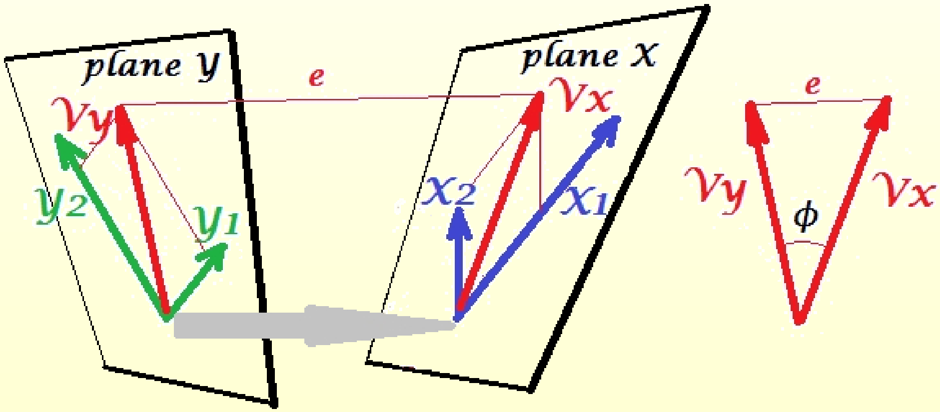
CS233: Geometric and Topological Data Analysis

Canonical Correlation Analysis (CCA)

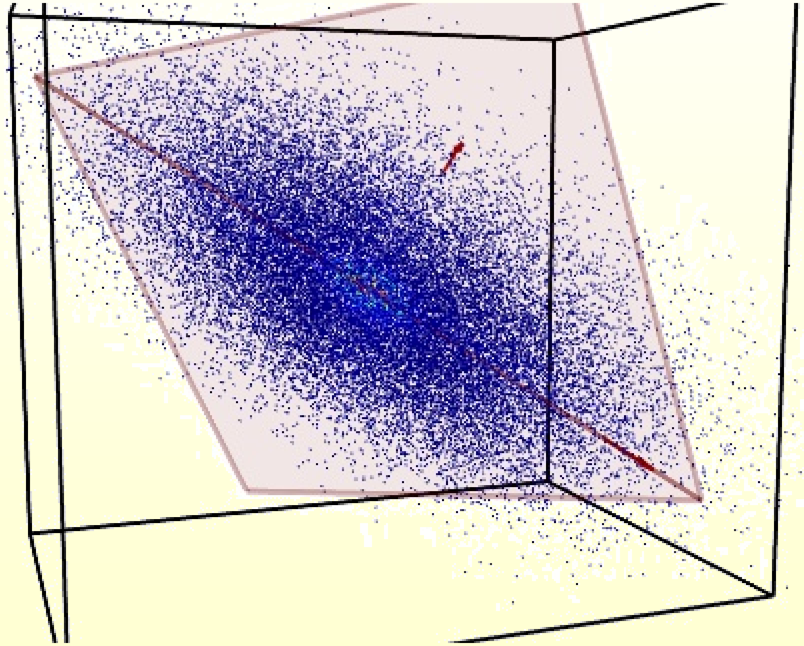
11 April 2018



Last Time: PCA

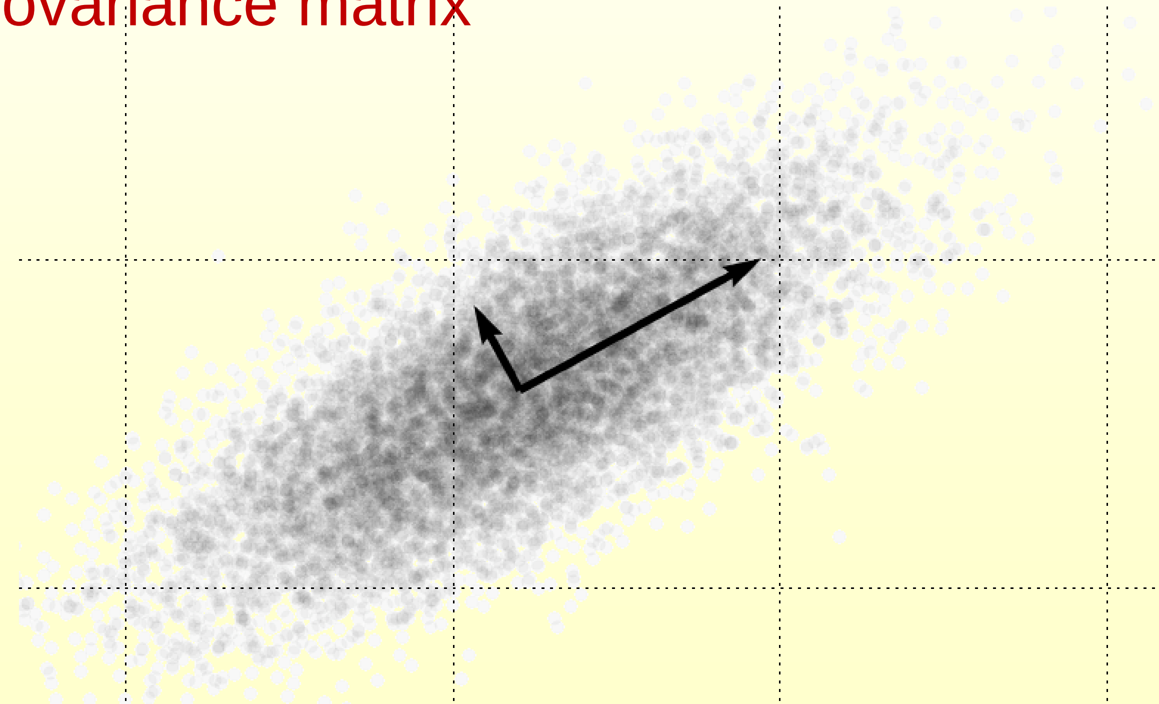
Principal Components Analysis (PCA)

- Introduced by Pearson (1901) and Hotelling (1933) to describe the variation in a set of multivariate data in terms of a set of uncorrelated variables.
- PCA looks for a single lower dimensional subspace that captures most of the variation in the data.
- Specifically, we aim to minimize the error introduced by projecting the data into this linear subspace.



Last time: PCA, KPCA

- Use spectral analysis of the covariance matrix *C* of the data
- For any integer *p*, the error-minimizing *p*-dimensional subspace is the one spanned by the first *p* eigenvectors of the covariance matrix

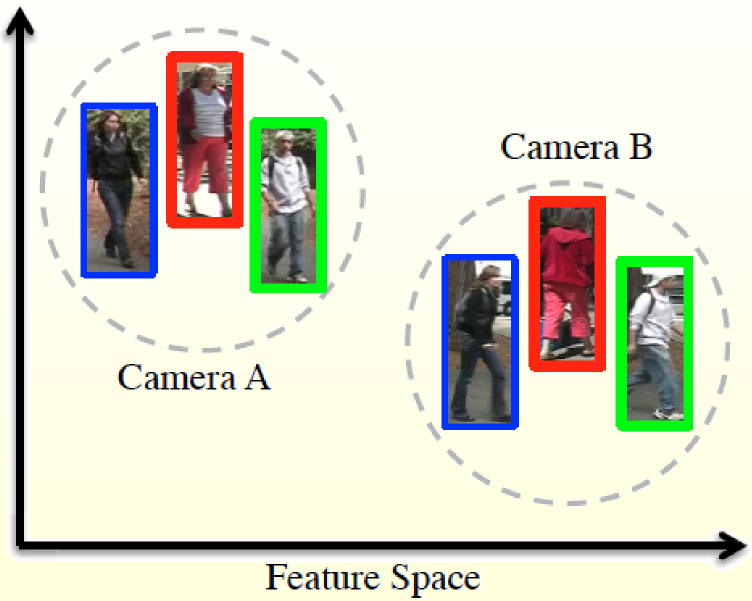


Kernel PCA (KPCA)

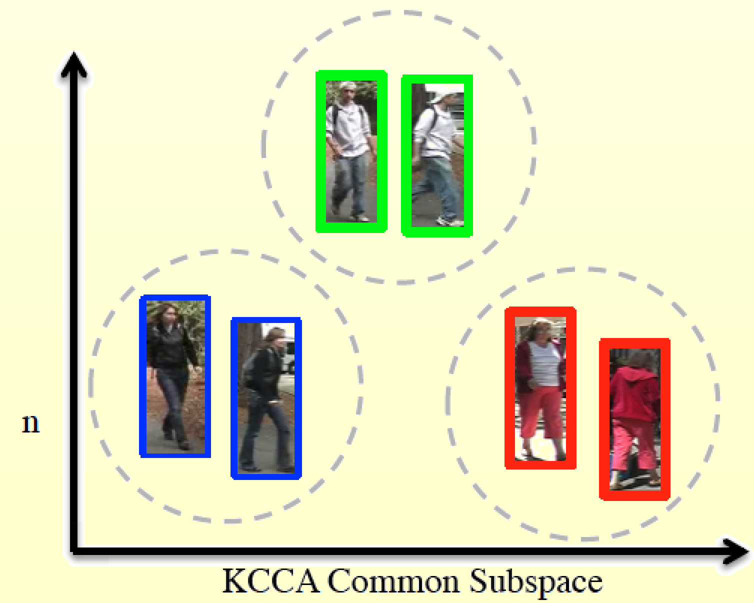
- Assumption behind PCA is that the data points **x** are well represented by a small-dimensional Euclidean subspace
- Often this assumption does not hold …
- However, it may still be possible that a non-linear transformation $\phi(\mathbf{x})$ "linearizes" the data -- then we can perform PCA in the space of $\phi(\mathbf{x})$
- Kernel PCA performs this "lifted" PCA; however, because of "kernel trick," it never computes the mapping $\phi(\mathbf{x})$ explicitly.

