

# Introduction to Alpha Shapes

SUMMARY

WARNING: MAY CONTAIN ERRORS!

Kaspar Fischer (kf@iaeth.ch)

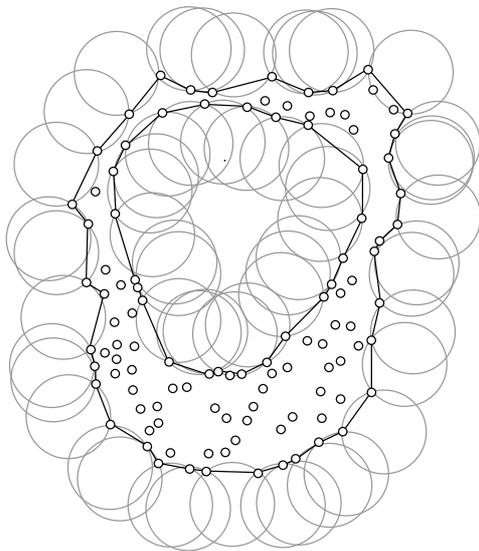
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## 1 Introduction

Assume we are given a set  $S \subset \mathbb{R}^d$  of  $n$  points in 2D or 3D and we want to have something like “the shape formed by these points.” This is quite a vague notion and there are probably many possible interpretations (cf. [3]), the  $\alpha$ -shape being one of them.

As mentioned in Edelsbrunner’s and Mücke’s paper [3], one can intuitively think of an  $\alpha$ -shape as the following. Imagine a huge mass of ice-cream making up the space  $\mathbb{R}^d$  and containing the points  $S$  as “hard” chocolate pieces. Using one of these sphere-formed ice-cream spoons we carve out all parts of the ice-cream block we can reach without bumping into chocolate pieces, thereby even carving out holes in the inside (eg. parts not reachable by simply moving the spoon from the outside). We will eventually end up with a (not necessarily convex) object bounded by caps, arcs and points. If we now straighten all “round” faces to triangles and line segments, we have an intuitive description of what is called the  $\alpha$ -shape of  $S$ . Here’s an example for this process in 2D (where our ice-cream spoon is simply a circle):



And what is  $\alpha$  in the game?  $\alpha$  is the radius of the carving spoon. A very small value will allow us to eat up all of the ice-cream except the chocolate points  $S$  themselves. Thus we already see that the  $\alpha$ -shape of  $S$  degenerates to the point-set  $S$  for  $\alpha \rightarrow 0$ . On the other hand, a huge value of  $\alpha$  will prevent us even from moving the spoon *between* two points since it's way too large. So we will never spoon up ice-cream lying in the inside of the convex hull of  $S$ , and hence the  $\alpha$ -shape for  $\alpha \rightarrow \infty$  is the convex hull of  $S$ .

In the following I will (i) summarize the definitions from [3] to state the above concepts (spoon, etc.) more precisely and (ii) present a result from [1] which allows one to easily compute  $\alpha$ -shapes from the Delaunay triangulation of  $S$ . Finally, I will very briefly give an overview of the problems and approaches with  $\alpha$ -shapes when applied to surface reconstruction.

## 2 Definition

In the following we assume that the points of  $S$  are in general position. This will allow us to desist from special cases. In this context, general position means that no 4 points of  $S$  lie on a common plane and no 5 points lie on a common sphere (GP1); furthermore we assume that, for any fixed  $\alpha$  the smallest sphere through any 2, 3 or 4 points of  $S$  has a radius different from  $\alpha$  (GP2).<sup>1</sup> Of course these assumptions aren't good news if you ever want to use  $\alpha$ -shapes for practical problems. However, there's a technique called SoS described in [2] and touched in [3] which "simulates an infinitesimal perturbation of the points [so that they are in general position afterwards] on the level of predicates and relieves the programmer from the otherwise necessary case analysis." (More I don't know, Mr./Miss Britannica...)

Notice that the general position assumption guarantees that for every  $T \subset S$  with  $|T| = k + 1 \leq d + 1$ , the polytope  $\Delta_T = \text{conv } T$  has exactly dimension  $k$

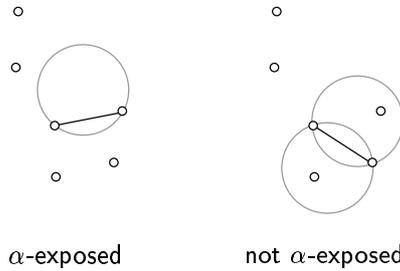
<sup>1</sup>(For their implementation in [3], Edelsbrunner and Mücke require even more conditions...)

and therefore is a  $k$ -simplex. (By the way: Whenever I talk about a “ $k$ -simplex” in the following, I mean a  $k$ -simplex  $\Delta_T$  for some  $T \subset S$  such that  $|T| = k + 1$ .)

First, we need the notion of a  $\lambda$ -ball which will stand for the ice-cream spoon mentioned in the introduction.

**Definition 1.** For  $0 < \lambda < \infty$  let an  $\lambda$ -ball be an open ball with radius  $\lambda$ . Furthermore, a 0-ball is a point for us and an  $\infty$ -ball is an open half-space. Now, a certain  $\lambda$ -ball  $b$  (at a given location) is called empty if  $b \cap S = \emptyset$ . With this, a  $k$ -simplex  $\Delta_T$  is said to be  $\alpha$ -exposed if there exists an empty  $\alpha$ -ball with  $T = \partial b \cap S$  where  $\partial b$  is the surface of the sphere (for  $d = 3$ ) or the circle ( $d = 2$ ) bounding  $b$ , respectively.

In the figure below you can see an example of an  $\alpha$ -exposed simplex (a line segment) for the case  $d = 2$ .



But how to define our “ $\alpha$ -shape”? Coming back to the ice-cream scenario from the introduction we notice that a face is on the boundary of our intuitive  $\alpha$ -shape (to be defined) if the ice-cream spoon hits against one or more of the points in  $S$ . But this simply means that the simplex spanned by these points is  $\alpha$ -exposed. This leads to the following definition of the “boundary” of the  $\alpha$ -shape. (Thereby, “boundary” is just a name for the time being—we will explain later that there is indeed a polytope having as its boundary the simplices of  $\partial\mathcal{S}_\alpha$ .)

