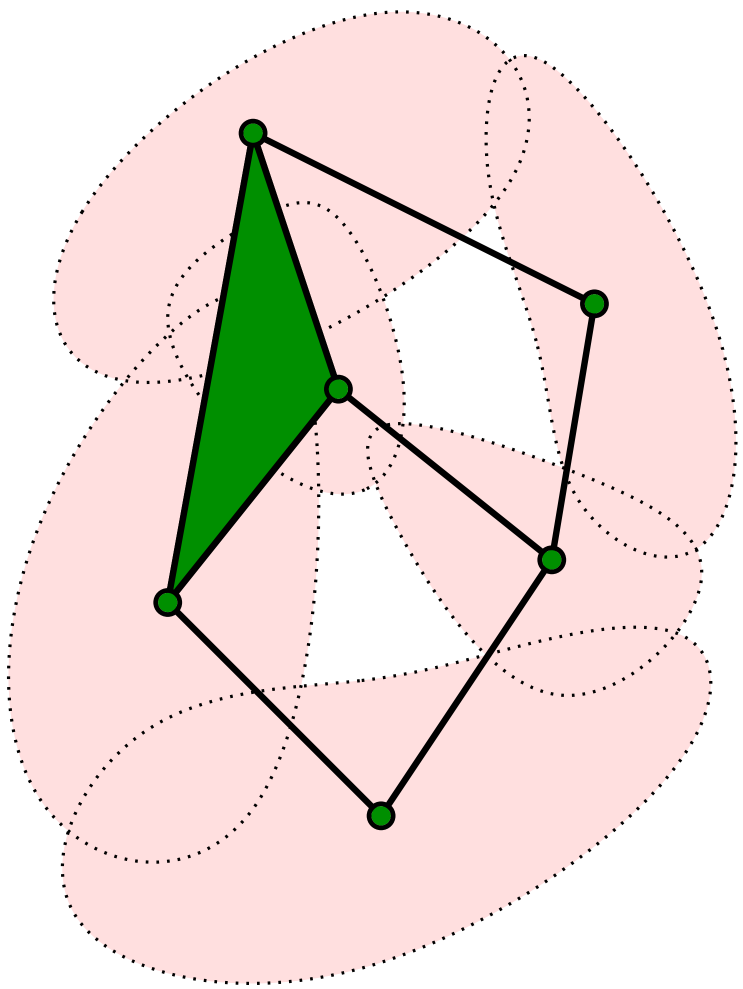

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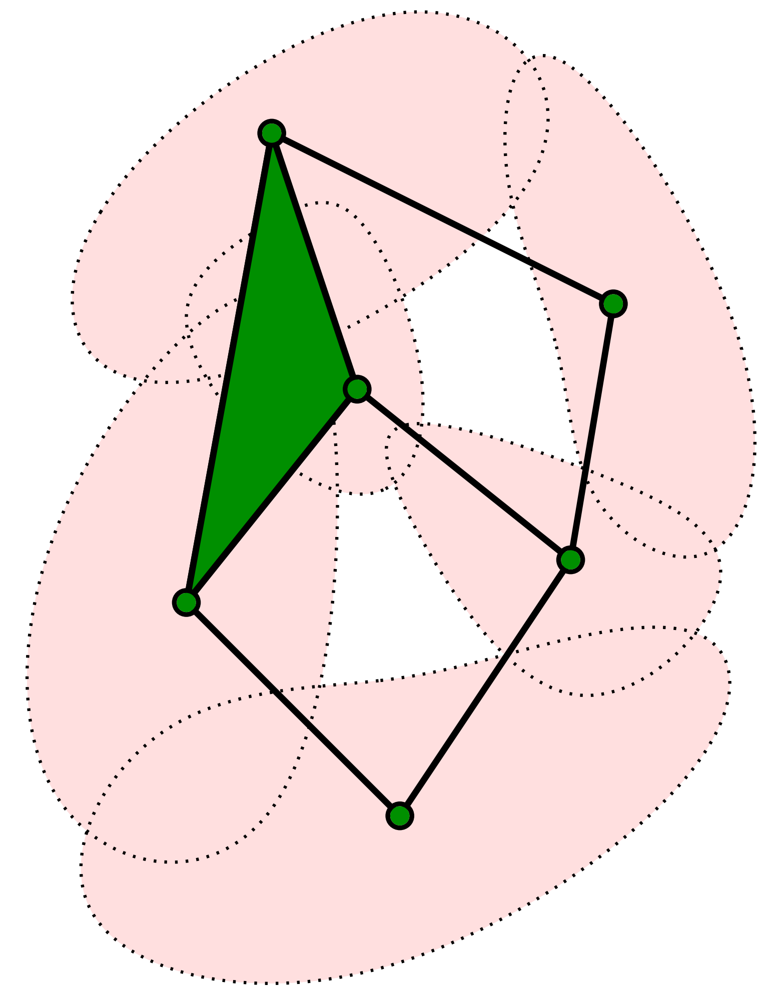
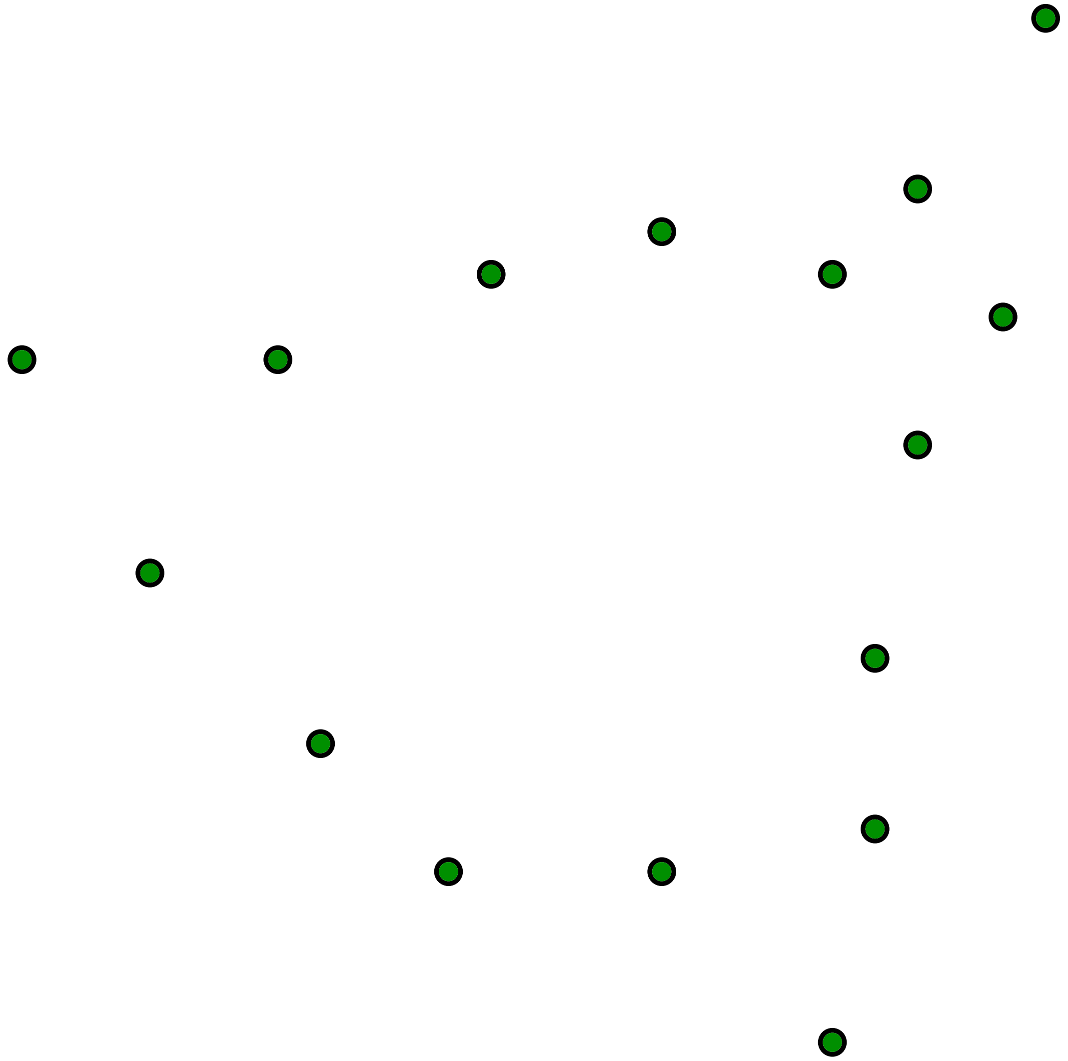


Nerve Theorem. If all the finite intersections of the sets in the collection are either empty or contractible, then the nerve of $\mathcal{U}$ and the union of the sets in $\mathcal{U}$ have the same homotopy type,

$$\mathrm{Nrv}\,\mathcal{U} \simeq \bigcup \mathcal{U}.$$

Čech Complex

$P \subset \mathbb{R}^d$



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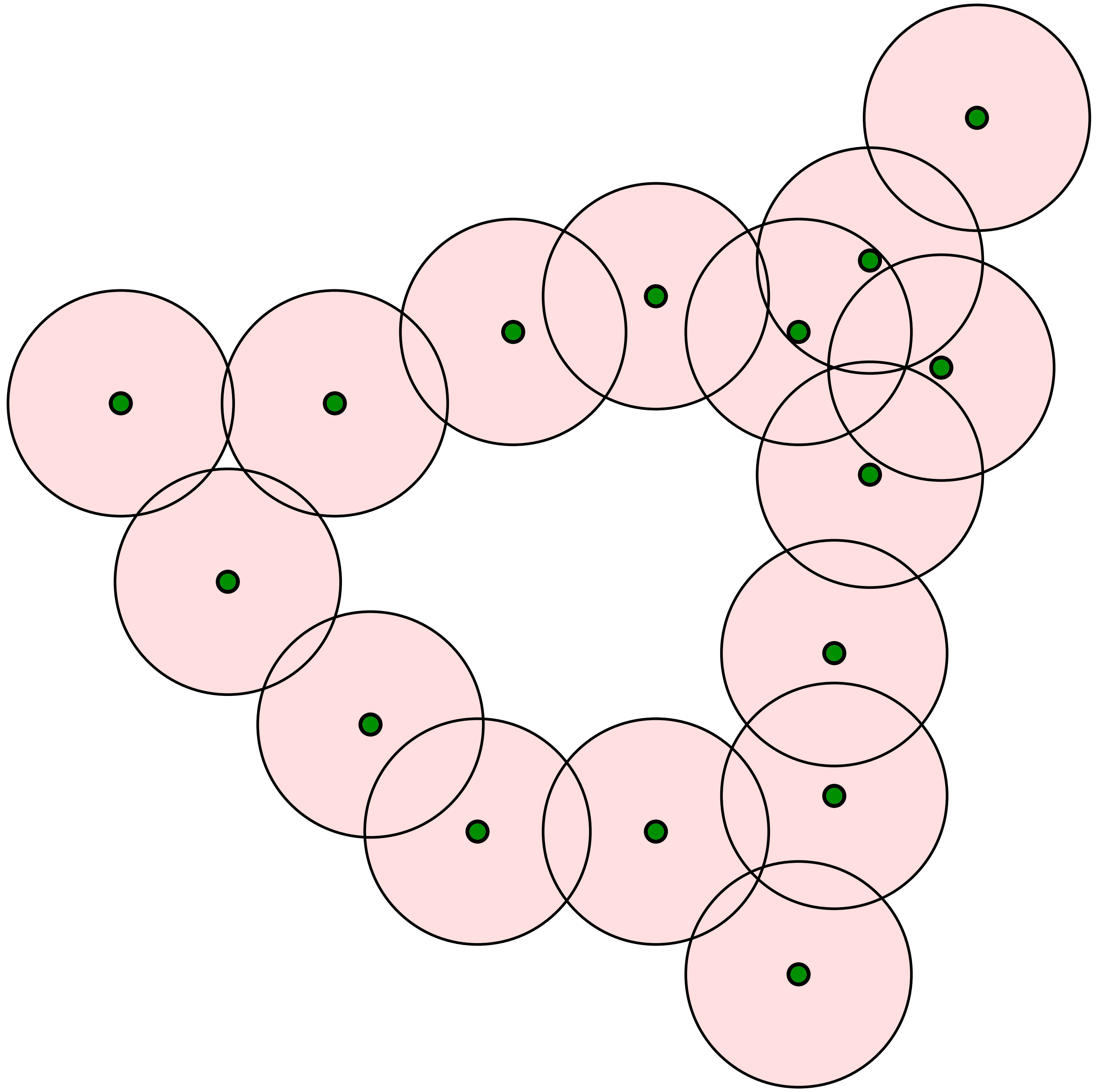