Shared Structure Across Different Data Sets

Different Views of the Same Data



Cluster by appearance similarity



Cluster by content similarity

Covariance and Correlation

Covariance and Correlation

- Pearson correlation coefficient between two random variables *X*, Y

$$\rho_{X,Y} = \mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_x)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$$

- Note that, by the Cauchy-Schwarz inequality

$$E[(X-\mu_x)(Y-\mu_Y)]^2 \le E[(X-\mu_x)^2]E[(Y-\mu_Y)^2]$$

so

$$-1 \le \rho_{x,Y} \le 1$$

$\mu_X = \bar{x}$
mean

σ_X
standard deviation

Covariance and Correlation

- Correlation measures a linear association between *X*, Y

$$\mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}}$$

- Note that, *X*, Y independent, then corr(*X*,*Y*) = 0
  - but the opposite is not true

    [U uniform in [0,2π], $X = \sin U$, $Y = \cos U$]

- High correlation not the same as causality

Empirical Correlation

- Empirical version, for *n* measurements x_i, y_i of *X* and *Y*

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n s_x s_y} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

$$r_{xy} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{n s_x s_y} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n\sum x_i^2 - (\sum x_i)^2}\sqrt{n\sum y_i^2 - (\sum y_i)^2}}$$

- We'll use centered versions, $\bar{x} = 0, \bar{y} = 0$

$$\frac{\sum x_i y_i}{\sqrt{\sum x_i^2}\sqrt{\sum y_i^2}} = \frac{< \boldsymbol{x}, \boldsymbol{y} >}{||\boldsymbol{x}||||\boldsymbol{y}||}$$

Canonical Correlations for Two Sets of Variates

Canonical Correlations

- Canonical correlation analysis seeks a pair of linear transformations, one for each of the sets of variables *X*, *Y*, such that when the set of variables is transformed, the corresponding coordinates are maximally correlated.

- Consider projections of *X* and *Y*

$$\boldsymbol{x} \rightarrow \quad < \boldsymbol{w}_x, \boldsymbol{x} >$$
$$\boldsymbol{y} \rightarrow \quad < \boldsymbol{w}_y, \boldsymbol{y} >$$

- So we get $\boldsymbol{S}_{x,\boldsymbol{w}_x} = (< \boldsymbol{w}_x, \boldsymbol{x}_1 >, < \boldsymbol{w}_x, \boldsymbol{x}_2 >, \ldots, < \boldsymbol{w}_x, \boldsymbol{x}_n >)$

$$\boldsymbol{S}_{y,\boldsymbol{w}_y} = (< \boldsymbol{w}_y, \boldsymbol{y}_1 >, < \boldsymbol{w}_y, \boldsymbol{y}_2 >, \ldots, < \boldsymbol{w}_y, \boldsymbol{y}_n >)$$

1st Canonical Correlation

- We choose w_x, w_y to maximize the correlation between these two vectors

$$\rho = (\rho_1 =) \quad \max_{\boldsymbol{w}_x, \boldsymbol{w}_y} corr(S_x \boldsymbol{w}_x, S_y \boldsymbol{w}_y) = \max_{\boldsymbol{w}_x, \boldsymbol{w}_y} \frac{< S_x \boldsymbol{w}_x, S_y \boldsymbol{w}_y >}{\|S_x \boldsymbol{w}_x\| \|S_y \boldsymbol{w}_y\|}$$

- Or, in empirical form, if we write $E[f(\boldsymbol{x}, \boldsymbol{y})] = \frac{1}{n} \sum_{i=1}^{n} f(\boldsymbol{x}_i, \boldsymbol{y}_i)$

We can re-write the correlation expression as

$$\rho = \max_{\boldsymbol{w}_x, \boldsymbol{w}_y} \frac{E[< \boldsymbol{w}_x, \boldsymbol{x} >< \boldsymbol{w}_y, \boldsymbol{y} >]}{\sqrt{E[< \boldsymbol{w}_x, \boldsymbol{x} >^2] E[< \boldsymbol{w}_y, \boldsymbol{y} >^2]}}$$

Covariance Formulation

- Can re-write as

$$\rho = \max_{\boldsymbol{w}_x, \boldsymbol{w}_y} \frac{E[\boldsymbol{w}_x^T \boldsymbol{x} \boldsymbol{y}^T \boldsymbol{w}_y]}{\sqrt{E[\boldsymbol{w}_x^T \boldsymbol{x} \boldsymbol{x}^T \boldsymbol{w}_{\boldsymbol{x}}] E[\boldsymbol{w}_y^T \boldsymbol{y} \boldsymbol{y}^T \boldsymbol{w}_{\boldsymbol{y}}]}}$$

- so that

$$\rho = \max_{\boldsymbol{w}_x, \boldsymbol{w}_y} \frac{\boldsymbol{w}_x^T E[\boldsymbol{x} \boldsymbol{y}^T] \boldsymbol{w}_y}{\sqrt{\boldsymbol{w}_x^T E[\boldsymbol{x} \boldsymbol{x}^T] \boldsymbol{w}_{\boldsymbol{x}} \boldsymbol{w}_y^T E[\boldsymbol{y} \boldsymbol{y}^T] \boldsymbol{w}_{\boldsymbol{y}}}}$$

- If we now write

$$C(\boldsymbol{x}, \boldsymbol{y}) = E\left[\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}^T\right] = \begin{bmatrix} C_{\boldsymbol{x}\boldsymbol{x}} & C_{\boldsymbol{x}\boldsymbol{y}} \\ C_{\boldsymbol{y}\boldsymbol{x}} & C_{\boldsymbol{y}\boldsymbol{y}} \end{bmatrix} = C$$

- we get

$$\rho = \max_{\boldsymbol{w}_x, \boldsymbol{w}_y} \frac{\boldsymbol{w}_x^T C_{\boldsymbol{x}\boldsymbol{y}} \boldsymbol{w}_y}{\sqrt{\boldsymbol{w}_x^T C_{\boldsymbol{x}\boldsymbol{x}} \boldsymbol{w}_{\boldsymbol{x}} \boldsymbol{w}_y^T C_{\boldsymbol{y}\boldsymbol{y}} \boldsymbol{w}_{\boldsymbol{y}}}}$$

Solution by Eigenanalysis

- Note that the he expression $$\frac{\boldsymbol{w}_x^T C_{\boldsymbol{xy}} \boldsymbol{w}_y}{\sqrt{\boldsymbol{w}_x^T C_{\boldsymbol{xx}} \boldsymbol{w}_{\boldsymbol{x}} \boldsymbol{w}_y^T C_{\boldsymbol{yy}} \boldsymbol{w}_{\boldsymbol{y}}}}$$

  is invariant to re-scalings of w_x or w_y

- We can therefore solve the optimization problem

  $$\max_{\boldsymbol{w}_x, \boldsymbol{w}_y} \quad \boldsymbol{w}_x^T C_{\boldsymbol{xy}} \boldsymbol{w}_y$$

  subject to the constraints

  $$\boldsymbol{w}_x^T C_{\boldsymbol{xx}} \boldsymbol{w}_x = 1 \text{ and } \boldsymbol{w}_y^T C_{\boldsymbol{yy}} \boldsymbol{w}_y = 1.$$

Lagrange Multipliers

$$L(\lambda, \boldsymbol{w}_x, \boldsymbol{w}_y) = \boldsymbol{w}_x^T C_{\boldsymbol{xy}} \boldsymbol{w}_y - \frac{\lambda_x}{2}(\boldsymbol{w}_x^T C_{\boldsymbol{xx}} \boldsymbol{w}_x - 1) - \frac{\lambda_y}{2}(\boldsymbol{w}_y^T C_{\boldsymbol{yy}} \boldsymbol{w}_y - 1)$$

