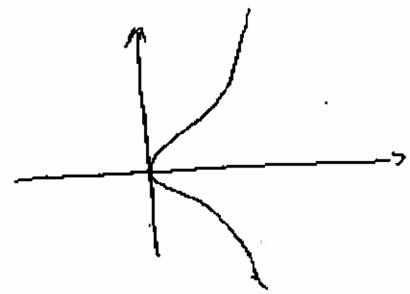
Differential Geometry of planar curves.

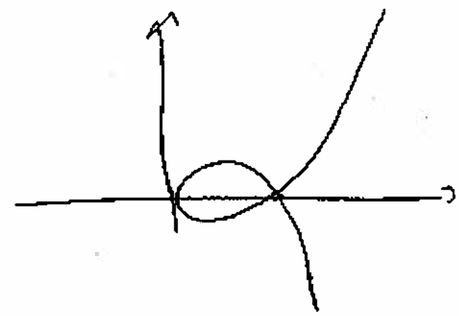
1 Humpy

$x(t) = t^2$
$y(t) = t^3 + t$



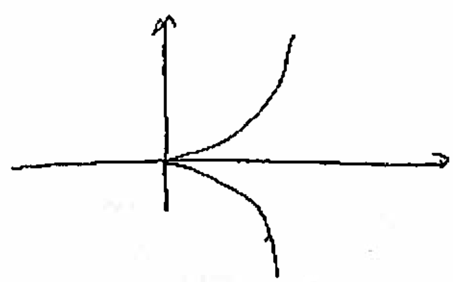
2. Loopy

$x(t) = t^2$
$y(t) = t^3 - t$



3. Pointy

$x(t) = t^2$
$y(t) = t^3$



In this lecture, we will only consider curves that are differentiable. I.e. $c'(t)$ exists.

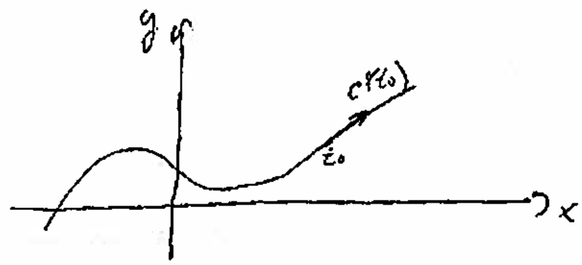
$$c = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \quad c'(t) = \frac{dc(t)}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

Tangent Vectors

Def: Given a parametric curve $c(t)$, let $c'(t)$ be its tangent vector.

Tangent vectors approximate $c(t)$ to first order: Taylor series approximating $c(t)$ around t_0:

$$c(t) = c(t_0) + c'(t_0)(t - t_0) + O((t - t_0)^2)$$



Around t_0, $c(t)$ looks like a line, ~~whose~~ in the direction $c'(t_0)$. Direction is much more important than magnitude.

For differential geometry it is important to define tangent vectors everywhere.

Singular point: a point t_0, such that $c'(t_0) = \underline{0}$.

Example: pointy cubic with $t_0 = 0$.

$$c'(t) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

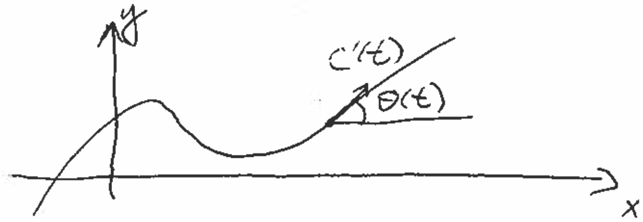
Regular curve: a curve without singular points.

$c'(t) \neq 0 \ \forall t.$ Loopy $c'(t) = \begin{pmatrix} 2t \\ 2t-1 \end{pmatrix} \neq 0$

Humpy $c'(t) = \begin{pmatrix} 2t \\ 2t+1 \end{pmatrix} \neq 0$

Signed Curvature

Let $\theta(t)$ be the angle between $c'(t)$ and the x-axis:



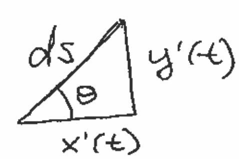
Then $\theta'(t)$ is the infinitesimal change in direction of the curve.

If ds is the infinitesimal change in length of the curve, then signed curvature:

$$\kappa(t) = \frac{d\theta}{ds} = \frac{d\theta}{dt} \cdot \frac{dt}{ds} = \theta'(t) \cdot \left(\frac{ds}{dt}\right)^{-1}$$

If $c(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, then

$$\theta(t) = \arctan\left(\frac{y'(t)}{x'(t)}\right)$$



$$\theta'(t) = \frac{1}{1 + \frac{y'(t)^2}{x'(t)^2}} \cdot \left(\frac{y''(t)\,x'(t) - x''(t)\,y'(t)}{x'(t)^2}\right)$$

$$= \frac{y''(t)\,x'(t) - x''(t)\,y'(t)}{x'(t)^2 + y'(t)^2}$$

And $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$

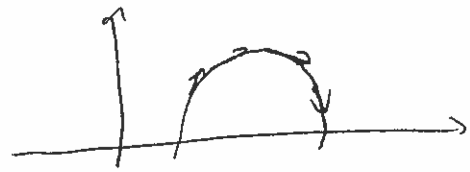
Thus: $$\kappa(t) = \frac{y''(t)\,x'(t) - x''(t)\,y'(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}$$

Curvature measures the change in the direction of tangent vectors. How much a curve bends.