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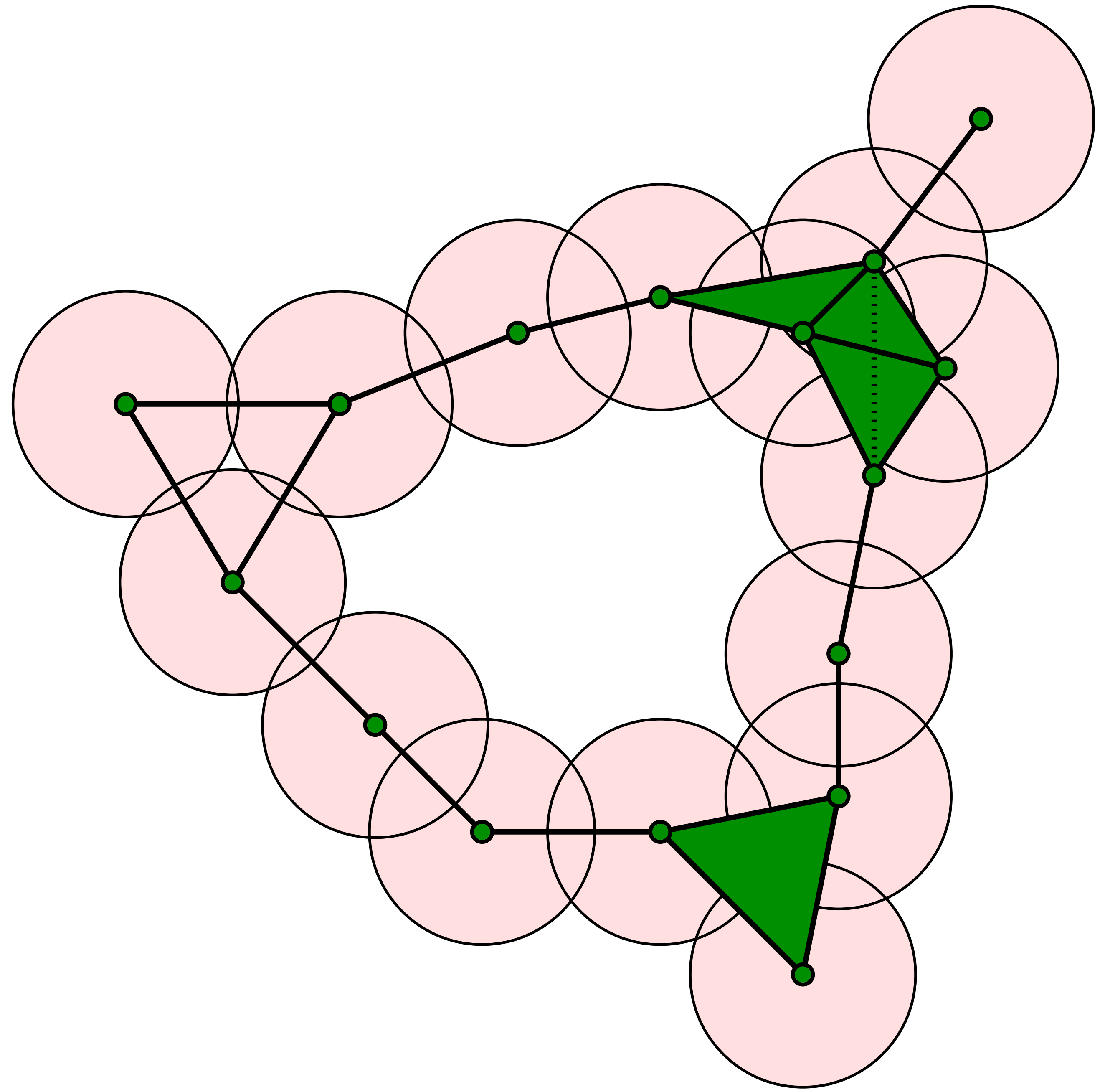

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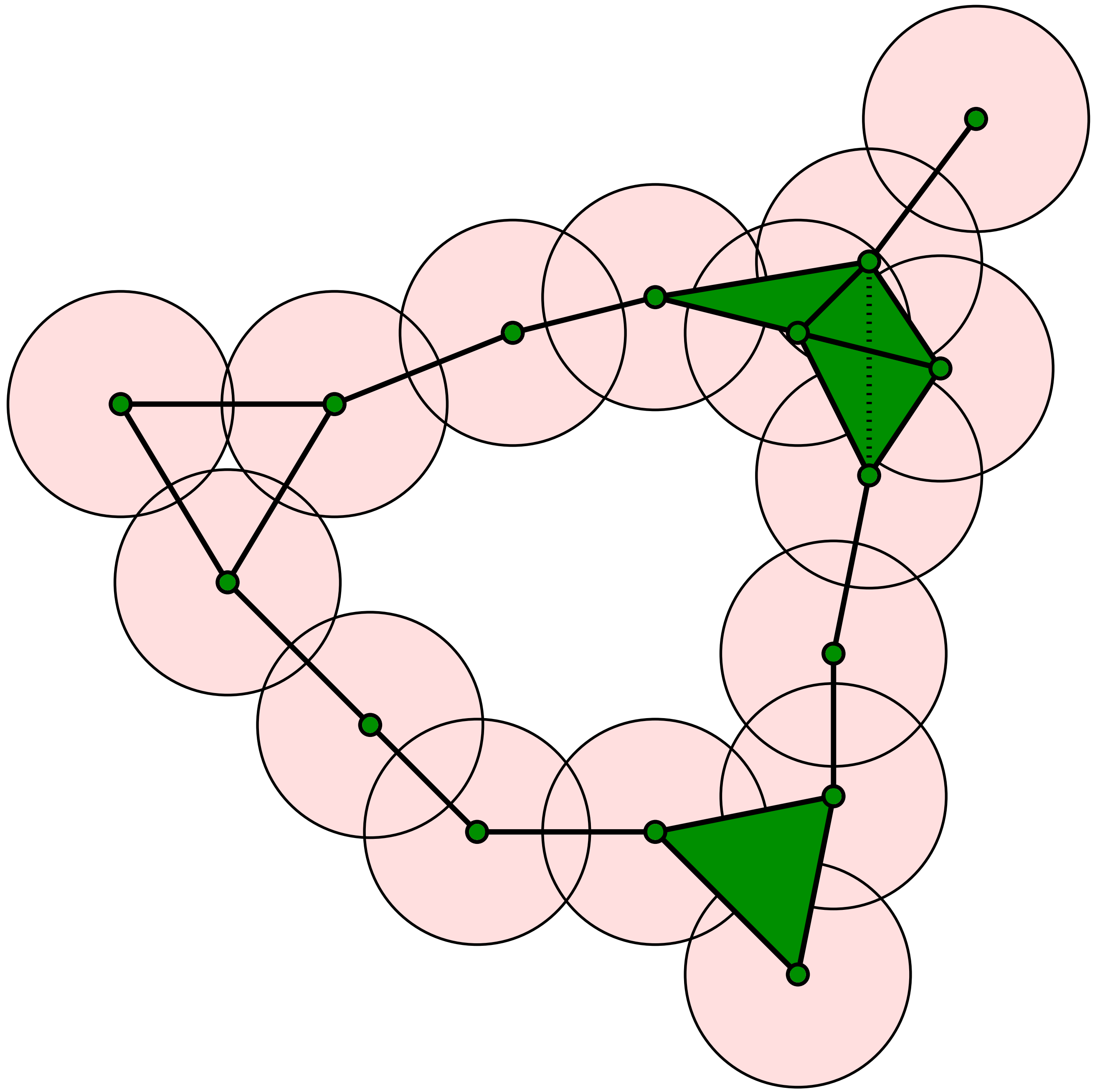
(why?)



Delaunay Triangulation

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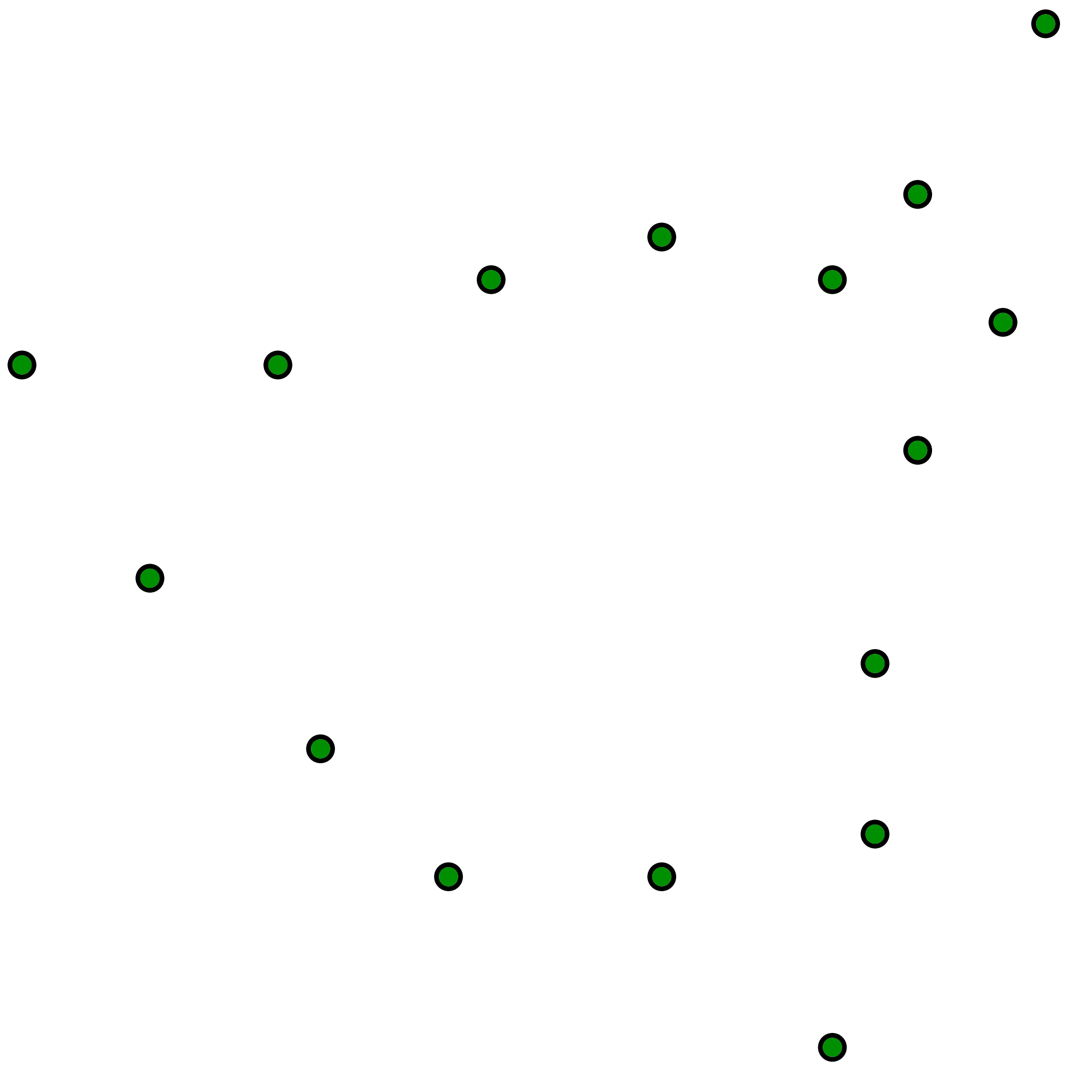

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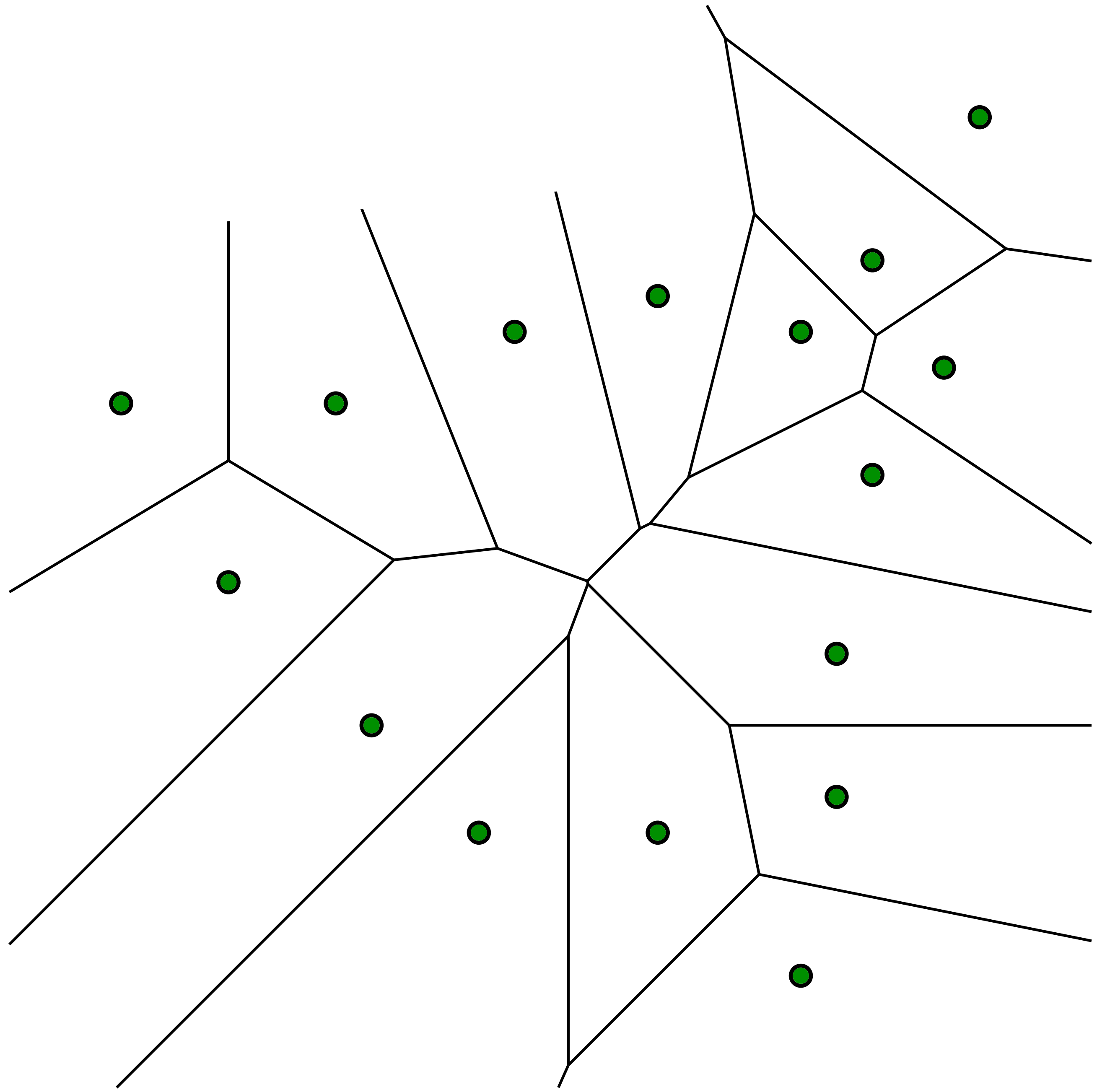