- Setting derivatives of *L* to 0 w.r.t. w_x, w_y we get (after some manipulation)

$$\frac{\partial L}{\partial \boldsymbol{w}_x} = C_{\boldsymbol{xy}} \boldsymbol{w}_y - \lambda_x C_{\boldsymbol{xx}} \boldsymbol{w}_x = 0 \quad \frac{\partial L}{\partial \boldsymbol{w}_y} = C_{\boldsymbol{yx}} \boldsymbol{w}_x - \lambda_y C_{\boldsymbol{yy}} \boldsymbol{w}_y = 0$$

multiply by w_x^T, multiply by w_y^T

subtract

$$0 = \boldsymbol{w}_x^T C_{xy} \boldsymbol{w}_y - \boldsymbol{w}_x^T \lambda_x C_{xx} \boldsymbol{w}_y - \boldsymbol{w}_y^T C_{yx} \boldsymbol{w}_x + \boldsymbol{w}_y^T \lambda_y C_{yy} \boldsymbol{w}_y$$

$$= \boldsymbol{w}_y^T \lambda_y C_{yy} \boldsymbol{w}_y - \boldsymbol{w}_x^T \lambda_x C_{xx} \boldsymbol{w}_y$$

Lagrange Multipliers

- Leading to $\lambda_x \boldsymbol{w}_x^T C_{\boldsymbol{xx}} \boldsymbol{w}_x = \lambda_y \boldsymbol{w}_y^T C_{\boldsymbol{yy}} \boldsymbol{w}_y \quad \text{and} \quad \lambda_x = \lambda_y (= \lambda)$

- Assuming C_{yy} is invertible, we can use the second derivative constraint above to get

$$\boldsymbol{w}_y = \frac{C_{\boldsymbol{yy}}^{-1} C_{\boldsymbol{yx}} \boldsymbol{w}_x}{\lambda}$$

- so now from the first derivate constraint we get

$$\frac{\partial L}{\partial \boldsymbol{w}_x} = C_{\boldsymbol{xy}} \boldsymbol{w}_y - \lambda_x C_{\boldsymbol{xx}} \boldsymbol{w}_x = 0$$

- or

Eigenvalue Problem

- A generalized eigenvalue problem

$$C_{\boldsymbol{xy}} C_{\boldsymbol{yy}}^{-1} C_{\boldsymbol{yx}} \boldsymbol{w}_x = \lambda^2 C_{\boldsymbol{xx}} \boldsymbol{w}_x$$

$$A\boldsymbol{x} = \lambda B\boldsymbol{x}$$

- Can symmetrically also get

$$C_{\boldsymbol{yx}} C_{\boldsymbol{xx}}^{-1} C_{\boldsymbol{xy}} \boldsymbol{w}_y = \lambda^2 C_{\boldsymbol{yy}} \boldsymbol{w}_y$$

- On can go back and forth between w_x and w_y

$$\boldsymbol{w}_y = \frac{C_{\boldsymbol{yy}}^{-1} C_{\boldsymbol{yx}} \boldsymbol{w}_x}{\lambda}$$

Solving the Eigenvalue Problem

- If C_{xx} is invertible, then can reduce to a standard symmetric eigenvalue problem

$$C_{\boldsymbol{xx}}^{-1} C_{\boldsymbol{xy}} C_{\boldsymbol{yy}}^{-1} C_{\boldsymbol{yx}} \boldsymbol{w}_x = \lambda^2 \boldsymbol{w}_x$$

- Numerically, all these inversions and multiplications lose precision
- Alternate approach:
  - C_{xx} and C_{xx} are symmetric positive definite; use complete Cholesky decomposition – R_{xx} lower triangular so that $C_{\boldsymbol{xx}} = R_{\boldsymbol{xx}} R_{\boldsymbol{xx}}^T$
  - Now let $\boldsymbol{u}_x = R_{\boldsymbol{xx}}^T \boldsymbol{w}_x$
  - Can re-write

$$C_{\boldsymbol{xy}} C_{\boldsymbol{yy}}^{-1} C_{\boldsymbol{yx}} R_{\boldsymbol{xx}}^{-T} \boldsymbol{u}_x = \lambda^2 R_{\boldsymbol{xx}} \boldsymbol{u}_x$$

$$R_{\boldsymbol{xx}}^{-1} C_{\boldsymbol{xy}} C_{\boldsymbol{yy}}^{-1} C_{\boldsymbol{yx}} R_{\boldsymbol{xx}}^{-T} \boldsymbol{u}_x = \lambda^2 \boldsymbol{u}_x$$

For more on the numerics, see the Weenik article

Centered Variables, CCA vs PCA

- For centered variates, the covariance $C_{\boldsymbol{xy}} = \boldsymbol{x}^T \boldsymbol{y}$
- Canonical correlation analysis attempts to answer the question "which directions accounts for most of the covariance between the two data sets?" The goal is to find directions w_x, w_y so as to maximize

$$\boldsymbol{w}_x^T C_{\boldsymbol{xy}} \boldsymbol{w}_y = (\boldsymbol{x}\boldsymbol{w}_x)^T (\boldsymbol{y}\boldsymbol{w}_y)$$

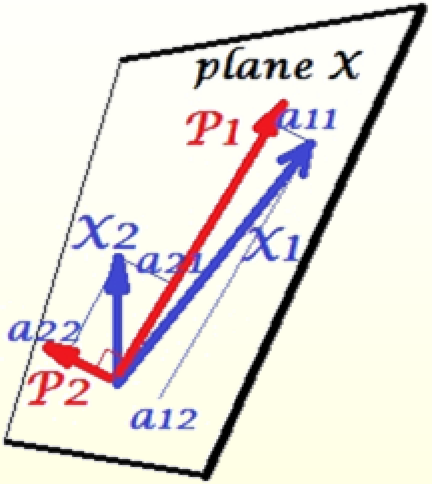
- subject to $\|\boldsymbol{x}\boldsymbol{w}_x\| = 1, \|\boldsymbol{y}\boldsymbol{w}_y\| = 1$

- In PCA we have a single variate and seek the direction that "maximizes the variance in the data"

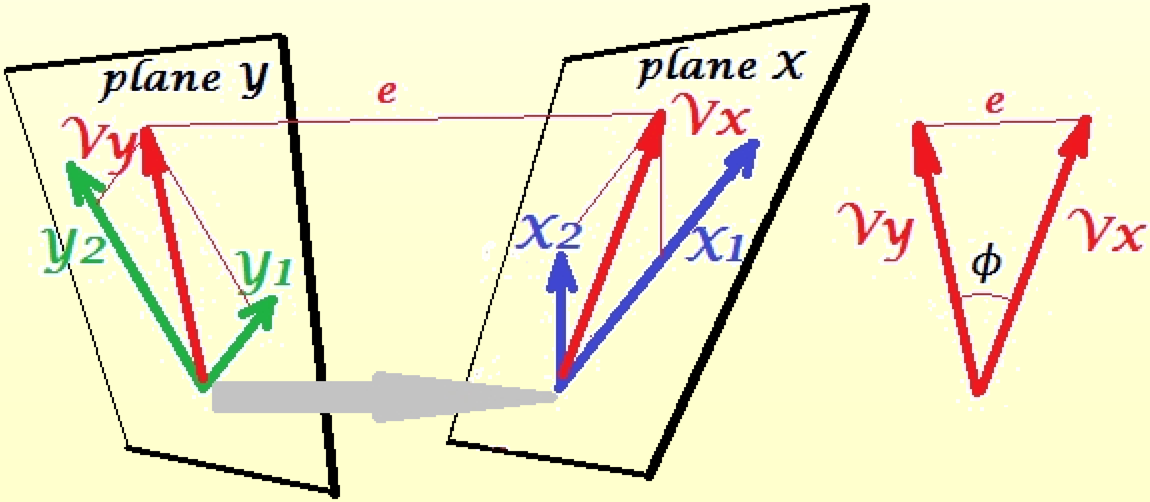
$$\boldsymbol{w}_x^T C_{\boldsymbol{xx}} \boldsymbol{w}_x = (\boldsymbol{x}\boldsymbol{w}_x)^T (\boldsymbol{x}\boldsymbol{w}_x)$$

- subject to $\|\boldsymbol{w}_x\| = 1$

CCA vs PCA



PCA



CCA

Example 1

CCA Example 1: Scores Data

Example: $n = 88$ students took tests in each of 5 subjects: mechanics, vectors, algebra, analysis, statistics. (From Mardia et al. (1979) "Multivariate analysis".) Each test is out of 100 points