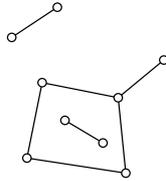
**Definition 2.** The boundary  $\partial\mathcal{S}_\alpha$  of the  $\alpha$ -shape of the point set  $S$  consists of all  $k$ -simplices of  $S$  for  $0 \leq k < d$  which are  $\alpha$ -exposed,

$$\partial\mathcal{S}_\alpha = \{\Delta_T \mid T \subset S, |T| \leq d \text{ and } \Delta_T \text{ } \alpha\text{-exposed}\}. \quad (1)$$

At this stage it’s not clear that there is a polytope  $P$  with  $\partial P = \partial\mathcal{S}_\alpha$ . That this is indeed the case will be explained in observation 7, so that we can say: The  $\alpha$ -shape  $\mathcal{S}_\alpha$  of a point set  $S$  is the polytope with boundary  $\partial\mathcal{S}_\alpha$ . (This polytope is not necessarily convex, it may even contain holes.<sup>2</sup>)

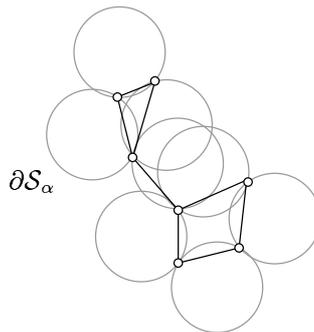
(Notice that  $|T| \leq d$  is equivalent to  $\dim \Delta_T < d$  because of the general position assumption. Also, you might wonder what the talk about the polytope  $P$  with  $\partial P = \partial\mathcal{S}_\alpha$  is all about. The “danger” is that the set  $\partial\mathcal{S}_\alpha$  is not a “boundary” at all.

<sup>2</sup>A polytope in this context is the underlying space of a simplicial complex.



These simplices do not form a boundary.

The above figure for instance shows a set of  $(d - 1)$ -simplices in  $\mathbb{R}^2$  which do *not* represent a boundary since one line-segment is contained in a “closed” area. In our case of the set  $\partial\mathcal{S}_\alpha$ , observation 7 will clarify that things like the above situation cannot occur here. Another question is whether there is more than one polytope with boundary  $\partial\mathcal{S}_\alpha$ . But here the answer is relatively simple, there are but two such polytopes: If  $P$  is one, so is  $\mathbb{R}^d \setminus P$  as you can easily verify. But we’re not really interested in the unbounded one among these two, since our point set “spans a bounded object.” So we will choose the other.) Finally, here’s another example of (the boundary of) an  $\alpha$ -shape.

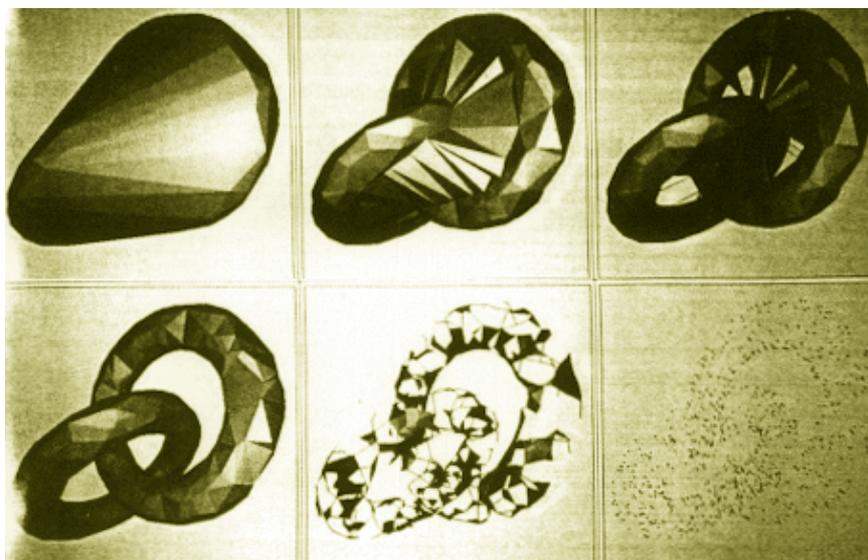


## 2.1 Alpha Shapes and the Convex Hull

Since an infinitely small ball exposes any point in  $S$  but none of the higher-dimensional simplices, and since an  $\alpha$ -ball with  $\alpha$  greater than the radius of the smallest enclosing circle of the points  $S$  doesn’t allow an interior simplex to be  $\alpha$ -exposed, we get  $\lim_{\alpha \rightarrow 0} \partial\mathcal{S}_\alpha = S$  and  $\lim_{\alpha \rightarrow \infty} \partial\mathcal{S}_\alpha = \partial \text{conv } S$ , respectively. And thus it’s plausible:

**Observation 1.**  $\lim_{\alpha \rightarrow 0} \mathcal{S}_\alpha = S$  and  $\lim_{\alpha \rightarrow \infty} \mathcal{S}_\alpha = \text{conv } S$ .

The following figure (originally published in [3] I think) illustrates some  $\alpha$ -shapes for different values of  $\alpha$  for a three-dimensional point set. The first image shows the  $\infty$ -shape, the last one the 0-shape.



## 2.2 Alpha Shapes and the Delaunay Triangulation

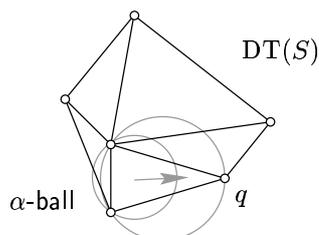
This section will show that the boundary  $\partial\mathcal{S}_\alpha$  of the  $\alpha$ -shape is, for any value  $0 \leq \alpha \leq \infty$ , a subset of the Delaunay triangulation of  $S$ . This is quite an useful observation since we know then that we only have to consider the faces of the Delaunay triangulation  $DT(S)$  as candidates of the  $\alpha$ -shape.