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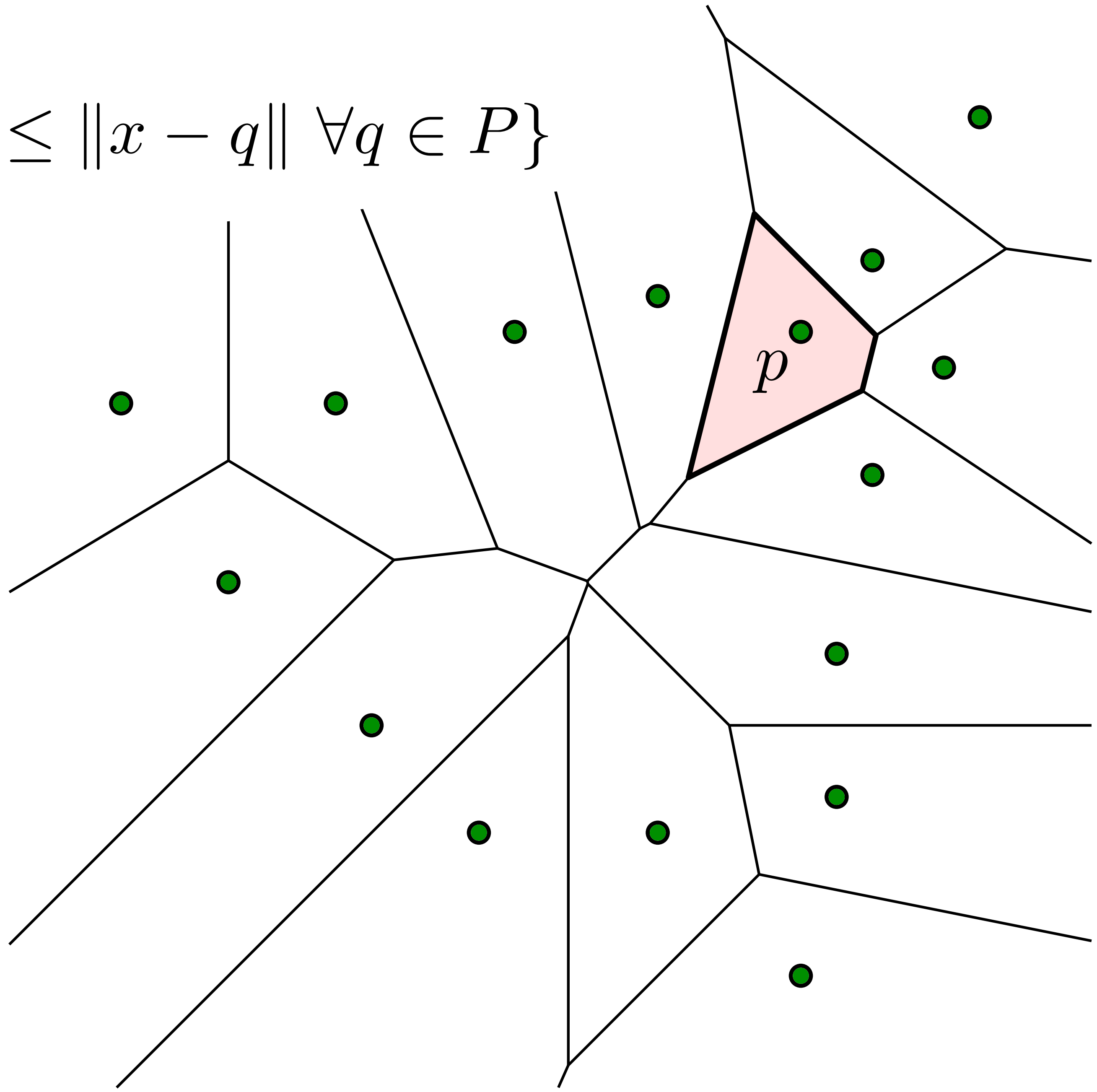

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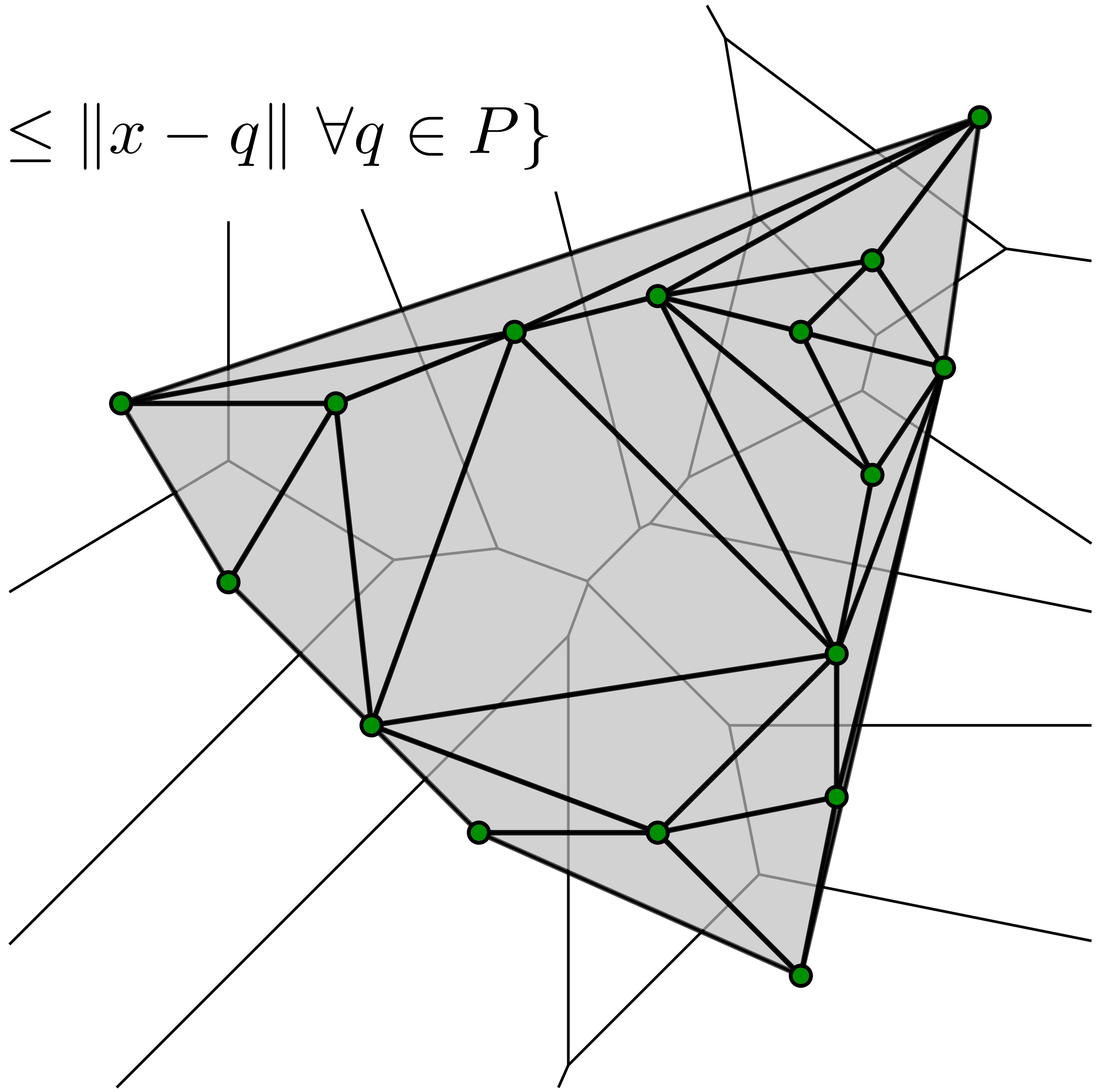

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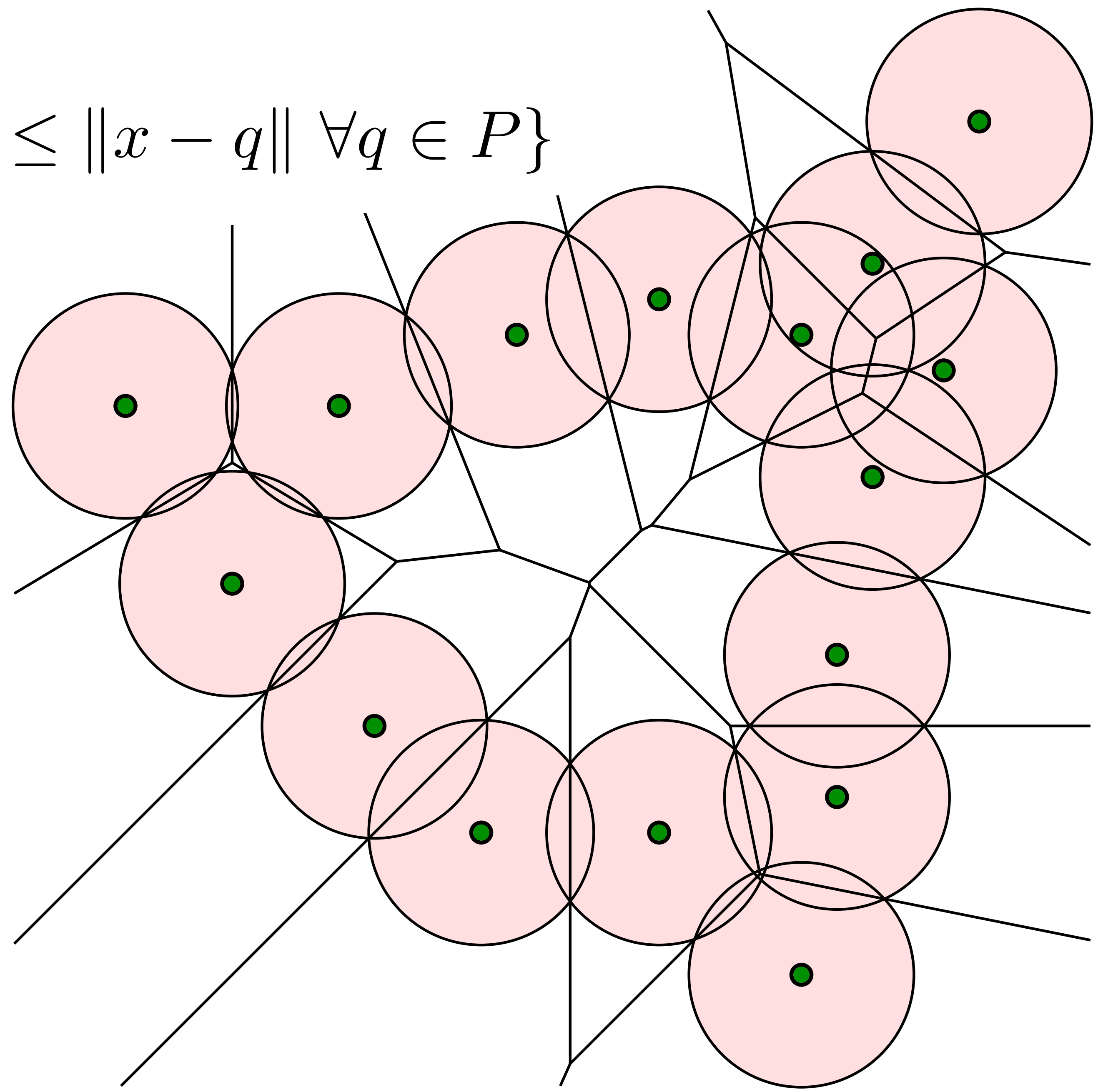

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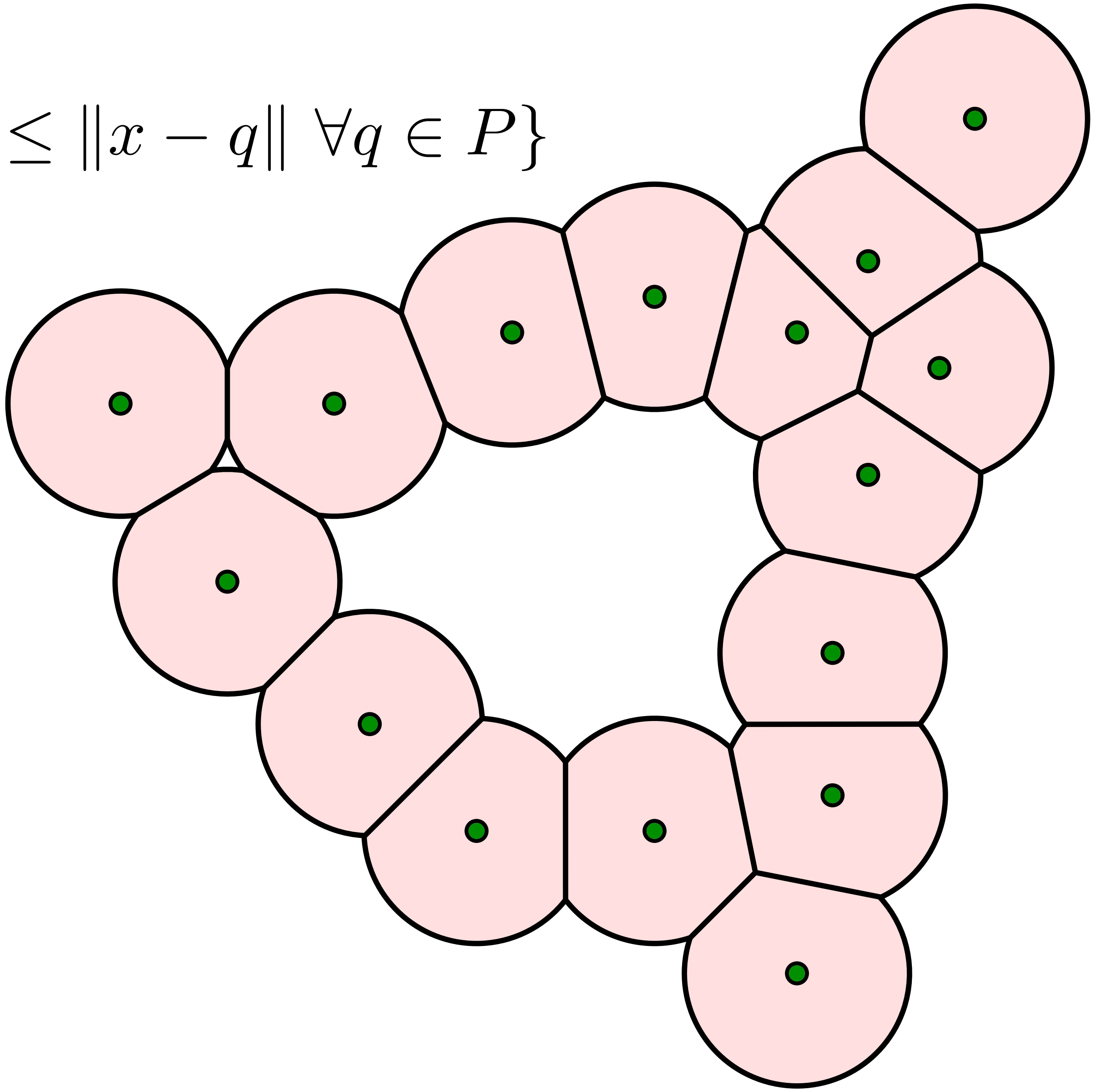