The tests on mechanics, vectors were closed book and those on algebra, analysis, statistics were open book. There's clearly some correlation between these two sets of scores:

	alg	ana	sta
mec	0.547	0.409	0.389
vec	0.610	0.485	0.436

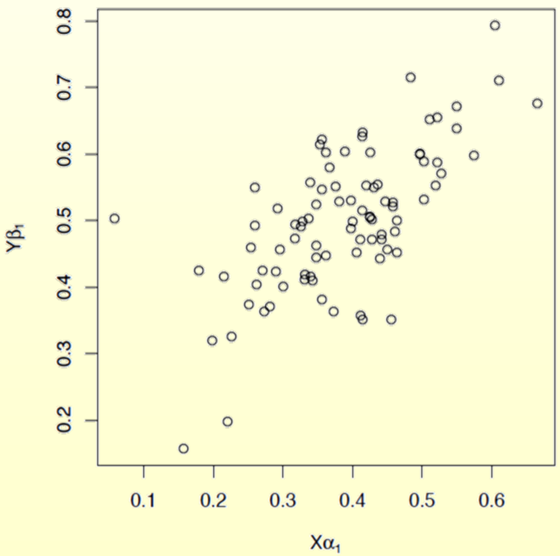
Canonical correlation analysis attempts to explain this phenomenon using the variables in each set jointly. Here X contains the closed book test scores and Y contains the open book test scores, so $X \in \mathbb{R}^{88\times 2}$ and $Y \in \mathbb{R}^{88\times 3}$

CCA Example 1: Scores Data

The first canonical directions (multiplied by 10^3):

$$\alpha_1 = \begin{pmatrix} 2.770 \\ 5.517 \end{pmatrix} \begin{matrix} \text{mec} \\ \text{vec} \end{matrix}, \quad \beta_1 = \begin{pmatrix} 8.782 \\ 0.860 \\ 0.370 \end{pmatrix} \begin{matrix} \text{alg} \\ \text{ana} \\ \text{sta} \end{matrix}$$

The first canonical correlation is $\rho_1 = 0.663$, and the variates:



Higher Order CCA

Higher-Order Canonical Correlates

- We defined the 1st canonical correlation ρ_1 through the projection vectors / directions

$$\boldsymbol{w}_x = \boldsymbol{w}_x^{(1)} \text{ and } \boldsymbol{w}_y = \boldsymbol{w}_y^{(1)}$$

- Given the first *k*-1 directions, the *k*-th canonical correlation is defined via vectors $\boldsymbol{w}_x^{(k)}$ and $\boldsymbol{w}_y^{(k)}$, so that we maximize

$$\max \quad (\boldsymbol{x}\boldsymbol{w}_x^{(k)})^T(\boldsymbol{y}\boldsymbol{w}_y^{(k)})$$

- but under orthogonality constraints to the previous directions

$$\|\boldsymbol{x}\boldsymbol{w}_x^{(k)}\| = 1, \|\boldsymbol{y}\boldsymbol{w}_y^{(k)}\| = 1$$

$$(\boldsymbol{x}\boldsymbol{w}_x^{(k)})^T(\boldsymbol{x}\boldsymbol{w}_x^{(j)}) = 0, j = 1, 2, \ldots, k-1$$

$$(\boldsymbol{y}\boldsymbol{w}_y^{(k)})^T(\boldsymbol{y}\boldsymbol{w}_y^{(j)}) = 0, j = 1, 2, \ldots, k-1$$

Other CCA Formulations, Computation

Equivalent CCA Formulations

$$\max_{\mathbf{a}_i,\mathbf{b}_i} \Sigma_i \mathbf{a}_i^T \mathbf{X}\mathbf{Y}^T \mathbf{b}_i \quad \text{s.t.} \quad \mathbf{a}_i^T \mathbf{X}\mathbf{X}^T \mathbf{a}_j = \delta_{i=j} \forall j \le i, \quad \mathbf{b}_i^T \mathbf{Y}\mathbf{Y}^T \mathbf{b}_j = \delta_{i=j} \forall j \le i$$

⟺

$$\max_{\mathbf{A},\mathbf{B}} \ \text{trace}\left(\mathbf{A}^T \mathbf{X}\mathbf{Y}^T \mathbf{B}\right) \quad \text{s.t.} \quad \mathbf{A}^T \mathbf{X}\mathbf{X}^T \mathbf{A} = \mathbf{I}_d, \quad \mathbf{B}^T \mathbf{Y}\mathbf{Y}^T \mathbf{B} = \mathbf{I}_d$$

⟺

$$\min_{\mathbf{A},\mathbf{B}} \ \frac{1}{2} \left\| \mathbf{A}^T \mathbf{X} - \mathbf{B}^T \mathbf{Y} \right\|_F^2 \quad \text{s.t.} \quad \mathbf{A}^T \mathbf{X}\mathbf{X}^T \mathbf{A} = \mathbf{I}_d, \quad \mathbf{B}^T \mathbf{Y}\mathbf{Y}^T \mathbf{B} = \mathbf{I}_d$$

⟺

$$\begin{bmatrix} \mathbf{0} & \mathbf{X}\mathbf{Y}^T \\ \mathbf{Y}\mathbf{X}^T & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{pmatrix} = \rho_i \begin{bmatrix} \mathbf{X}\mathbf{X}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}\mathbf{Y}^T \end{bmatrix} \begin{pmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{pmatrix}$$

Principal Angles Between Subspaces

- Goal: capture geometric configuration of two subspace with few and intuitive numbers
- Principal angles intuition
  - Measure angles between ‘most similar’ directions within subspaces
  - Capture relative ‘orientation’ of two subspaces
  - Recursive definitions with decreasing similarity
- Consider subspaces spanned by the cols of *X* and *Y*

- Definition

$$\cos\theta_i = \max_{\mathbf{u}_i \in \mathcal{Y}_1} \max_{\mathbf{v}_i \in \mathcal{Y}_2} \mathbf{u}_i^T \mathbf{v}_i$$

$$\text{s.t. } \|\mathbf{u}_i\|_2 = \|\mathbf{v}_i\|_2 = 1$$

$$\forall j < i : \mathbf{u}_i^T \mathbf{u}_j = \mathbf{v}_i^T \mathbf{v}_j = 0$$

$\Longleftrightarrow$

$$\max \ \text{trace}_{\boldsymbol{A},\boldsymbol{B}}(\boldsymbol{A}^T \boldsymbol{X}^T \boldsymbol{Y} \boldsymbol{B})$$

such that

$$\boldsymbol{A}, \boldsymbol{B} \in \mathcal{O}_p$$

Rotate subspaces to maximally ‘align’ them

CCA and Principal Angles

- CCA between RVs in column form → principal angles of row spaces spanned by data matrices

- For *i* = 1, 2, … $\rho_i = \cos\theta_i$

- Note 1: The number of canonical directions/variates is

$$r = \min\{\text{rank}(\boldsymbol{X}), \text{rank}(\boldsymbol{Y})\}$$

- Note 2:
  - If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are orthogonal ($\boldsymbol{X}^T\boldsymbol{Y} = 0$) then all principal angles are 90^o and the corresponding canonical correlations are 0
  - If $\boldsymbol{X}$ and $\boldsymbol{Y}$ intersect in a *d*-dimensional subspace, then the first *d* principal angles are 0

Sphering/Whitening

For any symmetric invertible matrix $A \in \mathbb{R}^{n \times n}$, there is a matrix $A^{1/2} \in \mathbb{R}^{n \times n}$, called the (symmetric) square root of A, such that $A^{1/2}A^{1/2} = A$

We write the inverse of $A^{1/2}$ as $A^{-1/2}$. Note $A^{-1/2}AA^{-1/2} = I$. (Why?)

Given centered matrices $X \in \mathbb{R}^{n \times p}$ and $Y \in \mathbb{R}^{n \times q}$,[2] we define $V_X = X^T X \in \mathbb{R}^{p \times p}$ and $V_Y = Y^T Y \in \mathbb{R}^{q \times q}$. Then

$$\tilde{X} = XV_X^{-1/2} \in \mathbb{R}^{n \times p} \text{ and } \tilde{Y} = YV_Y^{-1/2} \in \mathbb{R}^{n \times q}$$

are called the sphered versions of X and Y.[3] Note that the sample covariance of $\tilde{X}$ and $\tilde{Y}$ is

$$\mathrm{cov}(\tilde{X}) = I/n \text{ and } \mathrm{cov}(\tilde{Y}) = I/n$$

Transformed Problem

As suggested by the previous slide, we will take $\tilde{X} = XV_X^{-1/2}$ and $\tilde{Y} = YV_Y^{-1/2}$, and we'll solve the problem

$$\tilde{\alpha}_1, \tilde{\beta}_1 = \underset{\|\tilde{X}\tilde{\alpha}\|_2=1,\ \|\tilde{Y}\tilde{\beta}\|_2=1}{\operatorname{argmax}} (\tilde{X}\tilde{\alpha})^T(\tilde{Y}\tilde{\beta})$$

Recall that then $\alpha_1 = V_X^{-1/2}\tilde{\alpha}_1$ and $\beta_1 = V_Y^{-1/2}\tilde{\beta}_1$.