**Definition 3.** Given a set  $S \subset \mathbb{R}^d$  ( $d = 2, 3$ ) in general position, the Delaunay triangulation of  $S$  is the simplicial complex  $DT(S)$  consisting only of

- (i) all  $d$ -simplices  $\Delta_T$  with  $T \subset S$  such that the circumsphere of  $T$  (for  $d = 2$  this degenerates to the great circle of  $T$ ) does not contain any other points of  $S$ , and
- (ii) all  $k$ -simplices which are faces of other simplices in  $DT(S)$ .

**Observation 2.** If  $\Delta_T$  is an  $\alpha$ -exposed simplex of  $S$ , then  $\Delta_T \in DT(S)$ .

*Proof.* The statement definitely holds for  $d$ -simplices  $\Delta_T$  because in this case the  $\alpha$ -ball coincides with the circumsphere (or great circle, respectively) of  $T$ . So let  $\Delta_T$  be a  $k$ -simplex for  $k < d$  and assume  $\Delta_T \notin DT(S)$ . Move the center of the empty  $\alpha$ -ball continuously while adjusting the ball's radius so that the points of  $T$  always lie on its boundary. Since  $\Delta_T$  cannot lie on the boundary of the convex hull of  $S$  (for it would be in  $DT(S)$  then), the ball eventually moves to a position where it bumps on another point  $q \in S \setminus T$ . (The figure below shows a sketch for  $d = 2$  and a 1-simplex.)



If  $|T \cup \{q\}| = d + 1$  we have thus found an empty circumsphere of  $d + 1$  points which means that  $\Delta_{T \cup \{q\}}$  and its face  $\Delta_T$  lie in  $\text{DT}(S)$ .<sup>3</sup> However if  $|T \cup \{q\}| < d + 1$  we repeat the process until the  $\alpha$ -ball touches  $d + 1$  points.  $\circ$

Clearly, observation 2 shows:

**Observation 3.** *For any  $0 \leq \alpha \leq \infty$  we have  $\partial\mathcal{S}_\alpha \subset \text{DT}(S)$ .*

For instance, we know now that all the lines in the introductory figure on page 2 (which, together with some of the points of  $S$ , make up  $\mathcal{S}_\alpha$ ) *must* lie in the Delaunay complex.

### 3 Alpha Complexes

This section introduces  $\alpha$ -complexes which can be used to compute  $\alpha$ -shapes and which help in showing that there is indeed a polytope which has the set  $\partial\mathcal{S}_\alpha$  as its boundary.

In order to compute the  $\alpha$ -shape (more precisely, the boundary of the shape) for a given value of  $\alpha$  we could proceed as follows. Since  $\partial\mathcal{S}_\alpha(S) \subset \text{DT}(S)$  it suffices to inspect all simplices of  $\text{DT}(S)$ . Thus we inspect every triangle  $\Delta_T \in \text{DT}(S)$  (or every line segment in the case  $d = 2$  respectively) and check out whether one of its circumspheres with radius  $\alpha$  (there are two of them) is empty. If so, we accept such a  $(d - 1)$ -simplex. Lower-dimensional simplices would be treated in a similar fashion—but it's not quite clear how to implement this direct approach because there are, if you consider the inspection of a point  $s \in S$ , infinitely many  $\alpha$ -balls touching it.

The algorithm presented in [4] overcomes these problems by stating alternative conditions for a simplex  $\Delta_T \in \text{DT}(S)$  to be a member of  $\partial\mathcal{S}_\alpha(S)$ , namely the so-called *alpha test*. The algorithm has the further advantage that it implicitly computes a representation for *all* values of  $\alpha$ , so that there is no need to restart the whole computation if a new value of  $\alpha$  is specified.

Instead of directly computing  $\partial\mathcal{S}_\alpha(S)$  the algorithm first computes a structure called an  $\alpha$ -*complex*. The fact that the algorithm singles out simplices of  $\text{DT}(S)$  reflects itself in the definition of such a complex as a subset of  $\text{DT}(S)$ .

#### 3.1 The Alpha Complex of a Point Set

**Definition 4.** *Let  $\Delta_T = \text{conv}(T)$  be a  $k$ -simplex,  $0 \leq k \leq d$  spanned by some vertices of  $S$ . Set*

$$\begin{aligned}\sigma_T &= \text{radius of the circumsphere of } \Delta_T \\ \mu_T &= \text{center of the circumsphere of } \Delta_T\end{aligned}$$

where the circumsphere is simply the great circle in case of  $d = 2$ .

**Definition 5.** *For a given point set  $S \subset \mathbb{R}^d$  and  $0 \leq \alpha \leq \infty$ , the  $\alpha$ -complex  $\mathcal{C}_\alpha(S)$  of  $S$  is the following simplicial subcomplex of  $\text{DT}(S)$ . A simplex  $\Delta_T \in \text{DT}(S)$  is in  $\mathcal{C}_\alpha(S)$  if*

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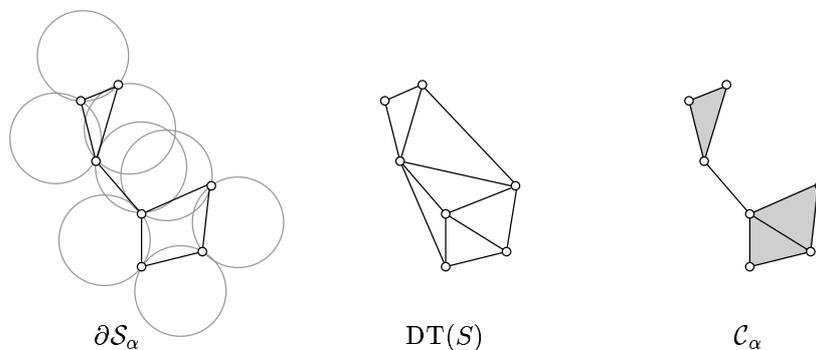
<sup>3</sup>Notice: Any  $d$ -simplex whose circumsphere is empty lies in  $\text{DT}(S)$ . This follows from the fact that the corresponding  $d$ -simplex in the lifting-map forms a face of the lower convex-hull.