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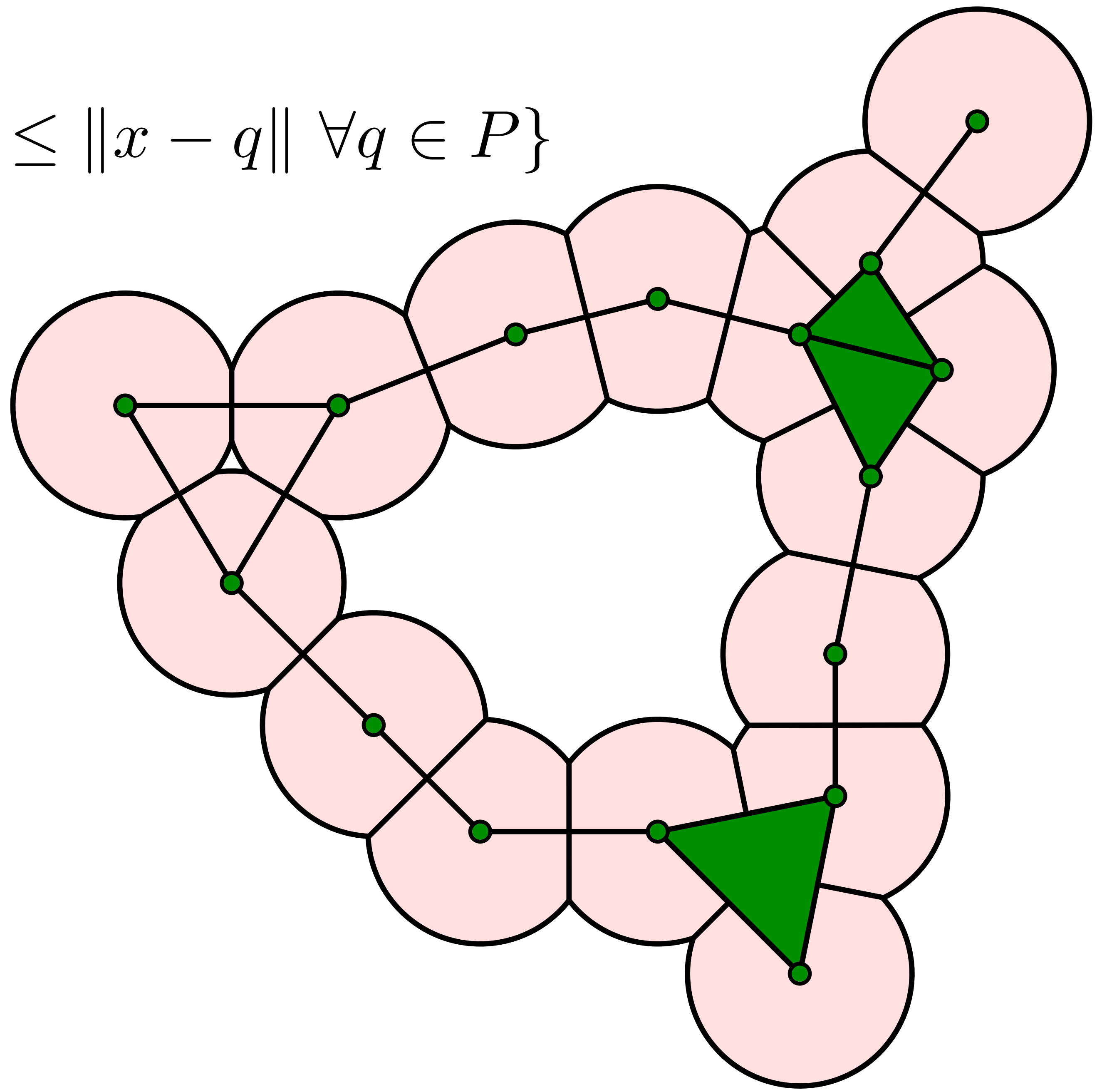

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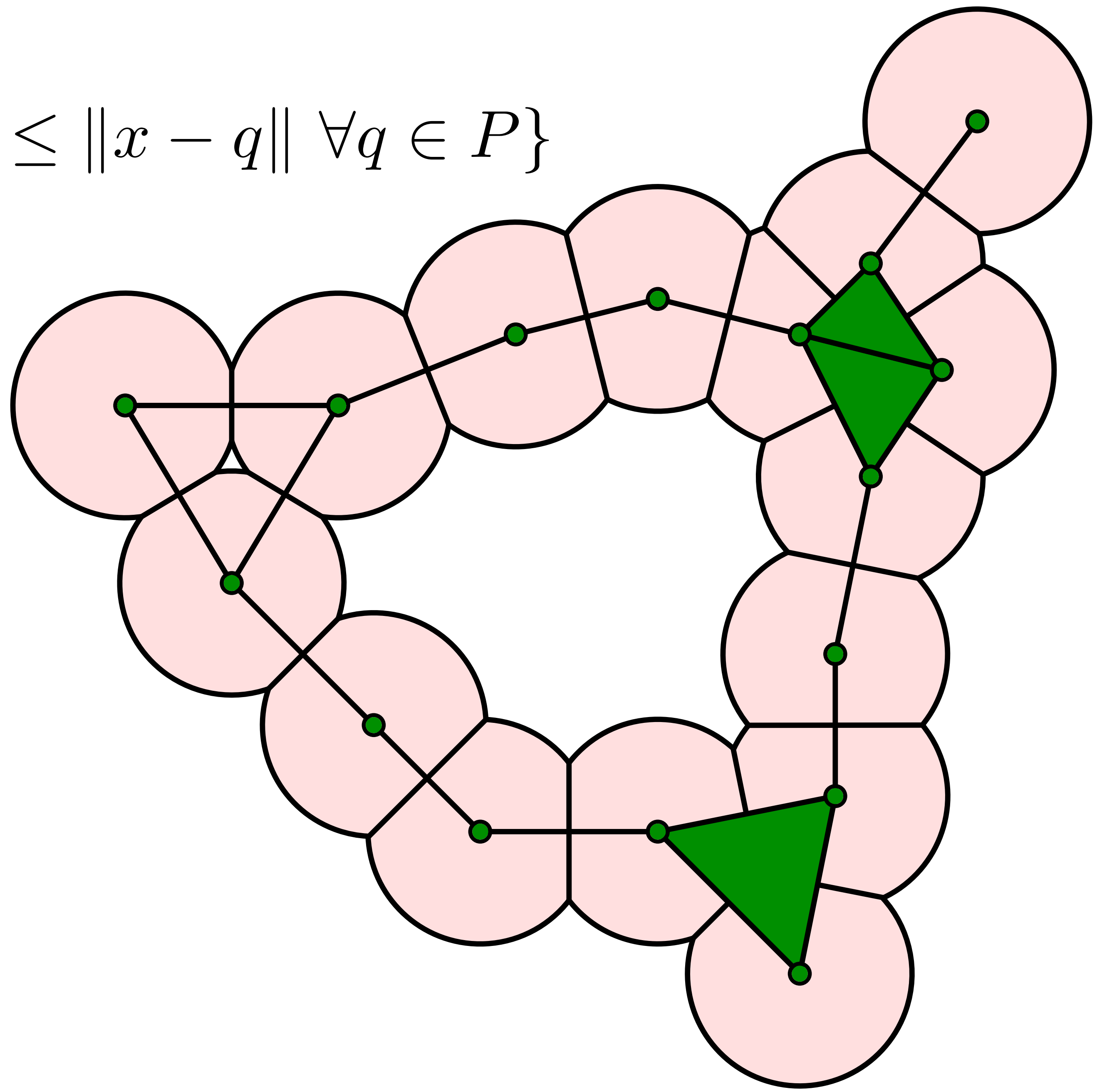

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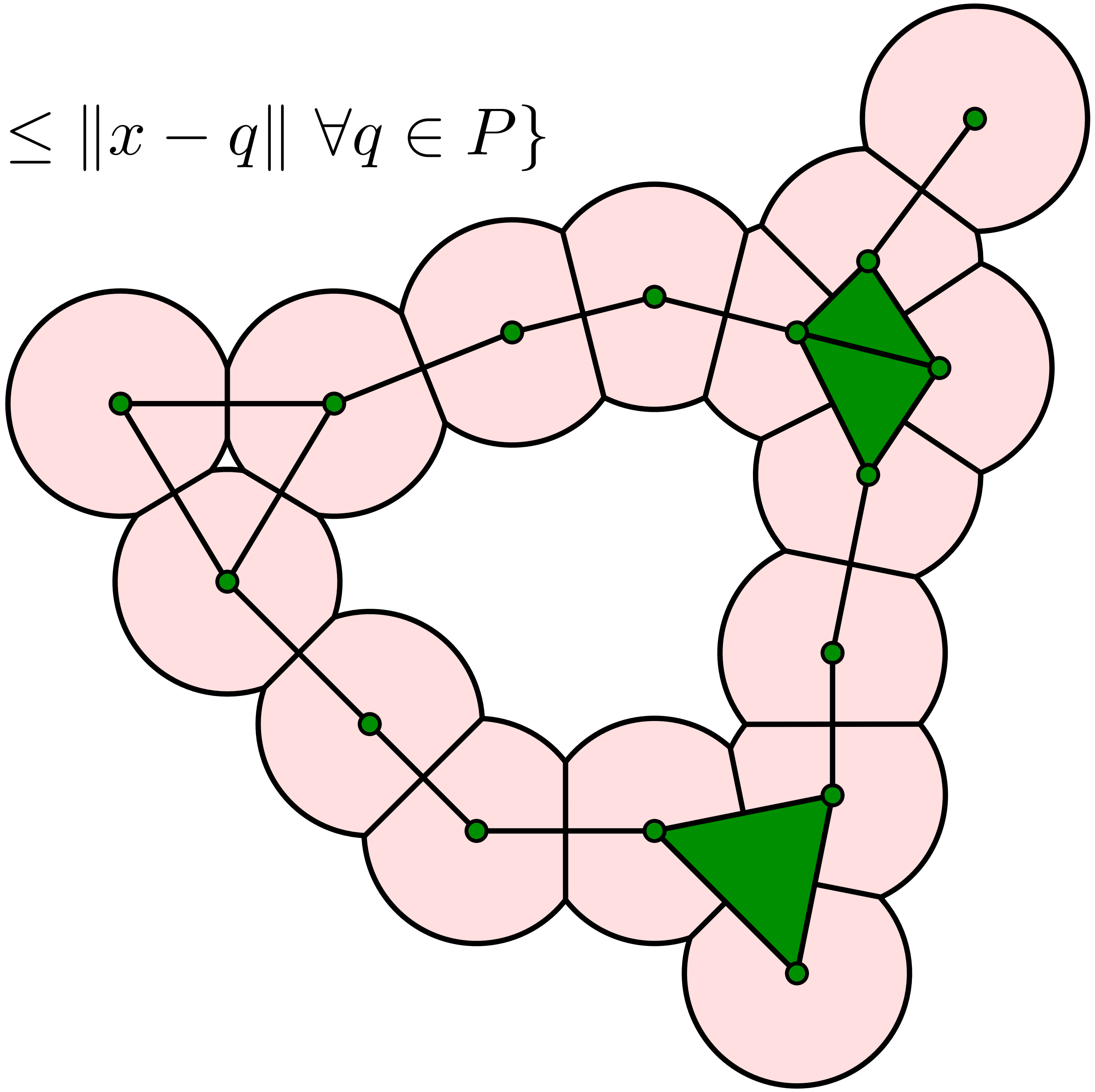

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$\mathrm{AS}(P, r) \subseteq \mathrm{Del}(P)$

$\mathrm{AS}(P, r) \simeq \bigcup_{p \in P} \mathrm{B}_r(p)$

(why?)

Invariants

- A (topological) invariant is a map f that assigns the same object to spaces of the same topological type.