So why is this simpler? Note that the constraint says

$$1 = (\tilde{X}\tilde{\alpha})^T(\tilde{X}\tilde{\alpha}) = \tilde{\alpha}^T V_X^{-1/2} X^T X V_X^{-1/2}\tilde{\alpha} = \tilde{\alpha}^T\tilde{\alpha}$$

i.e., $\|\tilde{\alpha}\|_2 = 1$. Similarly, $\|\tilde{\beta}\|_2 = 1$. Hence our problem can be rewritten as:

$$\tilde{\alpha}_1, \tilde{\beta}_1 = \underset{\|\tilde{\alpha}\|_2=1,\ \|\tilde{\beta}\|_2=1}{\operatorname{argmax}} \tilde{\alpha}^T M \tilde{\beta}$$

where $M = \tilde{X}^T\tilde{Y} = V_X^{-1/2} X^T Y V_Y^{-1/2} \in \mathbb{R}^{p\times q}$. The same is true for further directions

SVD to the Rescue

Now comes the singular value decomposition to the rescue (again!). Let $r = \min\{p, q\}$. Then we can decompose

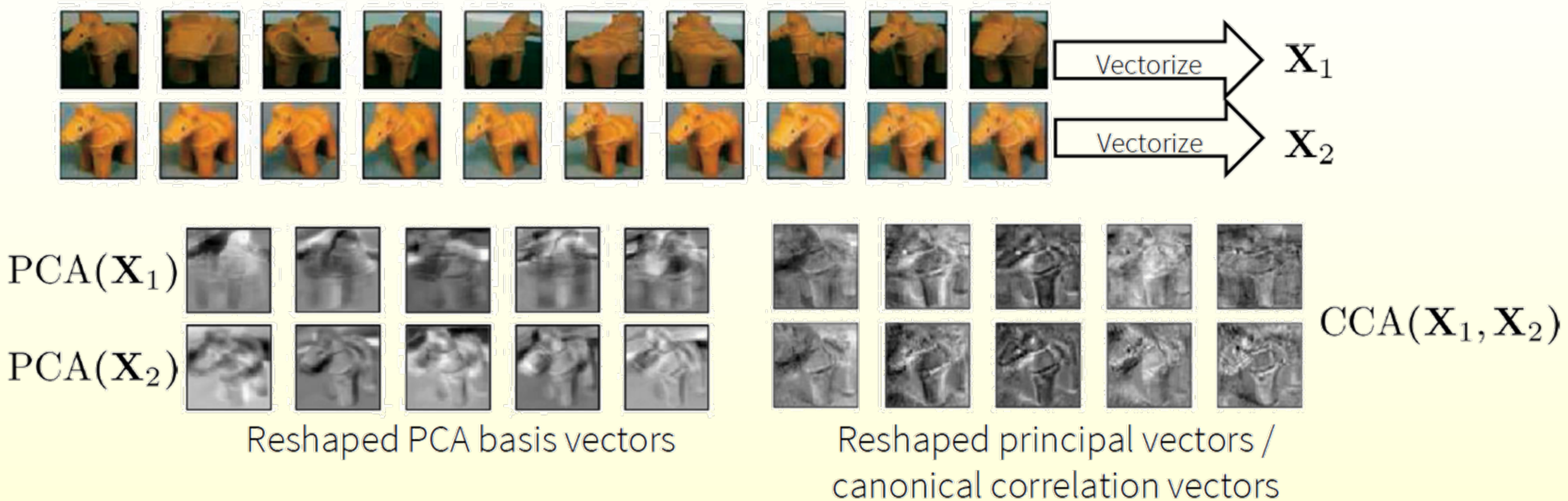
$$M = UDV^T$$

where $U \in \mathbb{R}^{p\times r}$, $V \in \mathbb{R}^{q\times r}$ have orthonormal columns, and $D = \mathrm{diag}(d_1, \ldots d_r) \in \mathbb{R}^{r\times r}$ with $d_1 \geq \ldots \geq d_r \geq 0$. Further:

- The transformed canonical directions $\tilde{\alpha}_1, \ldots \tilde{\alpha}_r \in \mathbb{R}^p$ and $\tilde{\beta}_1, \ldots \tilde{\beta}_r \in \mathbb{R}^q$ are the columns of U and V, respectively
- The canonical directions $\alpha_1, \ldots \alpha_r \in \mathbb{R}^p$ and $\beta_1, \ldots \beta_r \in \mathbb{R}^q$ are the columns of $V_X^{-1/2}U$ and $V_Y^{-1/2}V$, respectively;
- the canonical variates $X\alpha_1, \ldots X\alpha_r \in \mathbb{R}^n$ and $Y\beta_1, \ldots Y\beta_r \in \mathbb{R}^n$ are the columns of $XV_X^{-1/2}U \in \mathbb{R}^{n\times r}$ and $YV_Y^{-1/2}V \in \mathbb{R}^{n\times r}$, respectively
- The canonical correlations $\rho_1 \geq \ldots \geq \rho_r$ are equal to $d_1 \geq \cdots \geq d_r$, the diagonal entries of D

More Examples

CCA Example 2: Object Recognition

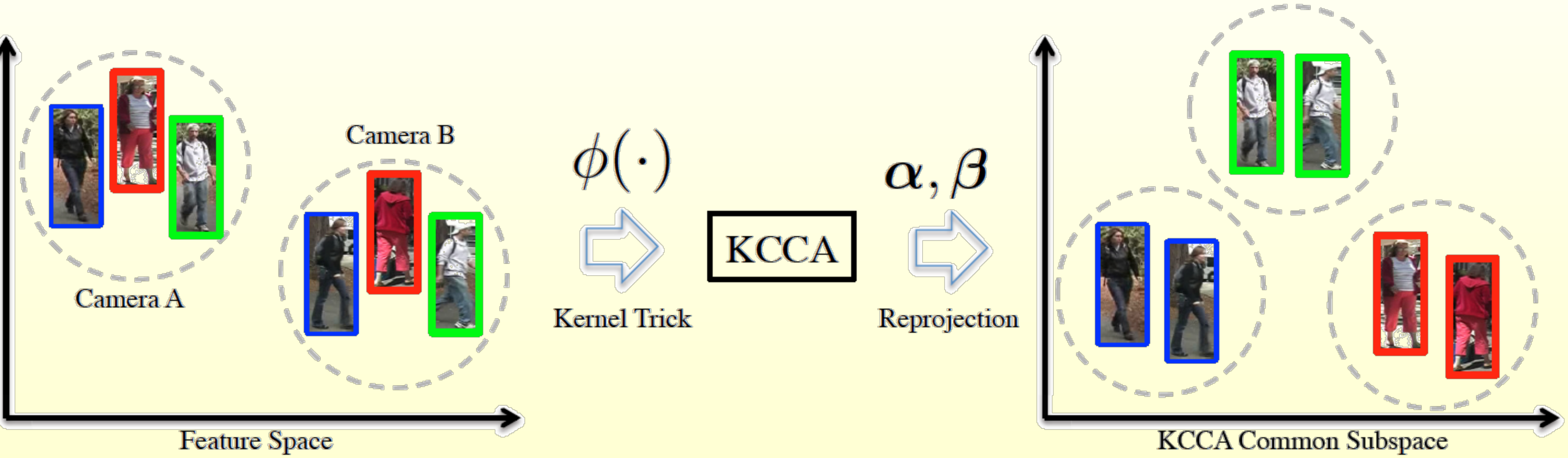


- "Discriminative Learning and Recognition of Image Set Classes Using Canonical Correlation" Kim, Kittler, Cipolla [PAMI 07]
- Idea: Find low-dimensional subspace embedding s.t.
  - within-class CCA is maximized
  - Between-class CCA is minimized

CCA Example 3: KCCA for Matching People

- Lisanti, Giuseppe, Iacopo Masi, and Alberto Del Bimbo. "Matching people across camera views using kernel canonical correlation analysis." Proceedings of the International Conference on Distributed Smart Cameras. ACM, 2014.

Common Subspace Through KCCA



Homework Policies

CS233 Homework Policies

- Collaboration groups allows, of up to three students
- Two free late periods available – after that 20% penanty for each late periof, but no more than two (100% penalty after that)
- All homeworks must be submitted by the last day of the course, Wed, June 6

The 1st Assignment

Assignment 1: PCA + CCA

- Problem 1 (PCA): Face Reconstruction/Recognition: Eigenfaces

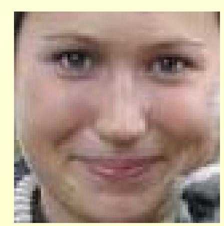
- Problem 2 (CCA): Word Image Recognition

PCA

Face Recognition

- Photo organization
- Surveillance
- …
- Has to be coupled with face detection

Detection

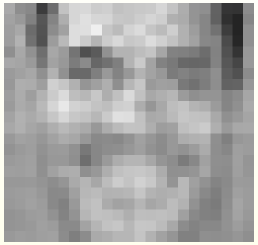


Recognition

“Sally”

Recognition: Embed!