(C1)  $\sigma_T < \alpha$  and the  $\sigma_T$ -ball located at  $\mu_T$  is empty, or

(C2)  $\Delta_T$  is a face of another simplex in  $\mathcal{C}_\alpha(S)$ .

Notice that this is actually the set of all simplices in  $\text{DT}(S)$  satisfying (C1), enlarged by as many faces of the latter as needed to make the set a simplicial complex. The condition (C1) is called the alpha test here. And: In contrast to  $\partial\mathcal{S}_\alpha$ , the  $\alpha$ -complex can contain  $d$ -dimensional simplices; on the other hand,  $\mathcal{C}_\alpha$  contains all points  $S$  by definition (which need not be the case for  $\mathcal{S}_\alpha$ .)

Before thinking about this new notion we apply it to an example. The figure below shows three times the same point-set. To the left you can see the boundary of the  $\alpha$ -shape. The figure in the middle shows the Delaunay triangulation and on the right you can see the  $\alpha$ -complex. And yes, you're right, the  $\alpha$ -complex looks suspiciously like the  $\alpha$ -shape  $\mathcal{S}_\alpha$  itself...!



This is not kind of an hazard: The main result from [1] is that the  $\alpha$ -complex's boundary makes up the  $\alpha$ -shape's boundary! And since the underlying space of a simplicial complex is a polytope, we immediately see that there *is* a polytope which has  $\partial\mathcal{S}_\alpha$  as its boundary, namely the underlying space itself. (So the problem in definition 2 is solved.)—I'm going to repeat the proof since it's not really obvious that this connection indeed exists (at least not for me, hm). Notice also that the condition for a simplex  $\Delta_T \in \text{DT}(S)$  to be in the  $\alpha$ -complex is quite simple. It involves *one* sphere only and is not something complicated like “if there exists *any*  $\alpha$ -balls touching the points  $T$ ...”

### 3.2 The Link Between the Complex and the Shape

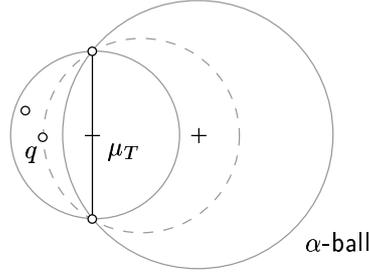
Let's start like this:

**Observation 4.** Let  $\Delta_T \in \partial\mathcal{S}_\alpha(S)$  be any simplex. Then  $\Delta_T \in \mathcal{C}_\alpha(S)$ .

*Proof.* Let  $\Delta_T$  be a simplex in  $\partial\mathcal{S}_\alpha(S)$ . Then  $\Delta_T$  has dimension smaller than  $d$  and it is  $\alpha$ -exposed. Now, if the  $\sigma_T$ -ball located at  $\mu_T$  is empty, then  $\Delta_T$  satisfies (C1) and the claim is true. So consider the case when the  $\sigma_T$ -ball is not empty but contains some points  $q_i$  instead. We will show that—though (C1) is not satisfied—the simplex  $\Delta_T$  is a face of another simplex  $\Delta_U$  satisfying (C1) and thus is contained in  $\mathcal{C}_\alpha$  because of (C2). For this, use induction on  $d$ :

- The simplex  $\Delta_T$  has dimension  $d - 1$ : Here, move the center of the  $\alpha$ -ball continuously to  $\mu_T$  while adjusting the ball's radius so that the points in  $T$

keep lying on its surface. Eventually, the ball will pop against one of the points  $q_i$ . At this moment we've found a  $d$ -simplex  $\Delta_U$  for  $U = T \cup \{q\}$  whose circumsphere is empty and which hence lies in  $\text{DT}(S)$ . But since we only decreased the ball's radius while moving it, we have  $\sigma_U < \alpha$ , implying  $\Delta_U \in \mathcal{C}_\alpha$  as needed. (This is illustrated for  $d = 2$  below.)

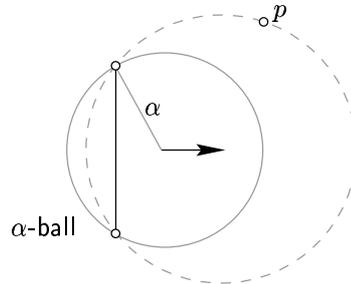


- The simplex  $\Delta_T$  has dimension  $k < d - 1$ : Inductively assume that all  $(k + 1)$ -simplices of  $\partial\mathcal{S}_\alpha$  are simplices of  $\mathcal{C}_\alpha$ . Move the center of  $\Delta_T$ 's  $\alpha$ -ball while keeping its radius fixed and the points  $T$  on its boundary. Since the  $\sigma_T$ -ball around  $\mu_T$  is not empty, the moving ball will eventually hit upon a point  $q$ . This shows that  $\Delta_V$  for  $V = T \cup \{q\}$  is an  $\alpha$ -exposed  $(k + 1)$ -simplex having  $\Delta_T$  as a proper face. Therefore  $\Delta_T \in \mathcal{C}_\alpha$  by (C2).

(Notice that  $\sigma_U = \alpha$  is not possible because of (GP2). Furthermore, all points in  $\mathcal{S}_\alpha$  lie trivially in  $\mathcal{C}_\alpha$ .)  $\circ$

**Observation 5.** *Let  $\Delta_T \in \partial\mathcal{S}_\alpha(S)$  be any simplex. Then  $\Delta_T \in \partial\mathcal{C}_\alpha(S)$ .*

*Proof.* We already know that such a simplex is in  $\mathcal{C}_\alpha$ . So consider the  $\alpha$ -ball which marks the simplex as being  $\alpha$ -exposed. It suffices to show that there is no  $d$ -simplex in  $\mathcal{C}_\alpha$  which is incidental to  $\Delta_T$  and which lies on the same “side” of  $\Delta_T$  as the  $\alpha$ -ball. Move the ball's center while adjusting its radius so that all points in  $T$  remain on its boundary and no point in  $S \setminus T$  enters its inside. If it's possible to increase the ball's radius beyond any finite bound,  $\Delta_T$  must lie on  $\partial \text{conv } S$ . Otherwise the ball eventually touches a point  $p$ , but according to observation 2 the simplex  $\Delta_W$  with  $W = T \cup \{p\}$  is part of the Delaunay triangulation and thus a face of a  $d$ -simplex in  $\text{DT}(S)$ .