- $\mathbb{X} \approx \mathbb{Y} \Rightarrow f(\mathbb{X}) = f(\mathbb{Y})$

- $f(\mathbb{X}) \neq f(\mathbb{Y}) \Rightarrow \mathbb{X} \not\approx \mathbb{Y}$ (contrapositive)

- $f(\mathbb{X}) = f(\mathbb{Y}) \Rightarrow$ nothing

Euler Characteristic

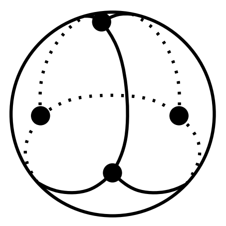
- K a simplicial complex with s_i simplices of dimension i.
- Euler characteristic $\chi(K)$ is

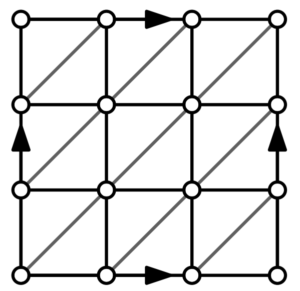
$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i = \sum_{\sigma \in K} (-1)^{\dim \sigma}.$$

- Invariant of $|K|$; any triangulation gives the same answer

From the last time:

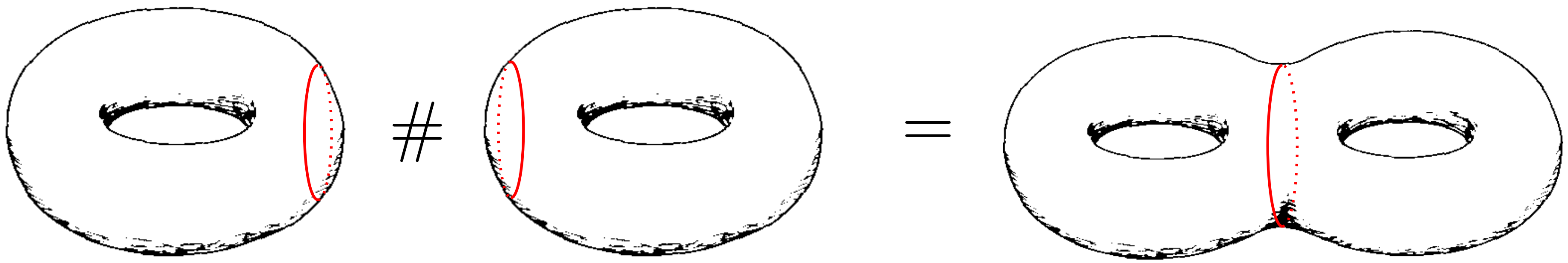
$\chi(\mathbb{N})$	$g(\mathbb{N})$	$\mathbb{N}$	$\mathbb{M}$	$g(\mathbb{M})$	$\chi(\mathbb{M})$
1	1	$\mathbb{P}^2$	$\mathbb{S}^2$	0	2
0	2	$\mathbb{P}^2 \# \mathbb{P}^2$	$\mathbb{T}^2$	1	0
-1	3	$\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$	$\mathbb{T}^2 \# \mathbb{T}^2$	2	-2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$





Connected sum

From last time connected sum $\mathbb{M}_1 \# \mathbb{M}_2$:



Theorem. For compact surfaces $\mathbb{M}_1$ and $\mathbb{M}_2$,

$$\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2. \qquad \text{(Why?)}$$

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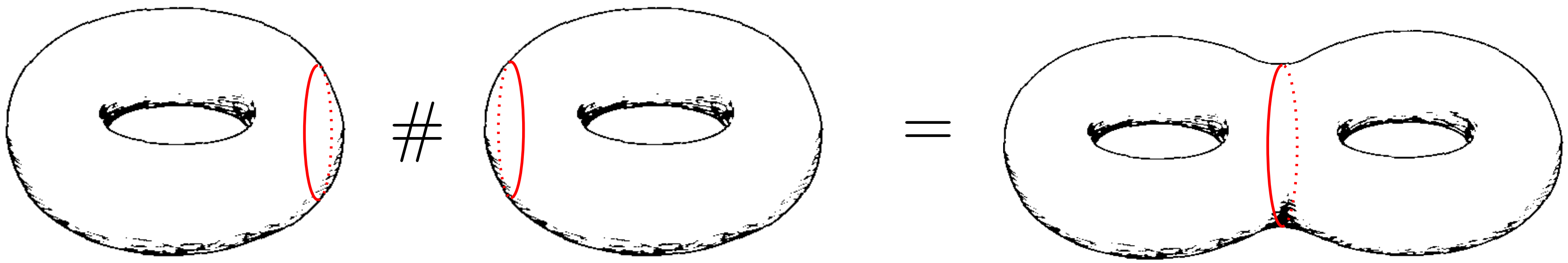

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Euler Characteristic for Compact 2-Manifolds. A sphere with g tubes has $\chi = 2 - 2g$ and a sphere with g cross-caps has $\chi = 2 - g$.

g = genus of $\mathbb{M}$; it is the maximum number of disjoint closed curves along which we can cut without disconnecting $\mathbb{M}$.

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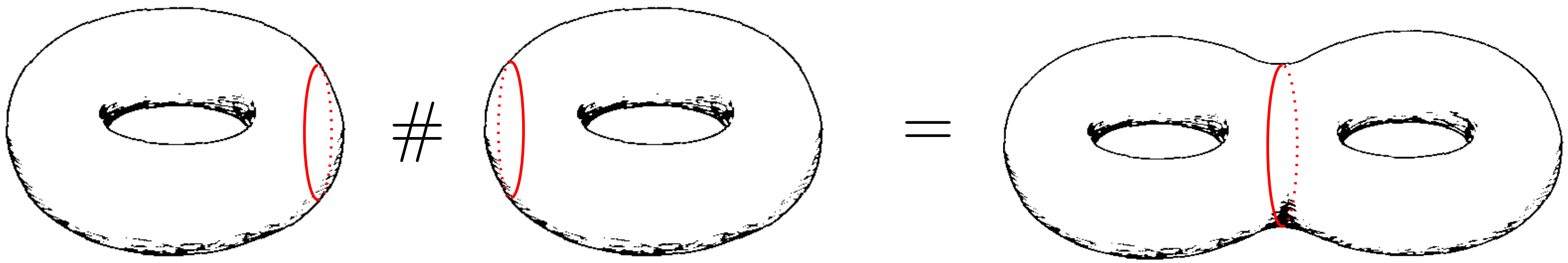

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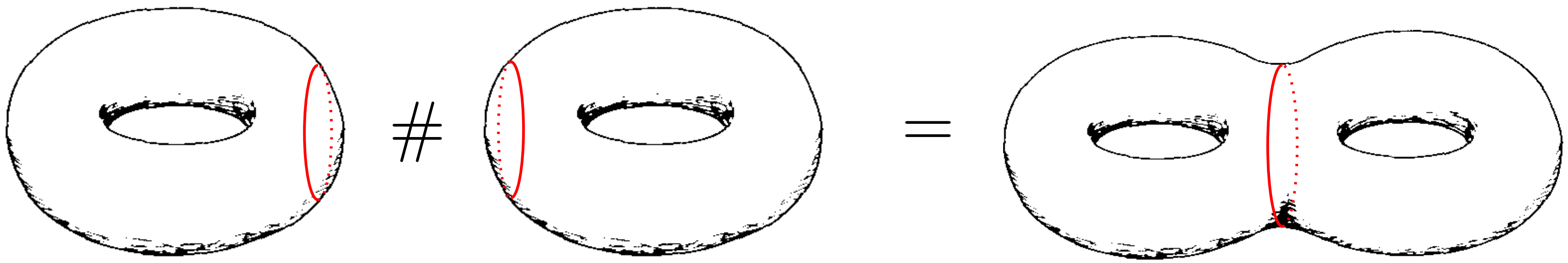

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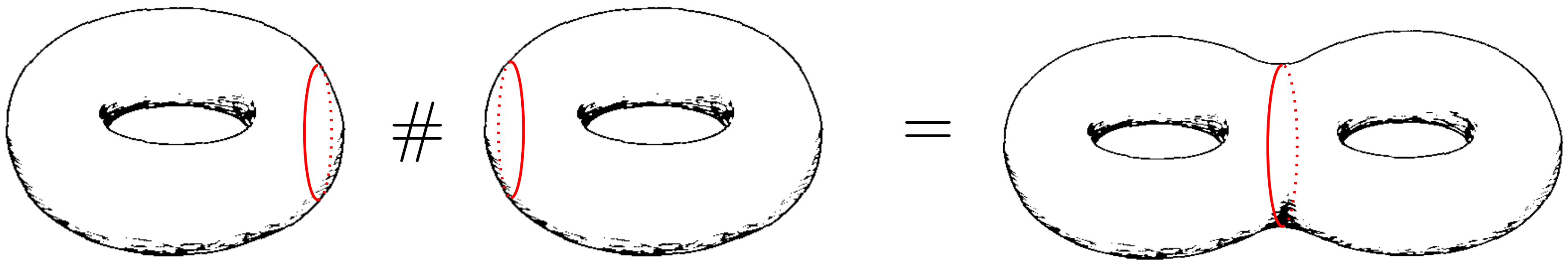

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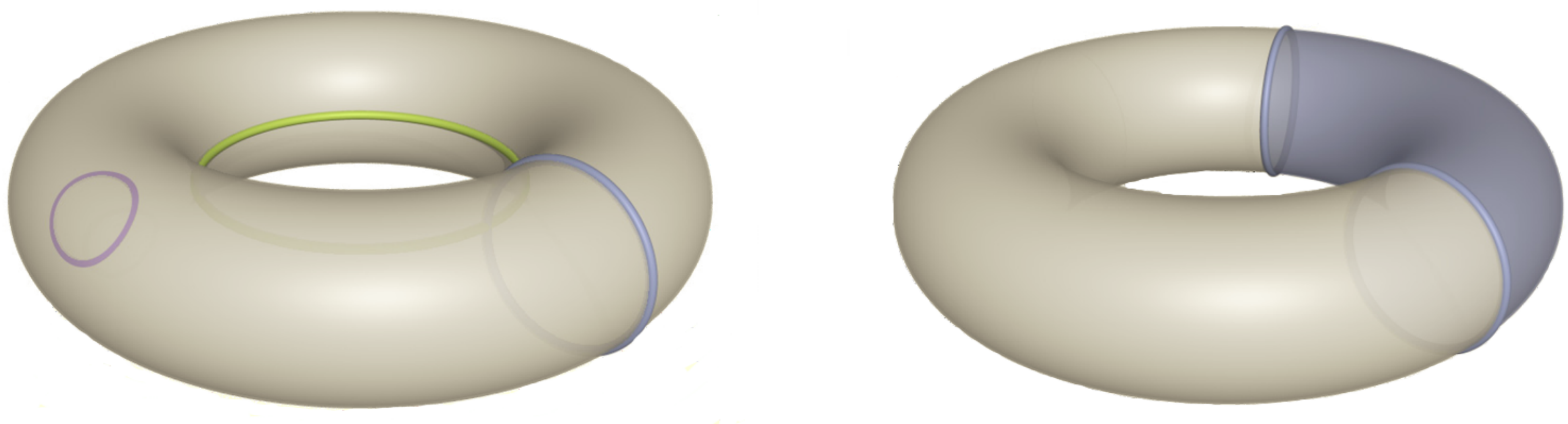
Euler characteristic + orientability are two invariants that completely classify compact 2-manifolds.

Homology



- How cells of dimension n attach to cells of dimension $n-1$
- Keeps track of “holes” in the space
- Combinatorial
- Finite description
- Computable

Chain Group

- Simplicial complex K
- k-chain: $c = \sum n_i[\sigma_i], n_i \in \mathbb{Z}, \sigma_i \in K$, $\dim \sigma_i = k$

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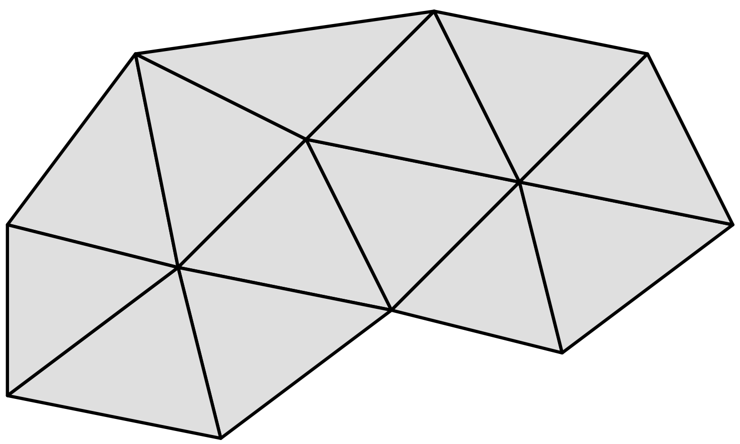

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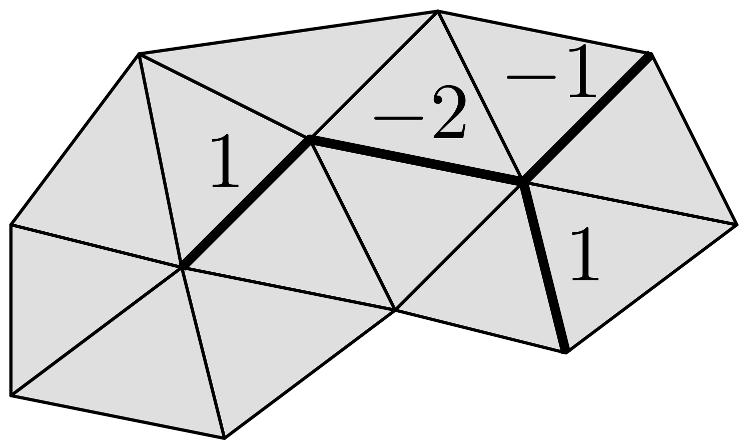

1-chain

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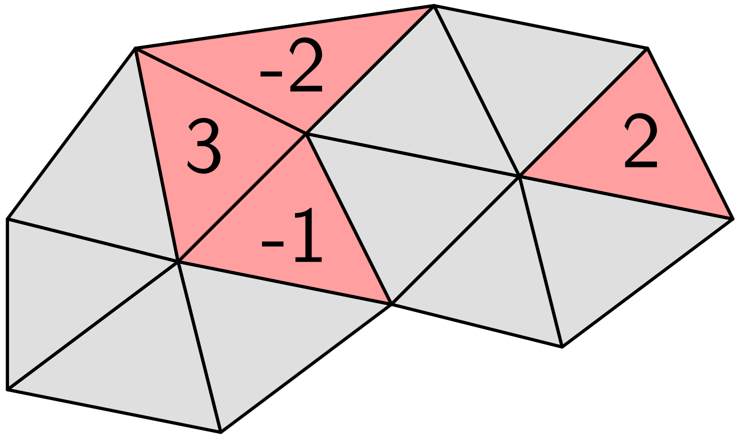

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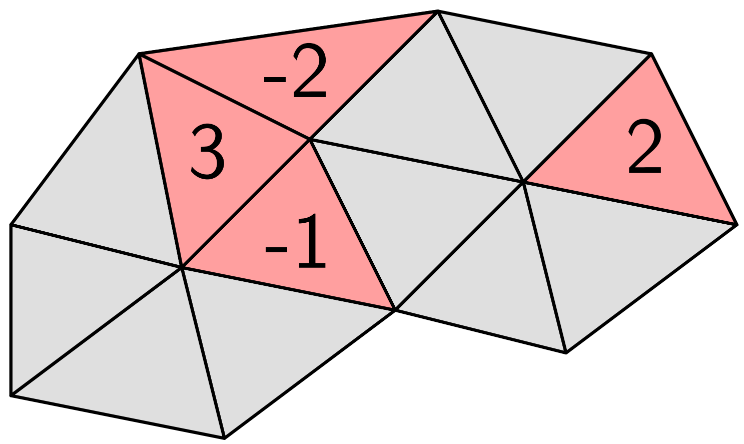

- kth chain group C_k of K is the free abelian group on the set of oriented k-simplices in K
- $\operatorname{rank} C_k = ?$

Boundary

- The boundary operator $\partial_k : C_k \to C_{k-1}$ is a linear homomorphism defined on a chain c by its action on any simplex $\sigma = [v_0, v_1, \dots, v_k] \in c$,

$$\partial_k \sigma = \sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k],$$

where $[v_0, \dots, \hat{v}_i, \dots, v_k]$ is the face of σ without vertex v_i.