- Treat image pixels as a long vector



$$\mathbf{x}$$

- Face recognition by nearest neighbor

$$\mathbf{y}_1 \ldots \mathbf{y}_n$$

$$k = \underset{k}{\operatorname{argmin}} \left\| \mathbf{y}_k{}^T - \mathbf{x} \right\|$$

Eigenfaces (PCA on Face Images)

- Compute covariance matrix of face images
- Derive the principal components ("eigenfaces")
  - K eigenvectors with largest eigenvalues
- Represent all face images in the dataset as linear combinations of eigenfaces
  - Perform nearest neighbor on these coefficients

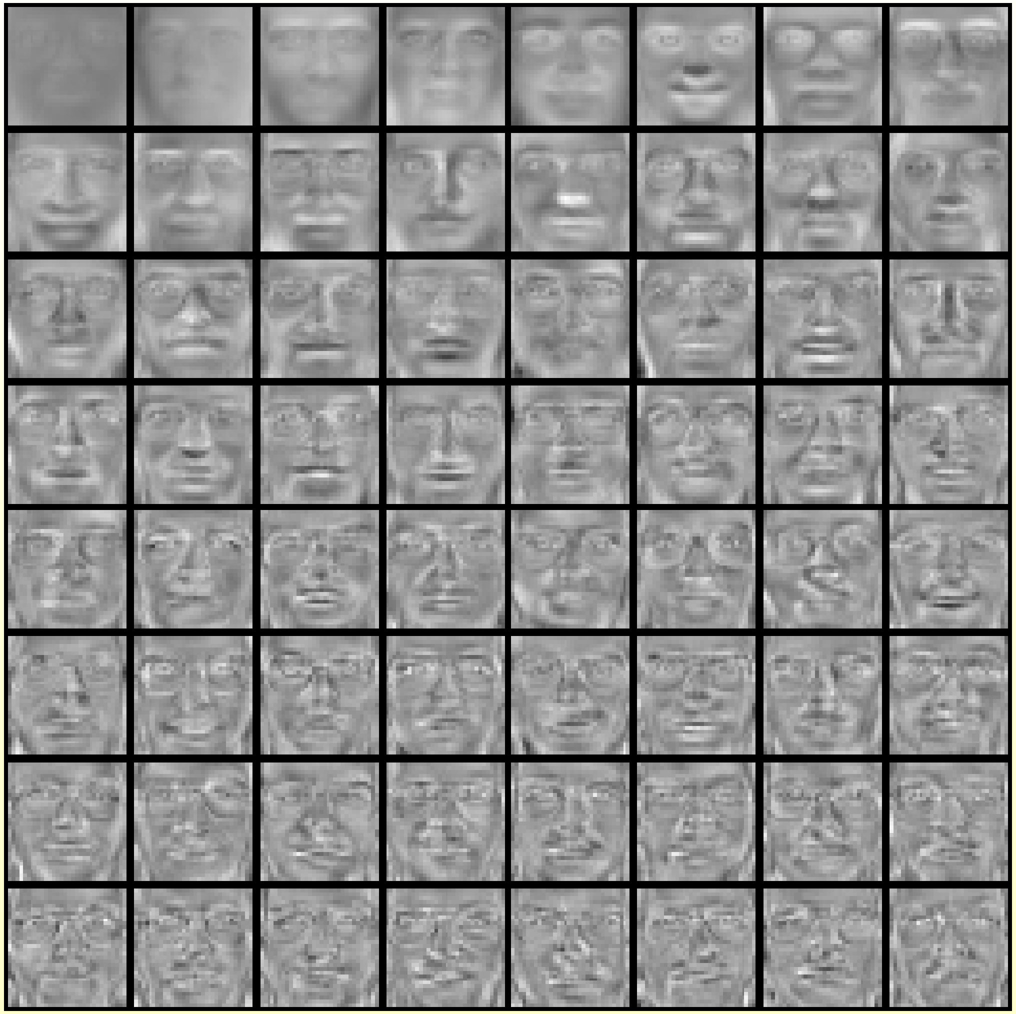
M. Turk and A. Pentland, Face Recognition using Eigenfaces, CVPR 1991

