

3D Transformations

3D Rotations about an axis through the origin
if not: conjugation trick

Orientation vs. Rotation

- orientation is a state
can be described by a rotation from a reference pose
- rotation is an action / motion / transformation
like point - vector
orientation - rotation
state - movement

Many representations for 3D rotations, sometimes inconsistently handled

- axis & angle "not a linear object"
- rotation matrix "circular"
- Euler angles $\vec{\theta} \equiv \vec{\theta} - \theta$
- quaternions

$$SO(3) \approx RP^3$$

Euler's Theorem: The general displacement of a rigid body with one point fixed is a rotation about some axis.

3 degrees of freedom (rot through origin)

5 degrees in general
4 = line + 1 = angle

$$M \quad \det(M) = +1$$

$$\begin{aligned} \det(M-I) &= \det((M-I)^T) = \det(M^T-I) = \\ \det(M^{-1}-I) &= \det(M^{-1}(I-M)) = \det(M^{-1}) \det(I-M) = \\ -\det(M^{-1}) \det(M-I) &= -\det(M-I) \Rightarrow \end{aligned}$$

$$\det(M-I) = 0$$

$$\det(M-\lambda I) = 0 \quad \text{for } \lambda=1$$

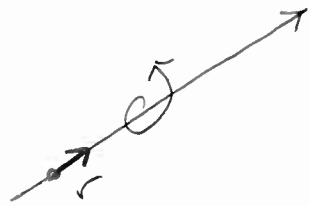
$M-I$ is singular & has 1 as an eigenvalue
 w corresponding eigenvector

$$\Leftrightarrow Mv = Iv: v \text{ is fixed by } M$$

Rotation representations

1. axis & angle

most intuitive



directional cosines

r a unit length vector at the origin

$$r = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \leftarrow \begin{matrix} r_x^2 + r_y^2 + r_z^2 = 1 \\ \text{at an angle } (\text{directions}) \end{matrix}$$

sometimes

$$\omega = \alpha r$$

difficult to apply rotations to points or vectors in this form

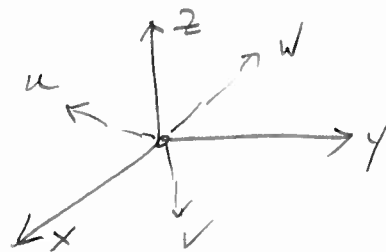
2. rotation matrices

easiest for applying to points or vectors

$$M = \begin{bmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad 4 \times 4$$

directional cosine matrix

cols of M are the unit vectors of the new frame after rotation



M is a real, orthogonal matrix

$$u_x^2 + u_y^2 + u_z^2 = 1$$

$$u_x v_x + u_y v_y + u_z v_z = 0$$

$$v_x^2 + v_y^2 + v_z^2 = 1$$

$$v_x w_x + v_y w_y + v_z w_z = 0$$

[col. relations]

$$w_x^2 + w_y^2 + w_z^2 = 1$$

$$w_x v_x + w_y v_y + w_z v_z = 0$$

9 unknowns - 6 relations = 3 degrees of freedom

Note that

$M^{-1} = M^T$ is also an orthogonal matrix,

so dual relations

$u_x^2 + v_x^2 + w_x^2 = 1,$ [row relations]

$MM^T = M^T M = I$

$u_x u_y + v_x v_y + w_x w_y = 0$

also hold

$\det(M) = +1$

$\det^2(M) = 1$

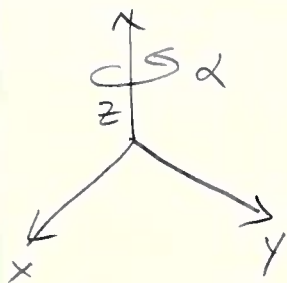
[$\det(M) = -1$ are the reflections / glide reflections]

Note that eigenvalues of M are $\{1, e^{\pm i\theta}\} = \{1, \cos\theta + i\sin\theta, \cos\theta - i\sin\theta\}$

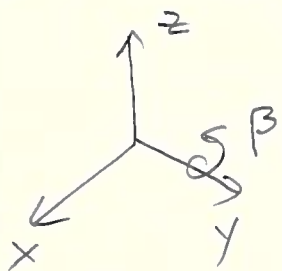
$O(3), SO(3)$

So what do these matrices look like?