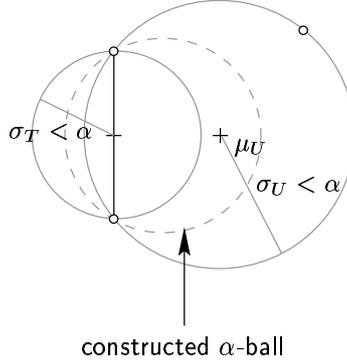


But this  $d$ -simplex cannot be in  $\mathcal{C}_\alpha$  because we've just seen that its radius  $\sigma_W$  is larger than  $\alpha$ . (The above figure shows a sketch for  $d = 2$ .)  $\circ$

The other direction is proved similarly. A *principal  $k$ -simplex* of  $\mathcal{C}_\alpha$  is a simplex which is not a proper face of another one in  $\mathcal{C}_\alpha$ . Every principal simplex must fullful (C1) (though the reverse is not always true). (Btw: Notice that a principal simplex  $\in \mathcal{C}_\alpha$  of dimension  $< d$  has to lie on the boundary of  $\mathcal{C}_\alpha$ .)

**Observation 6.** Let  $\Delta_T \in \partial\mathcal{C}_\alpha(S)$ . Then  $\Delta_T \in \partial\mathcal{S}_\alpha(S)$ .

*Proof.* If  $\Delta_T$  is principal, its circumsphere has radius  $\sigma_T < \alpha$  and is empty. This means, accoring to observation 2, that  $\Delta_T \in \text{DT}(S)$ . So let  $\Delta_U$  be an incident  $d$ -simplex in  $\text{DT}(S)$ . Since  $\Delta_U$  cannot be in  $\mathcal{C}_\alpha(S)$ , we have  $\sigma_U > \alpha$ .



Move the circumsphere of  $\Delta_T$  towards  $\mu_U$ , continuously adjusting the sphere's radius so that the points  $T$  remain on its boundary. Stop when the sphere takes on the radius  $\alpha$  which must happen since  $\sigma_T < \alpha < \sigma_U$ . What you have now is an empty  $\alpha$ -ball marking  $\Delta_T$  as  $\alpha$ -exposed.—If on the other hand  $\Delta_T$  is non-principal, we use induction on  $d$  once more:

- The simplex  $\Delta_T$  has dimension  $d - 1$ : If  $\Delta_T$  does not lie on the boundary of  $\text{conv } S$ , there are exactly two incident  $d$ -simplices  $\Delta_V$  and  $\Delta_W$ , one having  $\sigma_V < \alpha$ , the other  $\sigma_W > \alpha$  (for  $\Delta_T$  is non-principal). So we can play the same game as in the principal case and shrink the larger circumsphere until it has radius  $\alpha$  which unmasks  $\Delta_T$  as  $\alpha$ -exposed. The same procedure can be applied in the case when  $\Delta_T$  is a face of  $\text{conv } S$ .
- The simplex  $\Delta_T$  has dimension  $k < d - 1$ : Since  $\Delta_T$  is not principal, it is a face of  $(k + 1)$ -simplex  $\Delta_U$  which, by the induction hypothesis lies in  $\partial\mathcal{S}_\alpha$ . But this gives us an  $\alpha$ -ball exposing  $\Delta_U$ . By moving this ball just a tiny bit while keeping the points  $T$  on its boundary, we obtain an  $\alpha$ -ball which marks  $\Delta_T$  as  $\alpha$ -exposed.  $\circ$

Altogether we get:

**Observation 7.**  $\partial\mathcal{S}_\alpha(S) = \partial\mathcal{C}_\alpha(S)$ .

This result allows us to resolve the question brought up in definition 2: Observation 7 shows that there is indeed a polytope (which we will call  $\mathcal{S}_\alpha(S)$ ) having the simplices  $\partial\mathcal{C}_\alpha(S)$  as its boundary, namely the underlying space of  $\mathcal{C}_\alpha$ . But is it the only one? No, obviously there is exactly one more polytope having  $\partial\mathcal{C}_\alpha$  as its surface, namely the “inverse”, that is  $\mathbb{R}^d \setminus \|\mathcal{C}_\alpha(S)\|$  (which is unbounded).

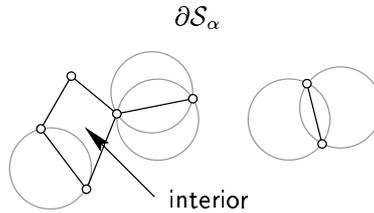
But since we're definitely interested in a bounded object we<sup>4</sup> “set”

$$\mathcal{S}_\alpha(S) := \|\mathcal{C}_\alpha(S)\|. \quad (2)$$

(If you want to, you can see this as an alternative definition of the  $\alpha$ -shape.)

### 3.3 The Interior of the Alpha Shape

Relax, we're over the hill, haha! There's just two more little things I'd like to mention. The first bit is that there's an easy way to find out on which side of a facet  $\Delta_T \in \partial\mathcal{C}_\alpha$  the interior of the  $\alpha$ -shape lies. What I mean is the following:



The line segment on the right does obviously not bound the interior of the shape, so the interior of the  $\alpha$ -shape is on neither of its sides. The same holds for the line in the middle of the figure. The remaining four lines (on the left of the image) however do bound the interior of the shape, so one side of  $\text{aff}(\Delta_T)$  is “inside” and the other “outside”.