Training Images

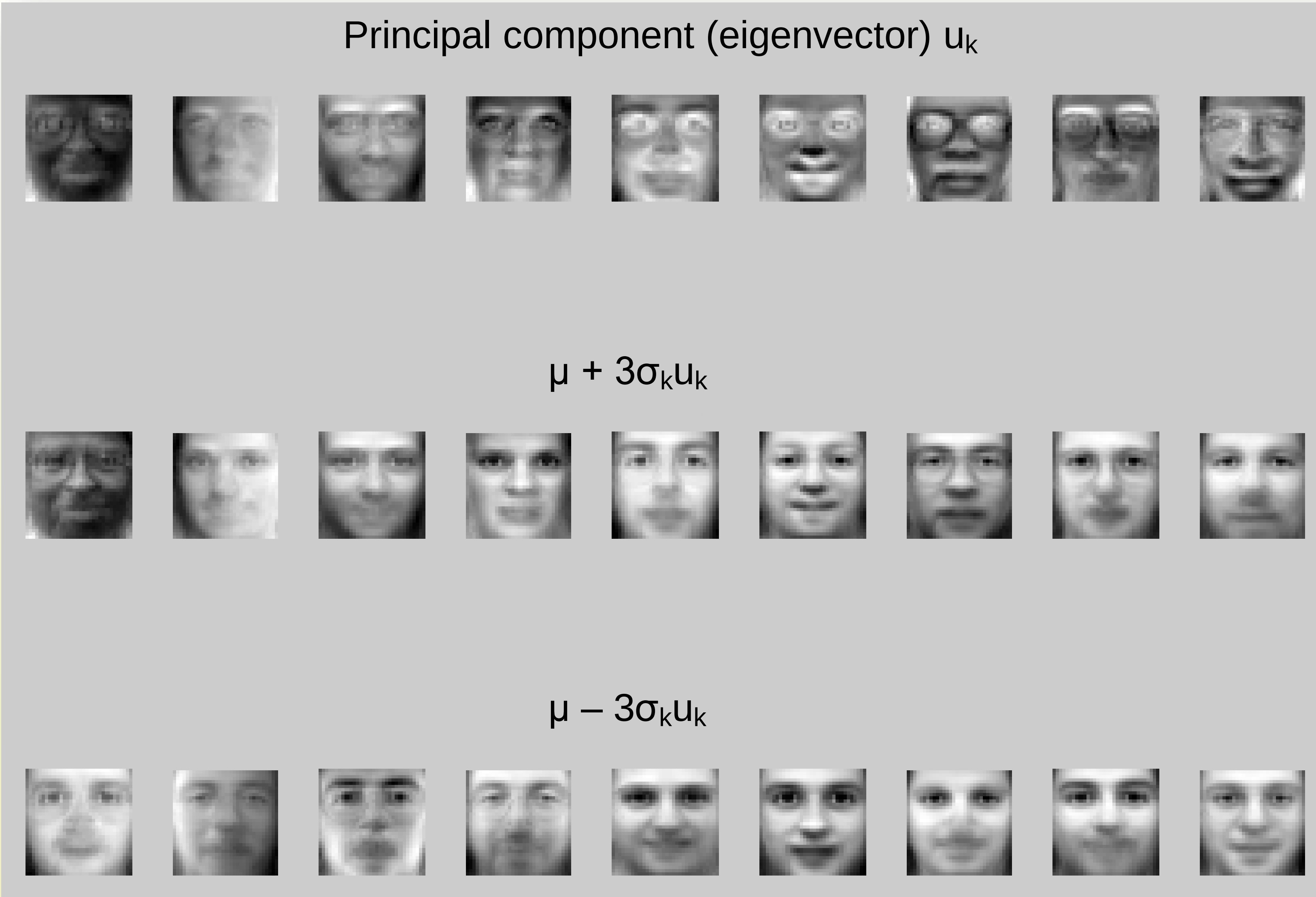
Eigenfaces

Mean: μ

Top eigenvectors: $u_1, \ldots u_k$



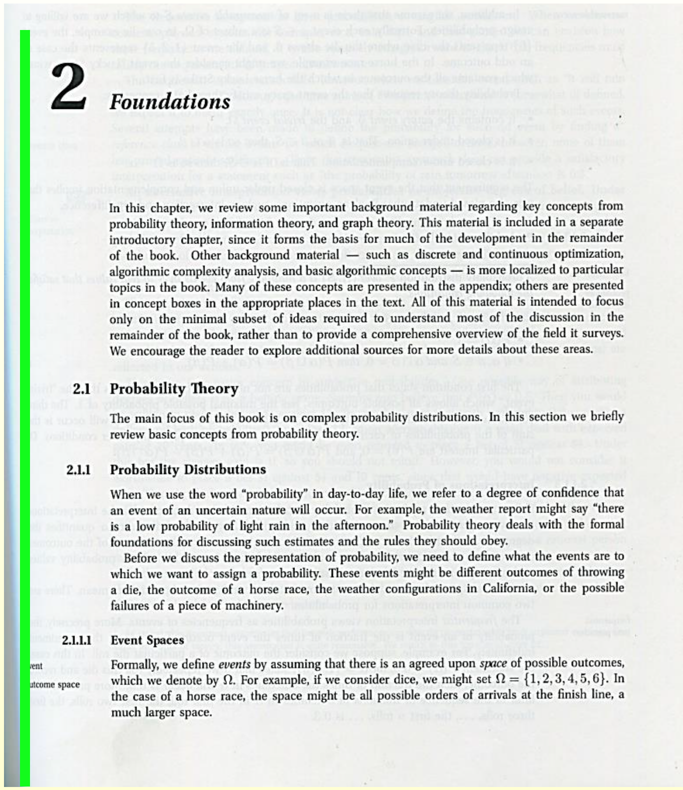
Eigenface Visualization



CCA

Camera Text Recognition

Scanned



2 *Foundations*

In this chapter, we review some important background material regarding key concepts from probability theory, information theory, and graph theory. This material is included in a separate introductory chapter, since it forms the basis for much of the development in the remainder of the book. Other background material — such as discrete and continuous optimization, algorithmic complexity analysis, and basic algorithmic concepts — is more localized to particular topics in the book. Many of these concepts are presented in the appendix; others are presented in concept boxes in the appropriate places in the text. All of this material is intended to focus only on the minimal subset of ideas required to understand most of the discussion in the remainder of the book, rather than to provide a comprehensive overview of the field it surveys. We encourage the reader to explore additional sources for more details about these areas.

2.1 **Probability Theory**

The main focus of this book is on complex probability distributions. In this section we briefly review basic concepts from probability theory.

2.1.1 **Probability Distributions**

When we use the word "probability" in day-to-day life, we refer to a degree of confidence that an event of an uncertain nature will occur. For example, the weather report might say "there is a low probability of light rain in the afternoon." Probability theory deals with the formal foundations for discussing such estimates and the rules they should obey.

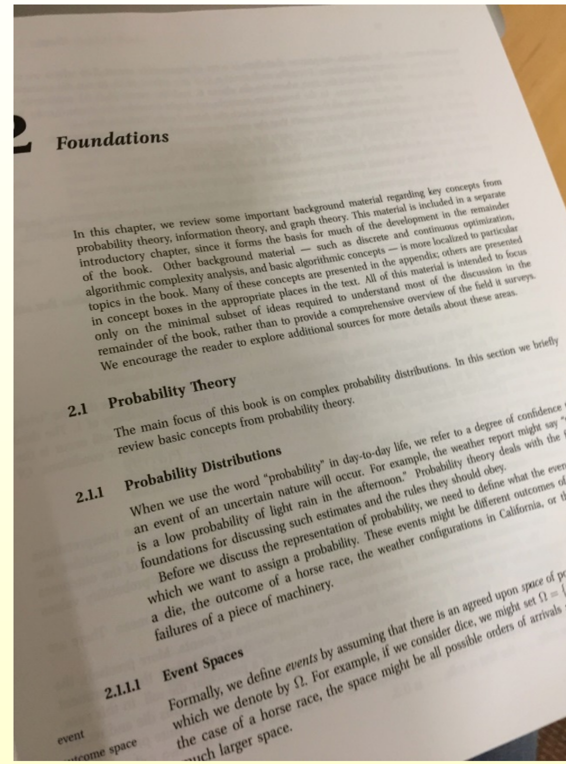
Before we discuss the representation of probability, we need to define what the events are to which we want to assign a probability. These events might be different outcomes of throwing a die, the outcome of a horse race, the weather configurations in California, or the possible failures of a piece of machinery.

2.1.1.1 **Event Spaces**

Formally, we define *events* by assuming that there is an agreed upon *space* of possible outcomes, which we denote by Ω. For example, if we consider dice, we might set $\Omega = \{1, 2, 3, 4, 5, 6\}$. In the case of a horse race, the space might be all possible orders of arrivals at the finish line, a much larger space.

In this chapter, we review some important background material regarding key concepts from probability theory, information theory, and graph theory…

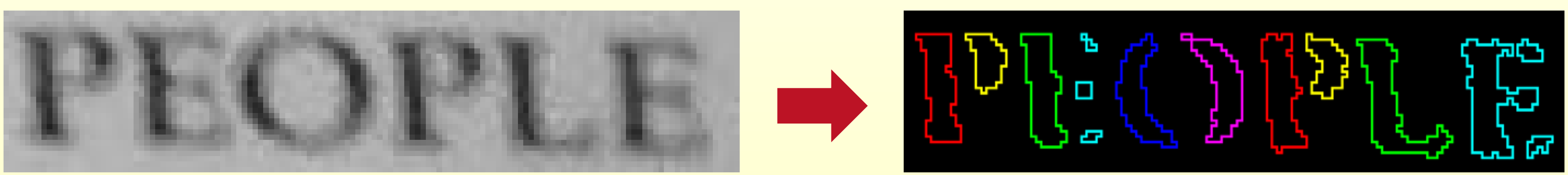
Camera



1n this chapter we FBViEW SUIHC important background mated-M regarding WY cnncepn 5mm probability theory information theory and Efaph theory…

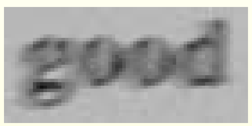
Why OCR Fails

- OCR reads text character by character, but character segmentation can be challenging.
  - View perspective distortions
  - Uneven lighting
  - Presence of noise and blur



Word Recognition via Image Retrieval

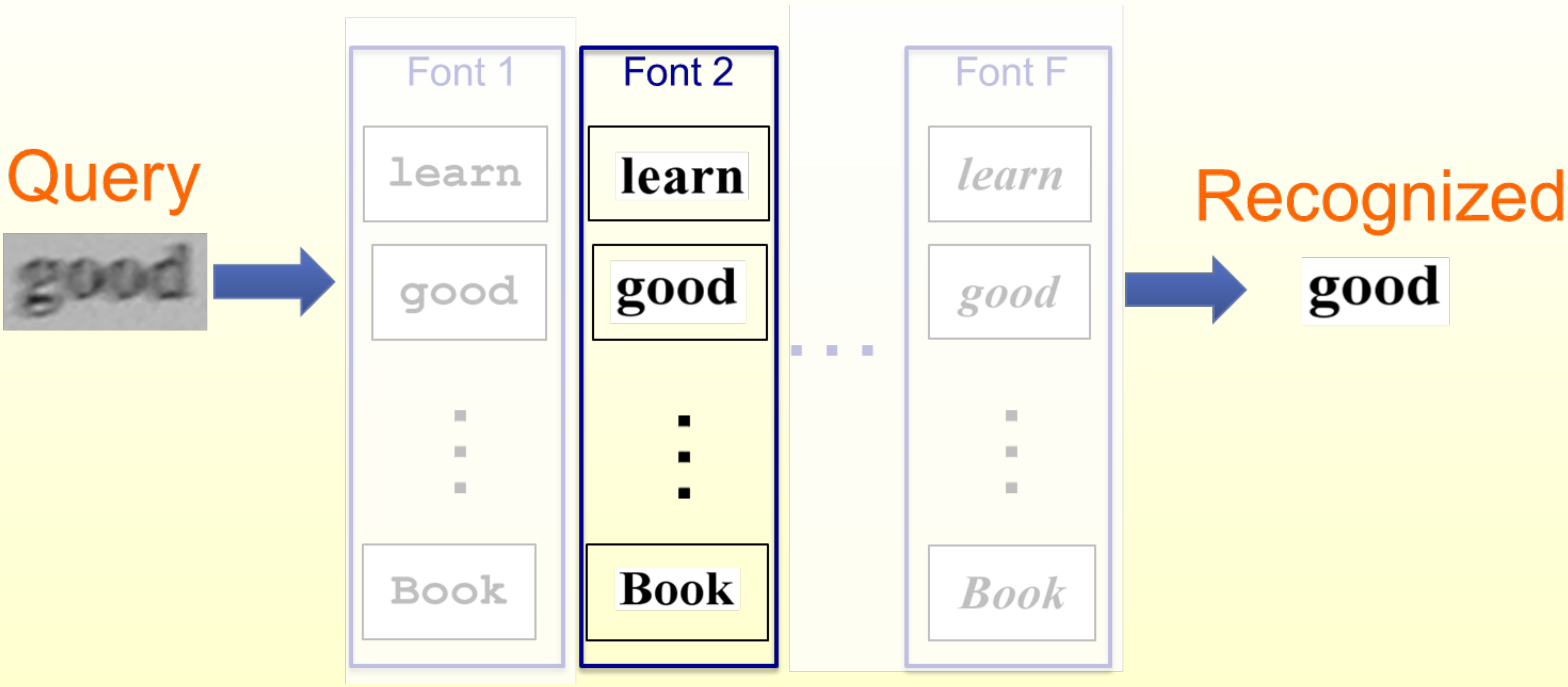
Query



Recognized

Huizhong Chen, Visual Word Recognition with Large-Scale Image Retrieval
Stanford EE Ph.D. Thesis, 2015

If Font Is Known, Task is Easier

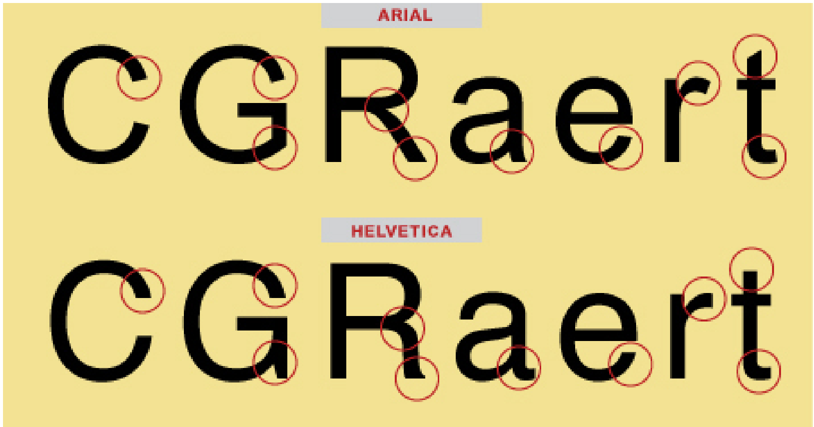


Inter-font Similarities

- Some fonts have visual similarities (e.g., Arial, Helvetica).