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- $\partial_1 [a, b] = b - a$
- $\partial_2 [a, b, c] = [b, c] - [a, c] + [a, b] = [b, c] + [c, a] + [a, b]$
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- $\partial_1 \partial_2 [a, b, c] = [c] - [b] - [c] + [a] + [b] - [a] = 0$

Boundary Theorem

Boundary Theorem. $\partial_{k-1}\partial_k = 0$, for all k.

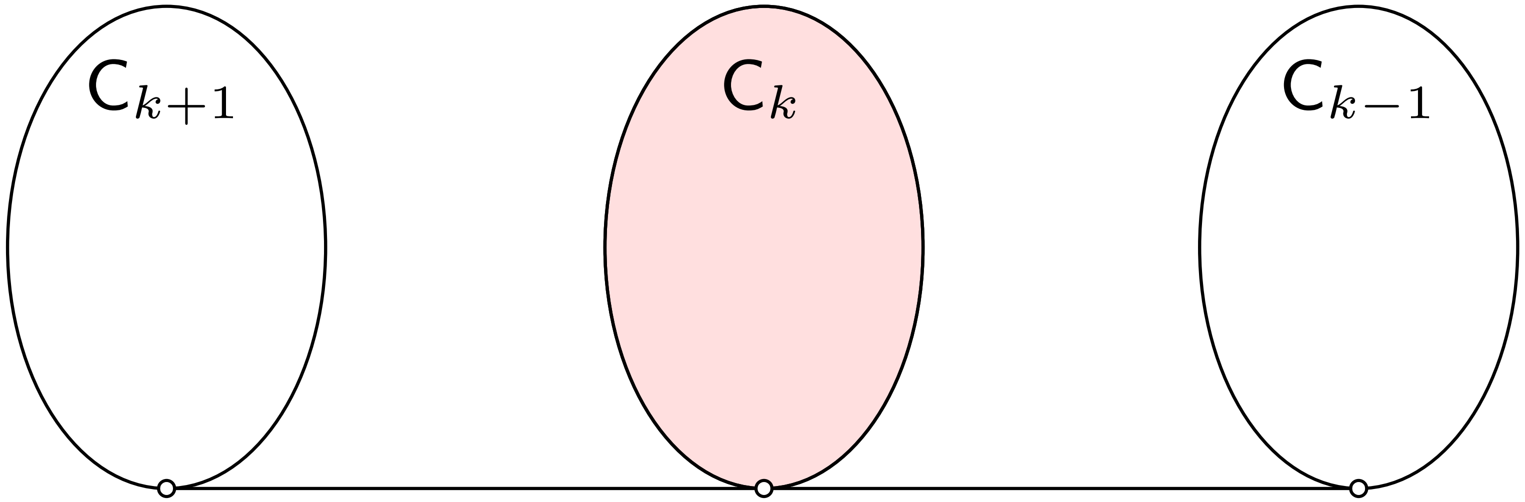
Proof:

$$\begin{aligned}
\partial_{k-1}\partial_k[v_0, v_1, \dots, v_k] &= \partial_{k-1} \sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k] \\
&= \sum_{j<i} (-1)^i (-1)^j [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_k] + \\
&\quad \sum_{j>i} (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_k] \\
&= 0
\end{aligned}$$

Chain Complex

- The boundary operator connects the chain groups into a chain complex C_*:

$$\ldots \to \mathsf{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathsf{C}_k \xrightarrow{\partial_k} \mathsf{C}_{k-1} \to \ldots$$



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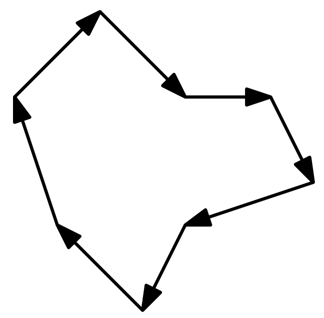

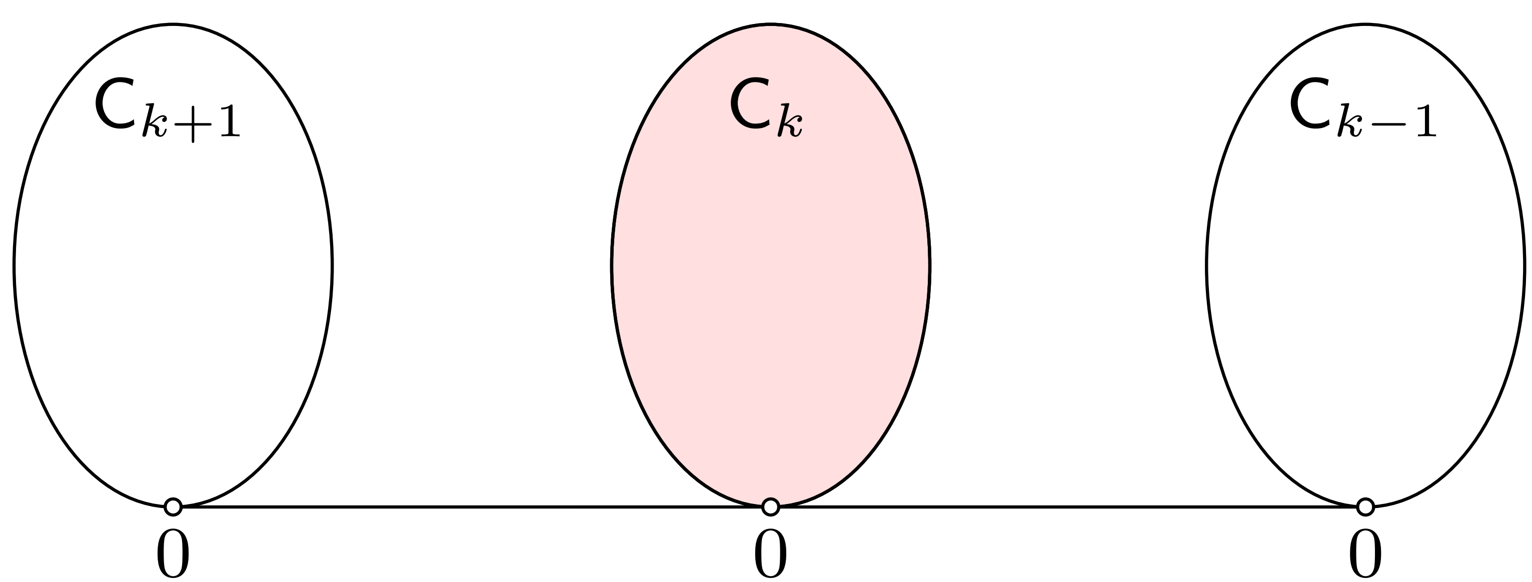

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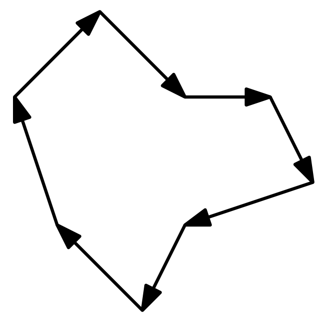
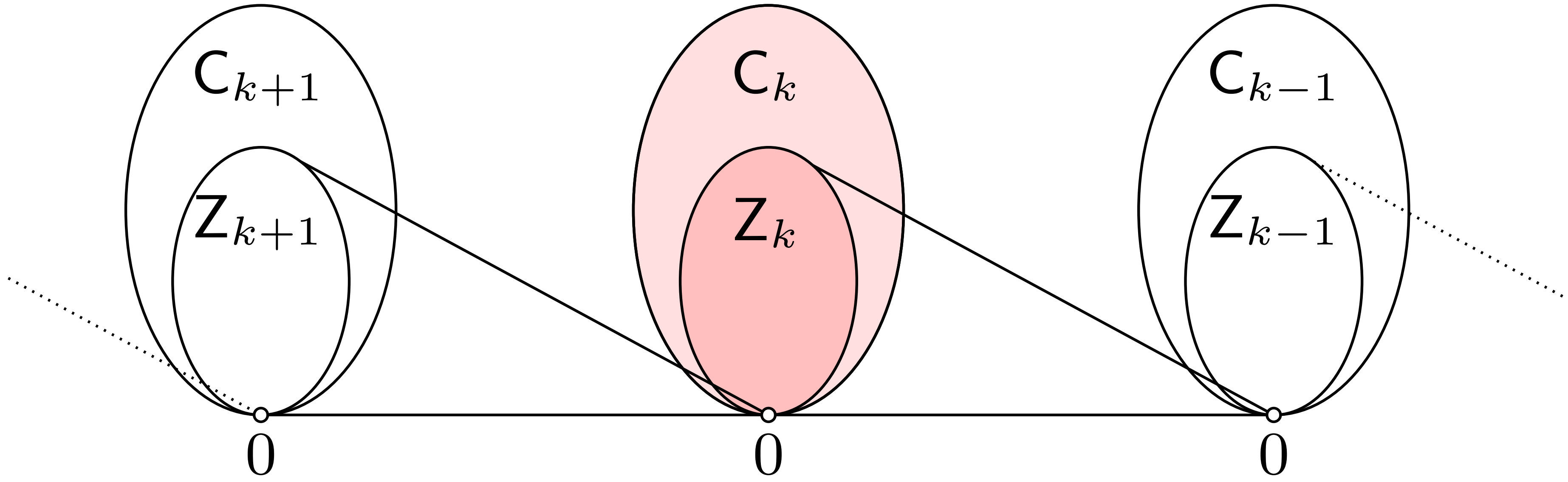


- The kth cycle group is

$$\mathsf{Z}_k = \ker \partial_k = \{c \in \mathsf{C}_k \mid \partial_k c = 0\}$$



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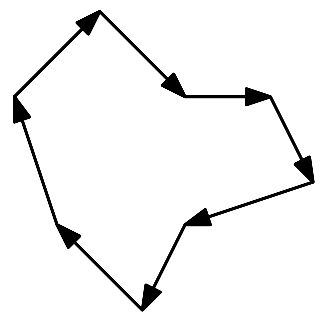
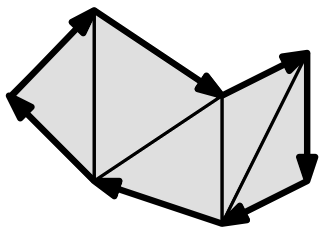
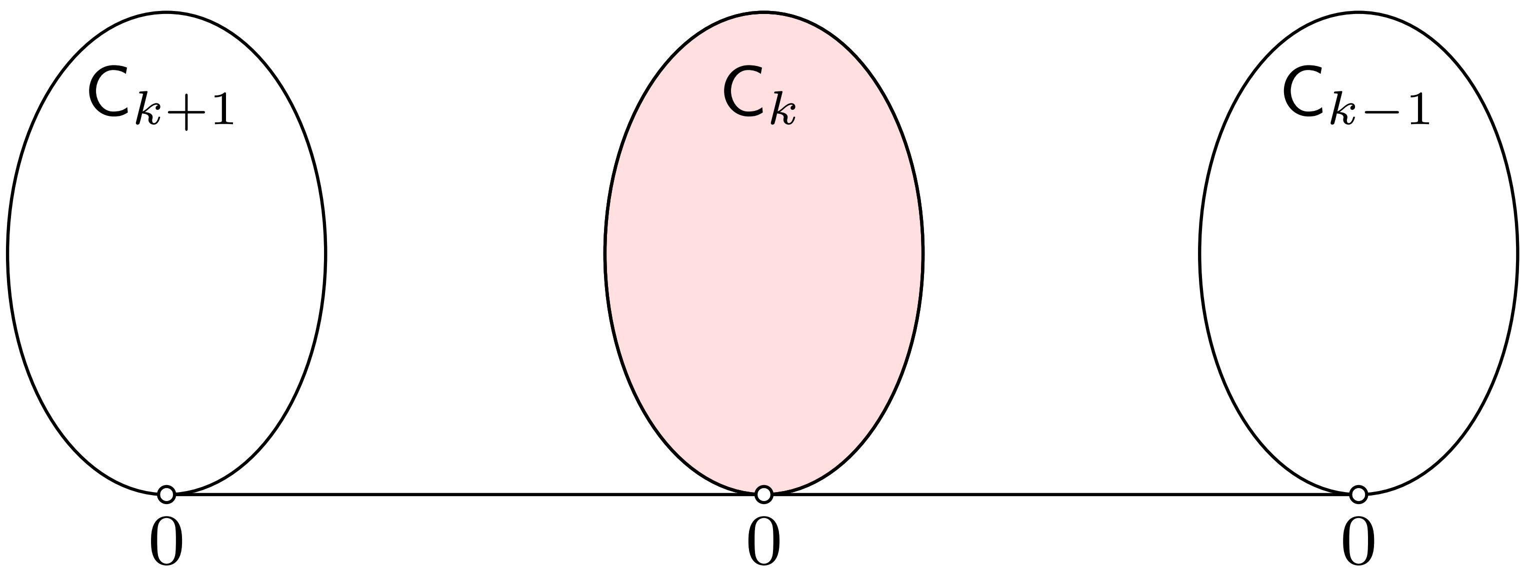


- The kth cycle group is

$$\mathsf{Z}_k = \ker \partial_k = \{c \in \mathsf{C}_k \mid \partial_k c = 0\}$$