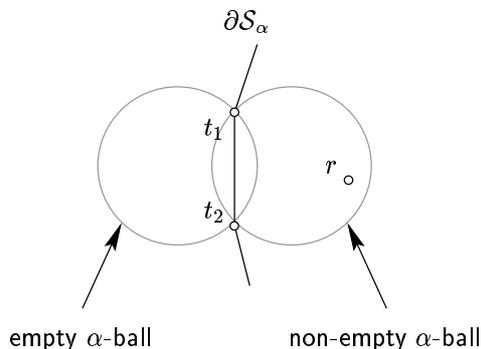
Of course we could solve the problem by simply inspecting the  $\alpha$ -complex (checking whether there is a  $d$ -simplex in the complex containing  $\Delta_T$ ). But here's another way:

**Observation 8.** *Let  $\Delta_T \in \partial\mathcal{C}_\alpha$  be a facet on the boundary of the  $\alpha$ -shape. Then:  $\Delta_T$  bounds the interior of the  $\alpha$ -shape iff but one of the two  $\alpha$ -balls  $b$  with  $T = \partial b \cap S$  is empty.*

*Proof.* The direction  $(\Rightarrow)$  can be shown as follows. Since  $\Delta_T \in \partial\mathcal{C}_\alpha = \partial\mathcal{S}_\alpha$ , one of the two  $\alpha$ -balls in question must be empty. So we'll have to show that the other is not. But if  $\Delta_T$  bounds the interior then there is a  $d$ -simplex  $\Delta_U$  in the  $\alpha$ -complex having  $\Delta_T$  as a face. Consequently, the circumsphere of  $\Delta_U$  has radius  $\sigma_U < \alpha$  which already shows that the  $\alpha$ -ball “on the side of  $\Delta_U$ ” cannot be empty.

The other direction is shown similarly, so take a look at the figure below. As already mentioned, one of the  $\alpha$ -balls must be empty. So we assume that the other is not, but contains a point  $r$  instead.

<sup>4</sup>Zwei moeglichkeiten???



Let  $\Delta_U$  be the  $d$ -simplex in  $\text{DT}(S)$  on the “side of the non-empty  $\alpha$ -ball,” having  $\Delta_T$  as a face (which means  $U = T \cup \{p\}$  for some  $p$ ). But  $\Delta_U$  as a member of  $\text{DT}(S)$  is empty, so the point  $p$  must lie somewhere in the inside of the non-empty  $\alpha$ -ball. So  $\sigma_U < \alpha$  which means  $\Delta_U \in \mathcal{C}_\alpha$  what was to be shown.  $\circ$

The second bit is the following:

**Observation 9.**  $\alpha_1 \leq \alpha_2 \Rightarrow \mathcal{C}_{\alpha_1}(S) \subset \mathcal{C}_{\alpha_2}(S) \Rightarrow \mathcal{S}_{\alpha_1}(S) \subset \mathcal{S}_{\alpha_2}(S)$ .

*Proof.* According to (C1)  $\alpha_1 \leq \alpha_2$  implies  $\mathcal{C}_{\alpha_1} \subset \mathcal{C}_{\alpha_2}$  which shows the statement by equation (2).  $\circ$

This last observation is quite intuitive if you recall the ice-cream scenario of introduction. Besides, it shows that for every simplex  $\Delta \in \text{DT}(S)$  there is an interval  $I = [a, \infty]$  such that the simplex is in  $\mathcal{C}_\alpha$  iff  $\alpha \in I$ .

### 3.4 Edelsbrunner’s Algorithm

At this stage it’s relatively straightforward to formulate an algorithm to obtain the  $\alpha$ -shape.

1. Compute the Delaunay triangulation of  $S$ , knowing that the boundary of our  $\alpha$ -shape is contained in it.
2. Then we determine  $\mathcal{C}_\alpha$  by inspecting all simplices  $\Delta_T$  in  $\text{DT}(S)$ : If the  $\sigma_T$ -ball around  $\mu_T$  is empty and  $\sigma_T < \alpha$  (—this is the alpha test—) we accept  $\Delta_T$  as a member of  $\mathcal{C}_\alpha$ , together with all its faces.
3. All  $d$ -simplices of  $\mathcal{C}_\alpha$  make up the interior of  $\mathcal{S}_\alpha$ . All simplices on the boundary  $\partial \mathcal{C}_\alpha$  form  $\partial \mathcal{S}_\alpha$ .

For this algorithm to work we need three things. First, the Delaunay triangulation (which isn’t a problem, there are algorithms to do that). Then, a test to check whether or not the  $\sigma_T$ -ball is empty. This too can be done, for instance by checking whether  $p$  lies in the said ball for every  $p \in S \setminus T$ . Finally, we need a way to see whether a simplex  $\Delta_T$  in  $\mathcal{C}_\alpha$  lies on the boundary. For this, let’s assume that the Delaunay triangulation algorithm returns (in addition to the triangulation) for every simplex whether or not it is on the boundary  $\partial \text{conv}(S)$  of the convex hull. Then:

**Observation 10.** *Let  $\Delta_T$  be a simplex in  $\mathcal{C}_\alpha(S)$ . If  $\Delta_T \in \partial \text{conv}(S)$ , then it is obviously on the boundary of  $\mathcal{C}_\alpha$ . Otherwise, it is in the interior of  $\mathcal{C}_\alpha$  iff all of the simplices in  $\text{DT}(S)$  properly containing  $\Delta_T$  lie in  $\mathcal{C}_\alpha$ , too.<sup>5</sup>*

The algorithm presented in [1] and [3] is an efficient implementation of the above procedure, with two additional advantages: First of all the algorithm does not run for a single value  $\alpha$  but computes an implicit representation instead which can be used to deduce  $\mathcal{S}_\alpha$  for any value of  $\alpha$ . More precisely, the algorithm computes for every simplex  $\Delta_T \in \text{DT}(S)$  an interval  $I = [a, \infty]$  with the interpretation that  $\Delta_T \in \mathcal{S}_\alpha$  iff  $\alpha \in I$ .<sup>6</sup> (That there is such an interval is a consequence of observation 9, as already mentioned.) Second, the algorithm not only distinguishes among interior and non-interior simplices of  $\mathcal{C}_\alpha$  (as we have done above), but makes three distinctions instead, one more among the non-interior simplices. (I didn't get why though, probably it's just useful to have the algorithm output some more information about the individual simplices of the  $\alpha$ -shape's boundary...?) In the following I will not make this finer differentiation.