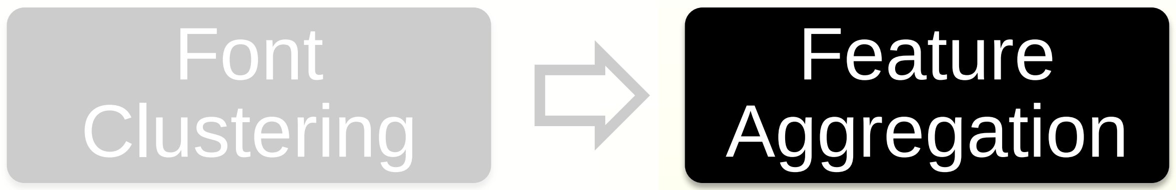
THE QUICK BROWN FOX JUMPS OVER THE LAZY DOG
THE QUICK BROWN FOX JUMPS OVER THE LAZY DOG

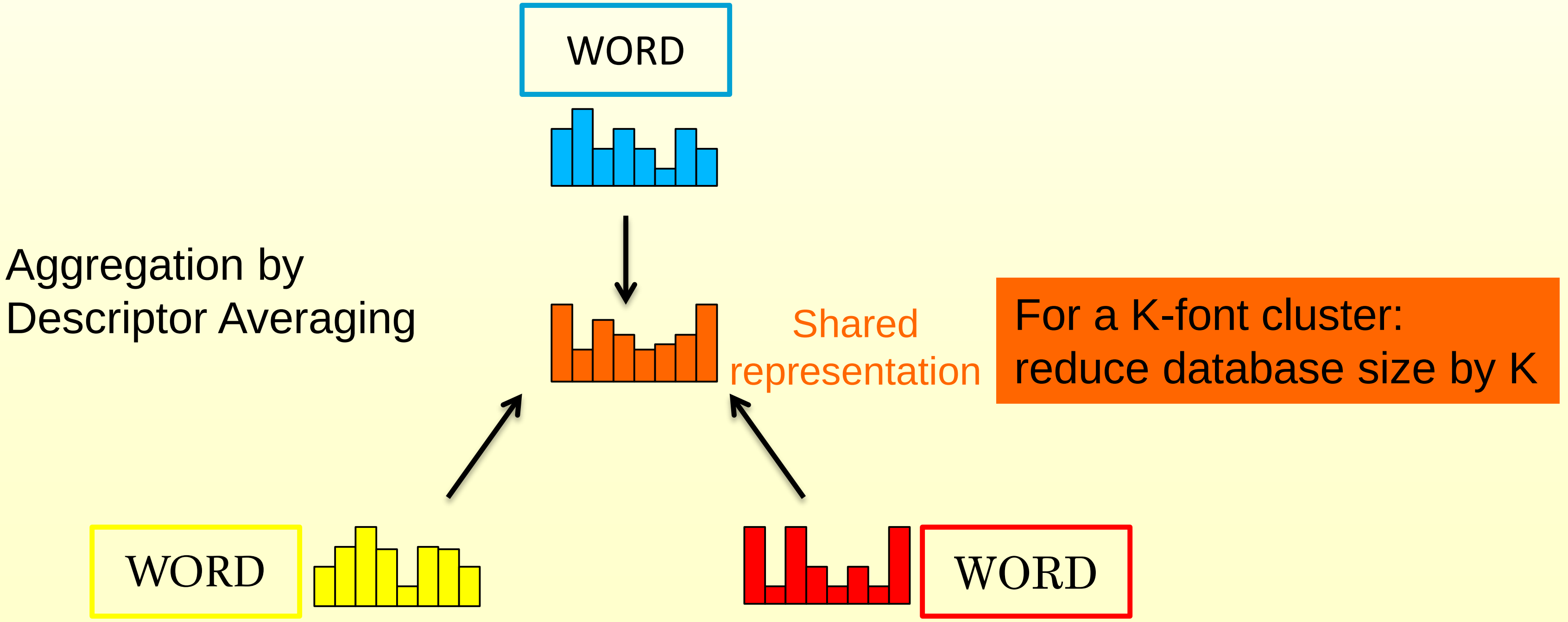
the quick brown fox jumps over the lazy dog
the quick brown fox jumps over the lazy dog



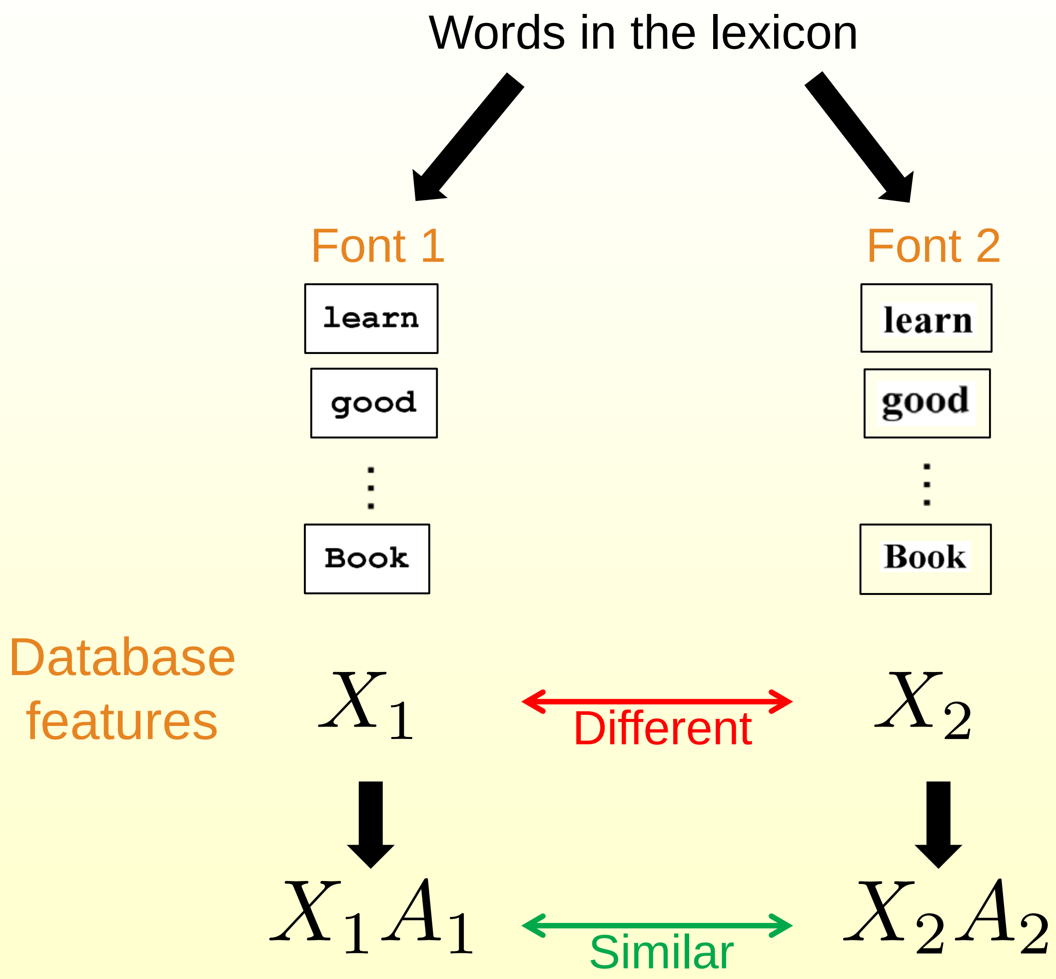
Source: www.fonts.com

Compact Data Base Representation





Motivation for CCA



Two View vs. Multi-View CCA

The End