- If a k-chain b is a boundary of a $(k+1)$-chain, it's a k-boundary





Chain Complex

- The boundary operator connects the chain groups into a chain complex C_*:

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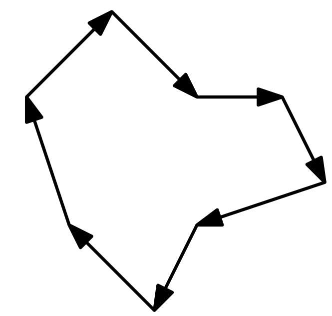

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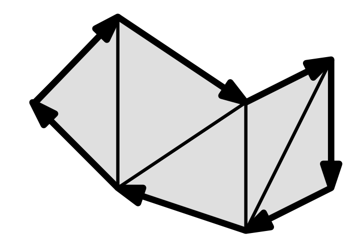
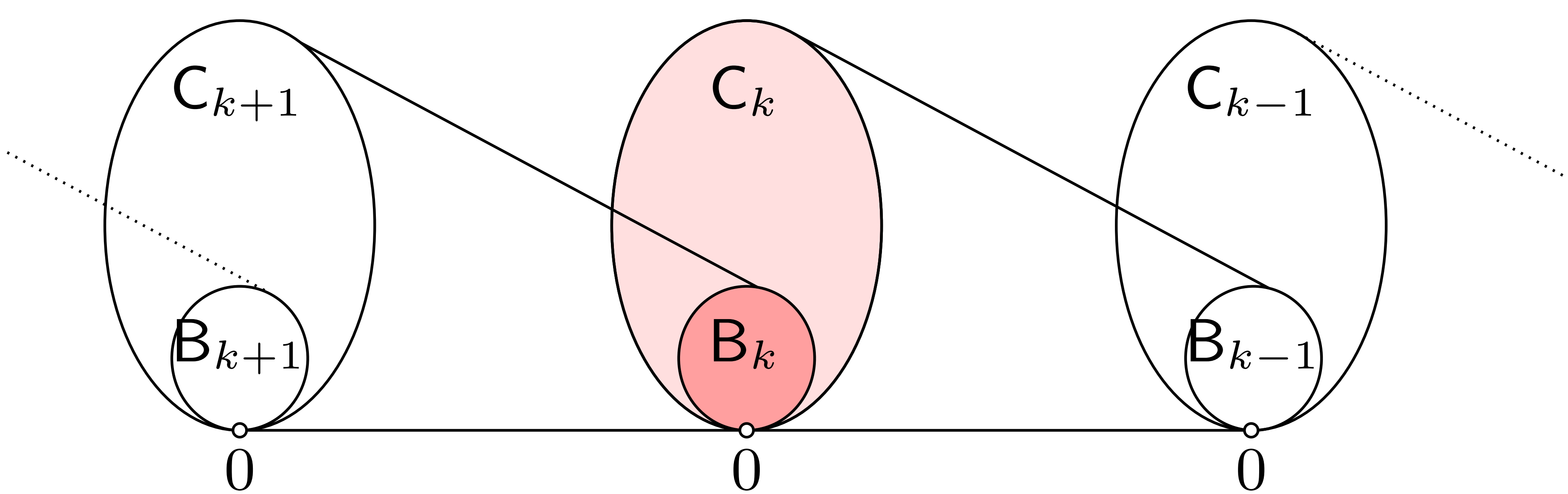


$$\mathsf{B}_k = \operatorname{im} \partial_{k+1} = \{b \in \mathsf{C}_k \mid \exists d \in \mathsf{C}_{k+1}, b = \partial_{k+1} d\}$$

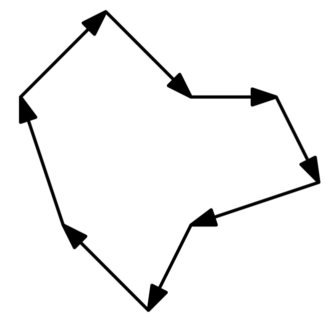


Chain Complex

- The bou ... ain complex C_*:

- If b is a k-boundary, then there is $c \in C_{k+1}$, such that $b = \partial_{k+1} c$
- $\partial_k b = \partial_k \partial_{k+1} c = 0$ (by the boundary theorem)

- If a k-ch ...
- The kth ...



$$Z_k = \ker \partial_k = \{c \in C_k \mid \partial_k c = 0\}$$

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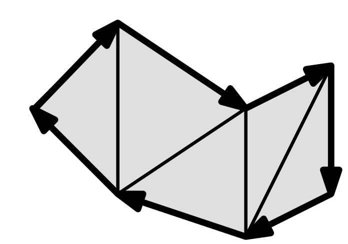

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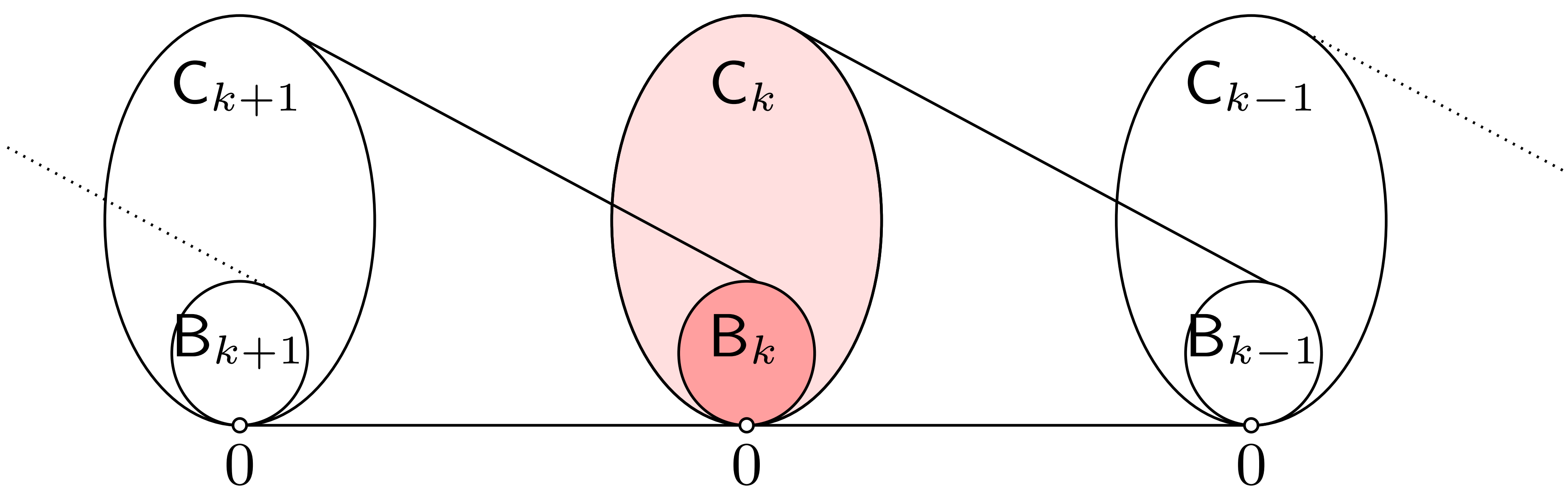

Chain Complex

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- Every boundary is a cycle: $\mathsf{B}_k \subseteq \mathsf{Z}_k \subseteq \mathsf{C}_k$

- The bou… ain complex C_*:
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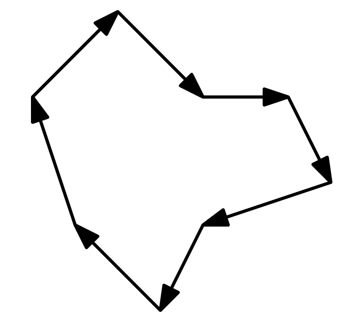

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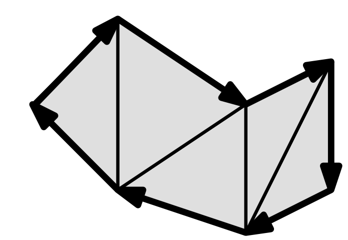

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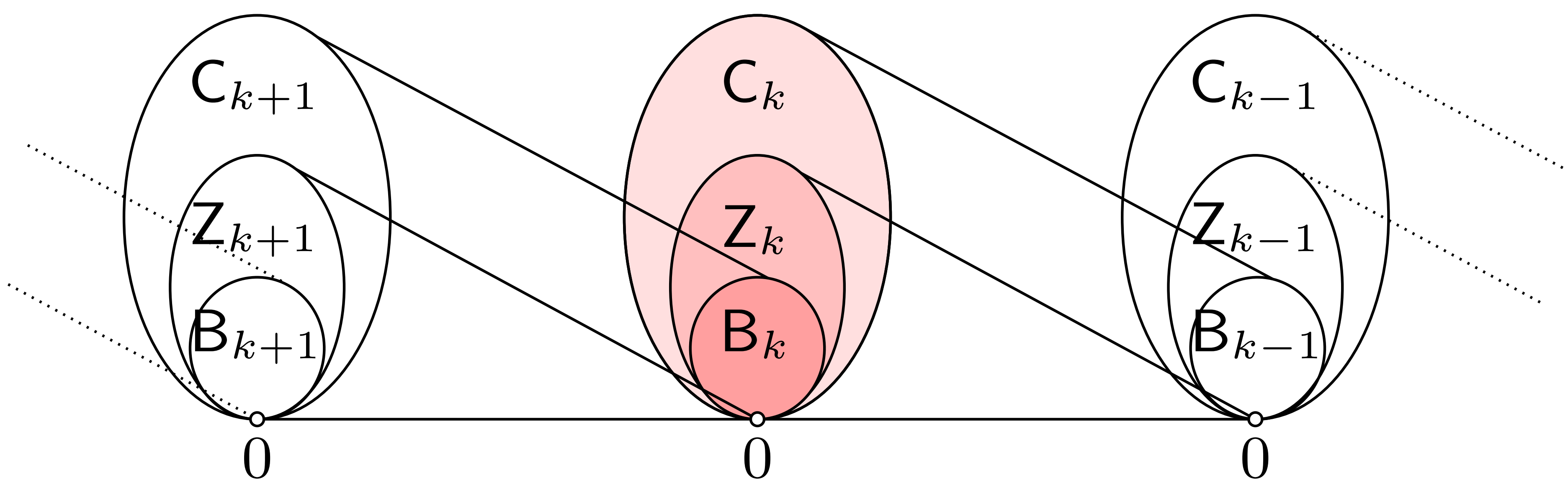
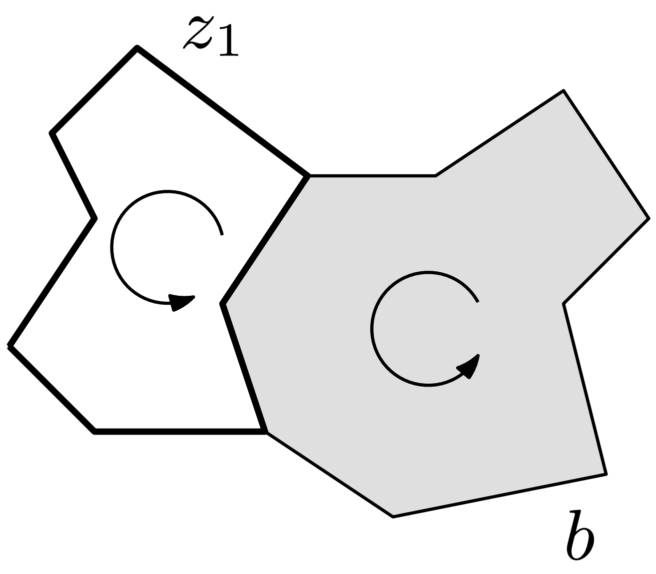


Homology

- If z is a k-cycle, b is a k-boundary; we would like to have $z + b$ be equivalent to z
- I.e. if $z_1 - z_2 = b$, where $b \in \mathsf{B}_k$, then $z_1 \sim z_2$



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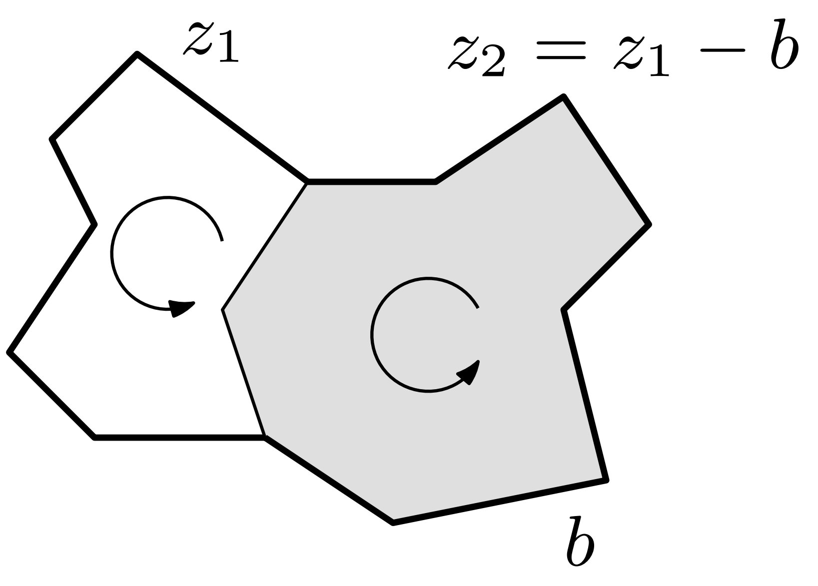

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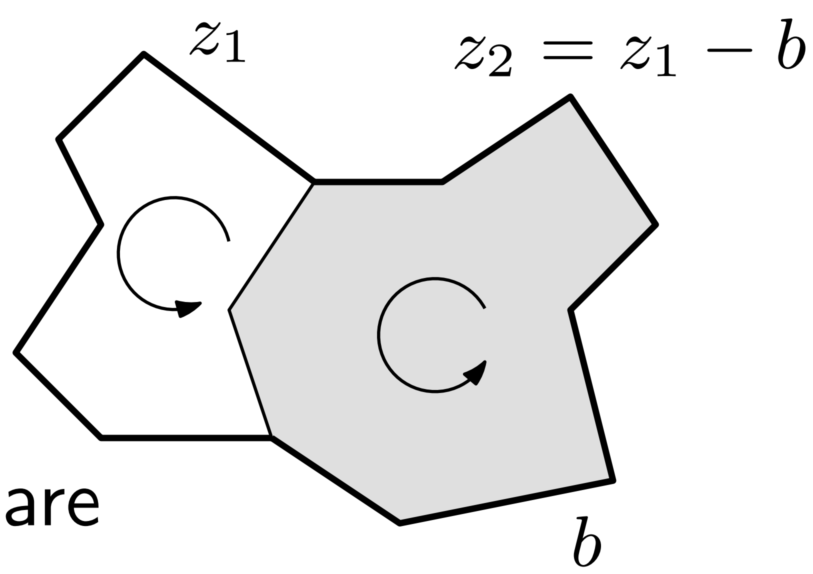
- The kth homology group is

$$\mathsf{H}_k = \mathsf{Z}_k / \mathsf{B}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}$$

- If $z_1 = z_2 + \mathsf{B}_k$, $z_1, z_2 \in \mathsf{Z}_k$, then z_1 and z_2 are homologous, $z_1 \sim z_2$.