When we increase  $\alpha$  continuously from 0 towards  $\infty$  and consider a simplex  $\Delta_T \in \text{DT}(S)$ , we see (using observation 9) that there are two (possibly empty) intervals  $(a, b)$  and  $(b, \infty)$  (with  $0 \leq a \leq b \leq \infty$ ) such that

$$\Delta_T \text{ is } \begin{cases} \text{not in } \mathcal{C}_\alpha & (\text{for } \alpha < a) \\ \text{in } \partial \mathcal{C}_\alpha & (\text{for } \alpha \in (a, b)) \\ \text{interior to } \mathcal{C}_\alpha & (\text{for } \alpha \in (b, \infty)) \end{cases} . \quad (3)$$

(Notice that there is no need to consider the boundaries of the interval (eg. the situations  $\alpha = a, b$ ) because of the general position assumption: Since the complex  $\mathcal{C}_\alpha$  only changes when a simplex is picked up by condition (C1), and since there are by assumption (GP2) no circumspheres with radius  $\alpha$ , we see that the complex cannot change when  $\alpha$  passes the values  $a$  and  $b$ .)

Our aim is to find  $a, b$  (eg. the intervals in (3)) for every simplex  $\Delta_T \in \text{DT}(S)$ . For a  $d$ -simplex in  $\text{DT}(S)$  the story is simple: Its circumsphere is empty by definition and the alpha-test indicates that it is in the  $\alpha$ -complex iff  $\sigma_T < \alpha$ . And since a  $d$ -simplex is always interior to  $\mathcal{C}_\alpha$  we get  $a = b = \sigma_T$ . We can thus write down the head of the algorithm as follows:

```

procedure AlphaShape( $S, d$ );
  {Given a point-set  $S \subset \mathbb{R}^d$ , computes a list  $R$  of simplices  $\Delta_T$  and}
  {two lists  $B, I$  of intervals such that  $\Delta_T \in \partial \mathcal{S}_\alpha$  if and only if  $\alpha \in B_T$ }
  {and  $\Delta_T \in \text{int}(\mathcal{S}_\alpha)$  if and only if  $\alpha \in I_T$ .}
  begin
     $R := \text{DT}(S)$ ;
    for each  $d$ -simplex  $\Delta_T \in R$  do
       $B_T := \emptyset$ ;  $I_T := (\sigma_i, \infty)$ ;
    end for;
     $\vdots$ 

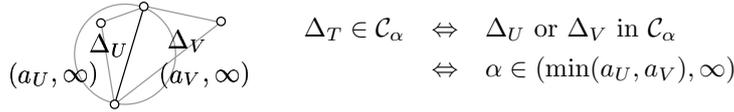
```

<sup>5</sup>Reason: Every simplex in the complex  $\text{DT}(S)$  is a face of a  $d$ -simplex, and thus, if one super-simplex of  $\Delta_T$  does not lie in  $\mathcal{C}_\alpha$  there's also a  $d$ -simplex containing  $\Delta_T$  not lying in  $\mathcal{C}_\alpha$  which means that  $\Delta_T$  is on the boundary.

<sup>6</sup>Notice that it's not really obvious how to alter our above procedure to output such an implicit representation. That's why we need another way...

After having computed the intervals for  $d$ -simplices we will proceed with the lower-dimensional ones. The idea is to compute the intervals of a  $k$ -simplex using the already computed intervals of the  $(k + 1)$ -simplices. So consider a  $k$ -simplex  $\Delta_T \in \text{DT}(S)$  now for  $k < d$ . If its circumsphere (the  $\sigma_T$ -ball with center  $\mu_T$ ) is empty then it belongs to  $\mathcal{C}_\alpha$  iff  $\sigma_T < \alpha$ . Otherwise,  $\Delta_T$  cannot be in  $\mathcal{C}_\alpha$  by (C1), so the only way for it to be accepted is by condition (C2). But this can only be the case if one of the “super-simplices”<sup>7</sup> of  $\Delta_T$  lies in  $\mathcal{C}_\alpha$ . Then,  $\Delta_T$  lies in  $\mathcal{C}_\alpha$  too, by (C2).

As a (two-dimensional) example consider the following figure where  $\Delta_T$  is the black line-segment.  $\Delta_T$  cannot be in the  $\alpha$ -complex by (C1) since its circumsphere is not empty. So it lies in the  $\alpha$ -complex if and only if one of the triangles  $\Delta_U$  or  $\Delta_V$  is in  $\mathcal{C}_\alpha$ . But we already know when this is the case for we’ve already computed the intervals  $(a_U, \infty)$  and  $(a_V, \infty)$  with  $\Delta_U \in \mathcal{C}_\alpha \Leftrightarrow \alpha \in (a_U, \infty)$  (and similarly for  $\Delta_V$ ). The condition for at least one of these super-simplices to lie in the  $\alpha$ -complex is that  $\alpha$  is bigger than the minimum of  $a_U$  and  $a_V$ , as indicated in the figure.



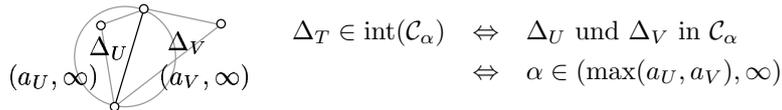
This idea leads to:

**Observation 11.** *Let  $\Delta_T$  be a  $k$ -simplex in  $\text{DT}(S)$  for  $k < d$ . Assume that we’ve already computed the intervals  $B_U$  for each  $(k + 1)$ -simplex  $\Delta_U \in \text{DT}(S)$  such that  $\Delta_U \in \mathcal{C}_\alpha \Leftrightarrow B_U$ , and let*

$$a = \min \{a_U \mid B_U = (a_U, b_u), \Delta_U \text{ (} k + 1\text{)-Simplex, } T \subset U\}$$

Then  $\Delta_T \in \mathcal{C}_\alpha$  if and only if  $\alpha \in (a, \infty)$ .