Positive curvature:

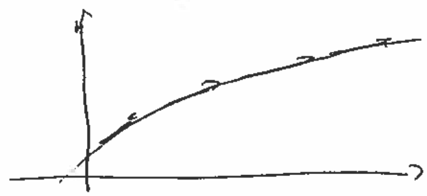
angle is increasing

Negative curvature:



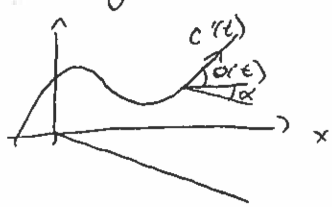
angle is decreasing

Small absolute value of curvature:



angle is almost constant. Curve is flat.

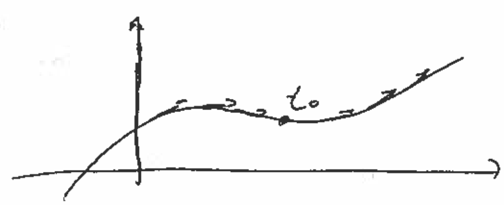
Note that the axis can be chosen arbitrarily:



$$\tilde{\theta}(t) = \theta(t) + \alpha$$
$$\tilde{\theta}'(t) = \theta'(t)$$

Orientation changes the sign.

Inflection points: points at which curvature changes sign:



Example

Inflection points of Loopy & Humpy cubics:

$$c(t) = \begin{pmatrix} t^2 \\ t^3+1 \end{pmatrix}, \quad c'(t) = \begin{pmatrix} 2t \\ 3t^2+1 \end{pmatrix}, \quad c''(t) = \begin{pmatrix} 2 \\ 6t \end{pmatrix}$$

$$K(t) = \frac{y''(t)\,x'(t) - x''(t)\,y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}$$

$$= \frac{6t \cdot 2t - 2\,(3t^2+1)}{(x'(t)^2 + y'(t)^2)^{3/2}}$$

The denominator is always positive, and thus does not matter for finding inflection points.

$$12t^2 - 6t^2 - 2 = 0 \quad \Rightarrow \quad 3t^2 - 1 = 0 \quad \Rightarrow$$

$t = \pm \frac{1}{\sqrt{3}}$ are the two inflection points.

Loopy: $c(t) = \begin{pmatrix} t^2 \\ t^3 - t \end{pmatrix}$, $c'(t) = \begin{pmatrix} 2t \\ 3t^2 - 1 \end{pmatrix}$, $c''(t) = \begin{pmatrix} 2 \\ 6t \end{pmatrix}$

$$y''(t)\,x'(t) - x''(t)\,y'(t) = 12t^2 - 6t^2 + 2 = 6t^2 + 2 \geq 0 \;\forall$$

No inflection point.

Fundamental Theorem of Planar curves

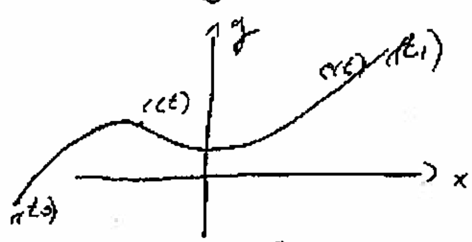
Given a differentiable function $\kappa(s)$ there exists a curve $c(t)$ whose curvature is $\kappa(s)$.
Moreover any other such curve can be obtained by rotation and translation of $c(t)$

Curvature determines $c(t)$ up to a rigid motion.
Second order property.

Note that the curve is also unique up to reparametrization. Is there a natural, "geometric" parametrization of a curve?