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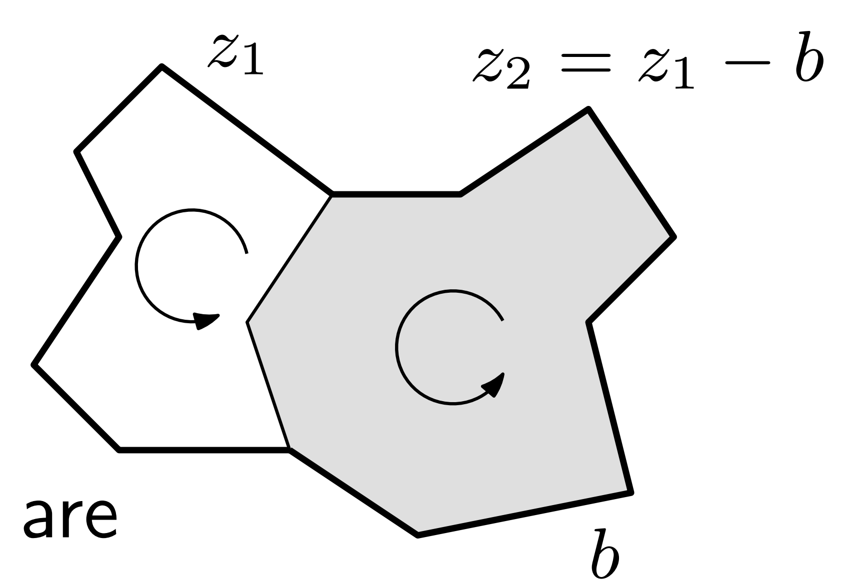

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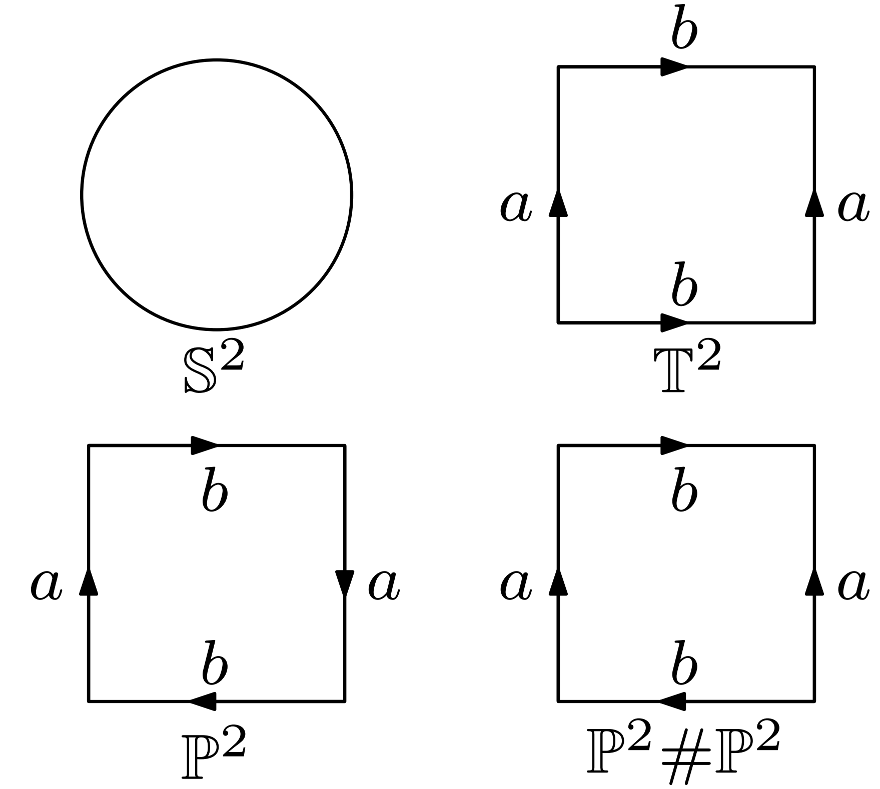
- Homology of 2-manifolds:

	H_0	H_1	H_2
sphere	$\mathbb{Z}$	0	$\mathbb{Z}$
torus	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$
projective plane	$\mathbb{Z}$	$\mathbb{Z}_2$	0
Klein bottle	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	0



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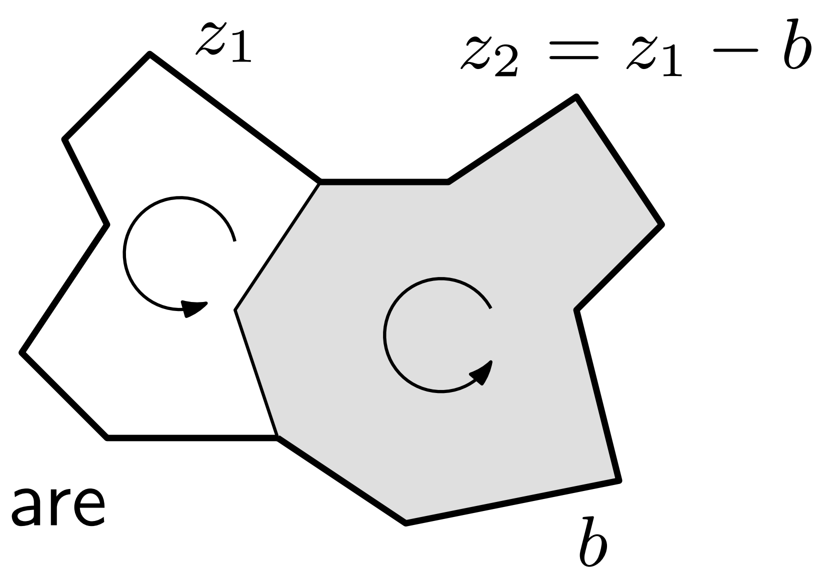

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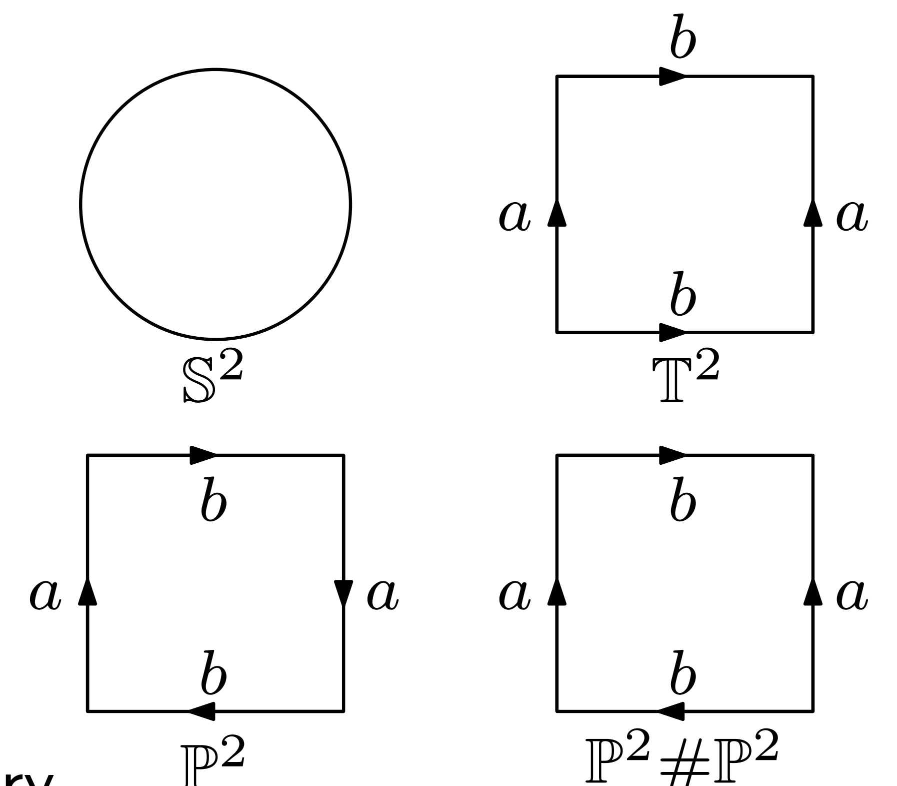
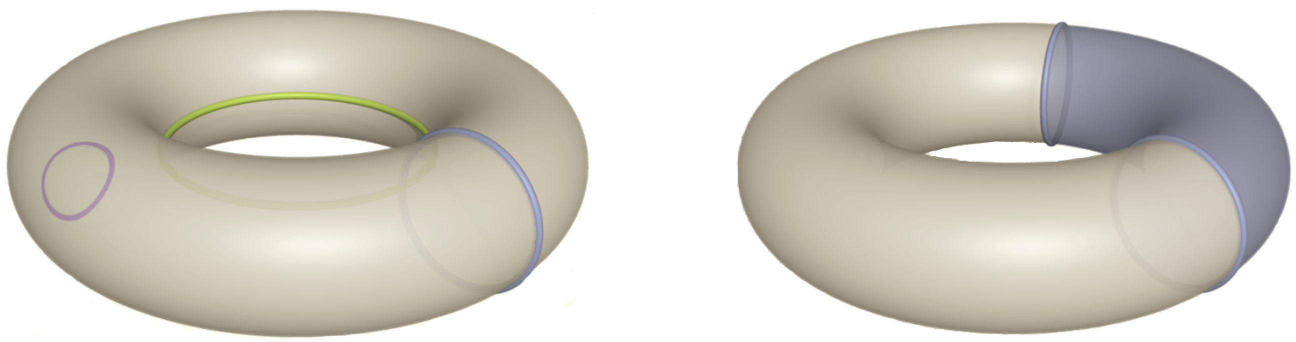
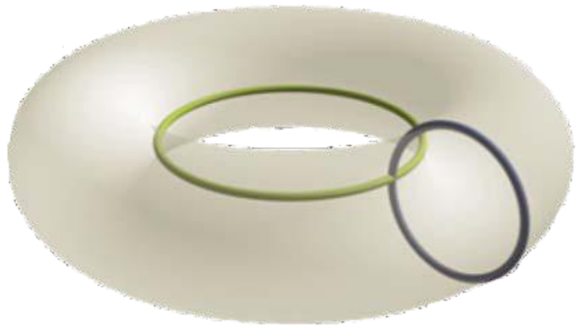
two copies of $a + b$ are a boundary



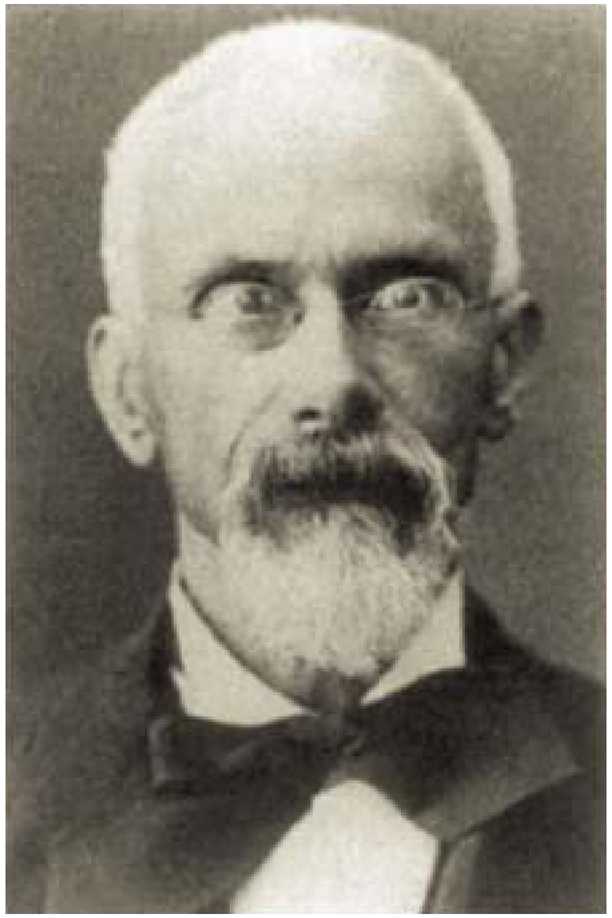
Betti numbers

- kth Betti number $\beta_k = \operatorname{rank} \mathsf{H}_k = \operatorname{rank} \mathsf{Z}_k - \operatorname{rank} \mathsf{B}_k$
- Geometric interpretation in $\mathbb{R}^3$:
  - β_0 is the number of components
  - β_1 is the number of tunnels
  - β_2 is the number of voids





$(1, 2, 1)$



Enrico Betti
(1823 – 1892)

Euler-Poincaré

- Let K be a simplicial complex
- Let $\mathsf{C}_*(K)$ be the chain complex of K
- $\mathrm{rank}(\mathsf{C}_i(K)) = |\{\sigma \in K \mid \dim \sigma = i\}|$
- Euler characterstic $\chi(K) = \chi(\mathsf{C}_*) = \sum_i (-1)^i \,\mathrm{rank}(C_i)$

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- Sphere: 2 $=$ 1 - 0 $+$ 1
- Torus: 0 $=$ 1 - 2 $+$1

Thank you for your time and attention!