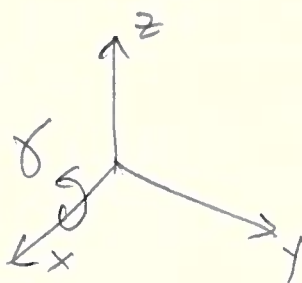
3x3



$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

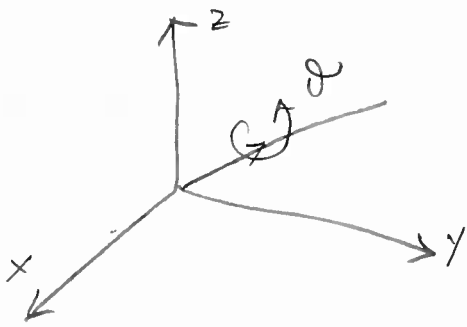


$$\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

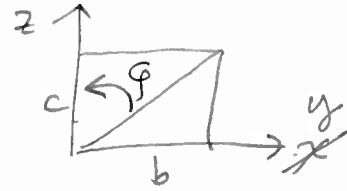


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

arbitrary axis case



$a, b, c \quad \sqrt{a^2+b^2+c^2}$

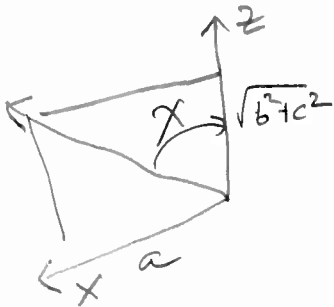


put axis on xz plane
by an x rotation

$$\cos \phi = \frac{c}{\sqrt{b^2+c^2}}$$

$$\sin \phi = \frac{b}{\sqrt{b^2+c^2}}$$

$$R_x^\phi$$



$$\cos \chi = \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}}$$

$$\sin \chi = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

align axis with z
by an y rotation

$$R_y^\chi$$

$$R_x^{-\phi} R_y^{-\chi} R_z^\theta R_y^\chi R_x^\phi$$

$$\leftarrow R_x(\theta)$$

General formula (Rodrigues)

$$c = \cos \theta \quad s = \sin \theta$$

$$R = cI + (1-c) r r^T + s r^\wedge$$

↑ identity

$$r^\wedge = \begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix}$$

skew symmetric
[$r^\wedge{}^T = -r^\wedge$]

$$r^\wedge v = r \times v = \begin{pmatrix} r_y v_z - r_z v_y \\ r_z v_x - r_x v_z \\ r_x v_y - r_y v_x \end{pmatrix}$$

From matrices to Euler angles

$$M = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

X - Y - Z fixed

α β γ

$$\beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

$$\alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/s\beta)$$

$$\gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta)$$

$$\text{Atan2}(y, x) \quad \tan^{-1}\left(\frac{y}{x}\right)$$

↑ ↑ uses both signs

single solution $-90 \leq \beta \leq 90$

Another Euler theorem

Any rotation about the origin can be expressed as the product of 3 rotations around the axes k, l, m , where $k \neq l, l \neq m$

Z-Y-X Euler

Rot $\begin{matrix} \sigma z & \alpha \\ \sigma y & \beta \\ \sigma x & \gamma \end{matrix}$

moving system

mostly used by
animation software

XYZ fixed \equiv ZYX Euler

Z-Y-Z Euler

$\begin{matrix} \sigma z & \alpha \\ \sigma y & \beta \\ \sigma z & \gamma \end{matrix}$

24 different

angle conventions

fixed/Euler

$\begin{matrix} & \circ & \circ & \circ \\ \nearrow \text{dir} & & \uparrow & \nwarrow \text{dir} \\ & & \text{different} & \end{matrix}$