Arc-Length Parametrization

Arc length of a curve:



$c(t)$; $t_0 \le t \le t_1$

Length of a curve between t_0 and t.

$$l(t) = \int_{t_0}^{t} \| c'(s) \|_2 \, ds = \int_{t_0}^{t} \left(c'(s)^T c'(s) \right)^{\frac{1}{2}} ds$$

$$= \int_{t_0}^{t} \left(x'(\xi)^2 + y'(\xi)^2 \right)^{\frac{1}{2}} ds$$

If $c'(s) \neq 0 \quad \forall s$

$\ell(t)$ is a monotonically increasing function. Has an inverse:

$$\ell^{-1}(\alpha) = t \quad \text{s.t.} \quad \int_{t_0}^{t} \| c'(s) \|_2 \, ds = \alpha \quad , \quad \begin{array}{l} 0 \le \alpha \le \ell(t_1) \\ t_0 \le t \le t_1 \end{array}$$

Define:

$$\tilde{c}(\alpha) = c(\ell^{-1}(\alpha)) \quad , \quad 0 \le \alpha \le \ell(t_1)$$

Note $\tilde{c}(\alpha) \in c(\cdot)$ - lies on the trace of c

Conversely, $\forall t \; \exists \alpha$ s.t. $\ell^{-1}(\alpha) = t$ so

$$\text{Trace}(c) = \text{Trace}(\tilde{c})$$

Also remember:

$$\int_{t_0}^{\ell^{-1}(\alpha)} \| c'(s) \|_2 \, ds = \alpha$$

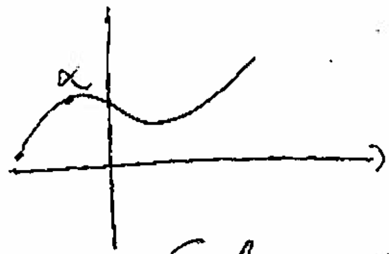
Differentiate both sides: w.r.t. α.

$$\| c'(\ell^{-1}(\alpha)) \|_2 = 1 \quad \forall \alpha$$

Or $\| \tilde{c}'(\alpha) \|_2 = 1$.

And $$\int_0^{\alpha} \| \tilde{c}'(s) \|_2 \, ds = \int_0^{\alpha} ds = \alpha$$

I.e. α is the length of $\tilde{c}$, between 0 and α.



Each point now gets a name α, which is the length of the curve between 0 and itself.

Arc-length parametrization $\tilde{c}$ exists for any regular curve c

Curvature $K(s)$ is defined as:

$$\boxed{\|\tilde{c}''(s)\|}$$

Purely geometric definition.

Extends to $\mathbb{R}^3$. Has very many nice properti

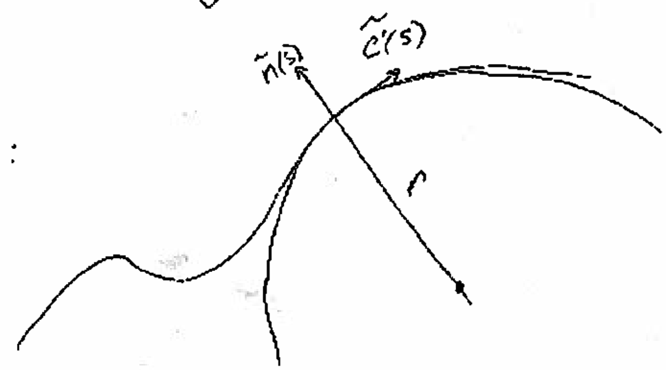
$\tilde{c}'(s)^T \tilde{c}'(s) = 1$ differentiate both sides w.r.t. s:

$\Rightarrow \quad \tilde{c}''(s)^T \tilde{c}'(s) = 0$ or

$\tilde{c}''(s)$ is perpendicular to $\tilde{c}'(s)$.

In the plane, the only vector perp. to $\tilde{c}'(s)$ is the normal. $\tilde{n}(s)$

Osculating circle:

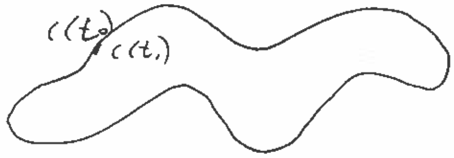


$r = \frac{1}{\|K(s)\|}$ or

$K(s) = \frac{1}{r} \Rightarrow \tilde{c}''(s) = \pm\frac{1}{r}\tilde{n}(s)$ depending on normal orientation.

Total Curvature

Suppose $c(t)$ is a closed regular curve parametrized by arc-length s.t. $c(t_0) = c(t_1)$ for some t_0, t_1

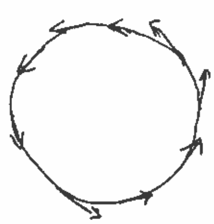


Then $\int_{t_0}^{t_1} \kappa(t)\, dt = 2\pi I$ for some integer I.

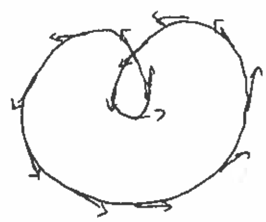
$$\int_{t_0}^{t_1} \kappa(t) dt = \int_{t_0}^{t_1} \theta'(t) \cdot \left(\frac{ds}{dt}\right)^{-1} dt = \int_{t_0}^{t_1} \theta'(t)\, dt$$

since $\dfrac{ds}{dt} = 1$ for arc-length parametrization.

$$= \theta(t_1) - \theta(t_0) = 2\pi I$$



$I = 1$



$I = 2$.