This gives us the first of the intervals in (3). But how do we get the other one, eg. the number  $b$  which differentiates when a simplex  $\Delta_T \in \mathcal{C}_\alpha$  lies on the boundary and when it is in the interior of  $\mathcal{C}_\alpha$ ? A similar idea as above will help. First notice that  $d$ -simplices always lie in the interior, so the second interval in (3) is empty in this case. In the case of a lower-dimensional simplex  $\Delta_T$  we can distinguish among two cases. If  $\Delta_T \in \partial \text{conv}(S)$  then it is obviously on the boundary of  $\mathcal{C}_\alpha$ . In the case of that  $\Delta_T$  does not lie on the boundary of the convex hull, we can argue like this (see the figure, with  $\Delta_T$  being the black line segment):  $\Delta_T$  lies in the interior if and only if all its  $d$ -dimensional super-simplices of  $\text{DT}(S)$  are elements of the  $\alpha$ -complex. For, if one such  $d$ -simplex is not in the  $\alpha$ -complex, the corresponding “side” of  $\Delta_T$  is “open”.



This leads us to:

---

<sup>7</sup>(A super-simplex of  $\Delta_T$  is a simplex  $\Delta_U$  with  $T \subsetneq U$ .)

**Observation 12.** Let  $\Delta_T$  be a  $k$ -simplex in  $\mathcal{C}_\alpha$  for  $k < d$ . Assume that we've already computed the intervals  $B_U$  for each  $d$ -simplex  $\Delta_U \in \text{DT}(S)$  such that  $\Delta_U \in \mathcal{C}_\alpha \Leftrightarrow B_U$ , and let

$$b = \max \{a_U \mid B_U = (a_U, b_u), B_U \text{ } d\text{-Simplex mit } T \subset U\}$$

Then  $\Delta_T \in \partial\mathcal{C}_\alpha$  if and only if  $\alpha \in (b, \infty)$ .

Altogether we get the following algorithm:

```

procedure AlphaShape( $S, d$ );
{Given a point-set  $S \subset \mathbb{R}^d$ , computes a list  $R$  of simplices  $\Delta_T$  and}
{two lists  $B, I$  of intervals such that  $\Delta_T \in \partial\mathcal{S}_\alpha$  if and only if  $\alpha \in B_T$ }
{and  $\Delta_T \in \text{int}(\mathcal{S}_\alpha)$  if and only if  $\alpha \in I_T$ .}
begin
  R:= DT( $S$ );
  for each  $d$ -simplex  $\Delta_T \in R$  do
     $B_T := \emptyset$ ;  $I_T := (\sigma_i, \infty)$ ;
  end for;
  for  $k := d - 1$  to 0 by  $-1$  do
    for each  $k$ -simplex  $\Delta_T \in R$  do
      if  $b_T$  is empty then
         $a := \sigma_T$ ;
      else
         $a := \min \{a_U \mid B_U = (a_U, b_u), \Delta_U \text{ } (k + 1)\text{-Simplex, } T \subset U\}$ ;
      if  $\Delta_T \in \partial \text{conv}(S)$  then
         $b := \infty$ ;
      else
         $b := \max \{a_U \mid B_U = (a_U, b_u), B_U \text{ } d\text{-Simplex mit } T \subset U\}$ ;
         $B_T := (a, b)$ ;  $I_T := (b, \infty)$ ;
      end for;
    end for;
  return ( $R, B, I$ );
end AlphaShape;

```

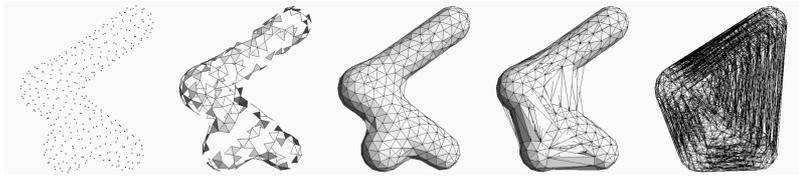
(Notice that the test  $\Delta_T \in \partial \text{conv}(S)$  can be achieved by “flagging” the simplices on  $\partial \text{conv}(S)$  during the calculation of the Delaunay triangulation.)

## 4 Limitation of “Classical” Alpha Shapes and Improvements

### 4.1 Limitations

The main application of  $\alpha$ -shapes is the reconstruction of objects which have been sampled by points. For instance, a 3D-scanner provides points on the surface of a human-being or so. The main problems with  $\alpha$ -shapes in surface reconstruction are (as mentioned in [5]):

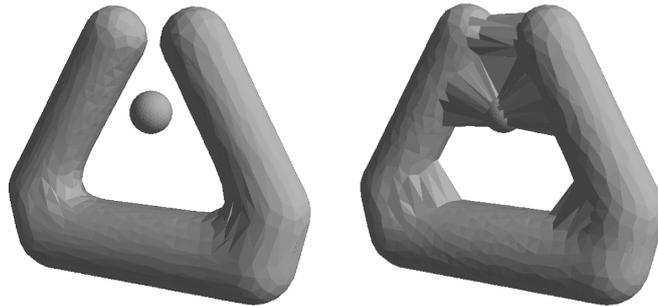
- How are we going to determine the “best”  $\alpha$ ? As far as I can see,  $\alpha$  is found out by trial-and-error, eg. you have something like an “interactive slider” where you can vary  $\alpha$  until the result “looks good.”



The figure, taken from [5], shows some  $\alpha$ -shapes; we would chose the middle one, intuitively.

- There are point-sets for which there is *no* satisfying  $\alpha$ , eg. all  $\alpha$ -shapes don't give intuitively good approximations of the object's surface. This is mostly the case when the points  $S$  are not uniformly sampled. The reason for the unsatisfactory result is that  $\alpha$ -shapes are based on distances between points in order to decide which points to connect by triangles or lines. Non-uniform point-set are thus relatively inappropriate.

I'd like to leave the first point and give an example for the second problem, together with some approaches given in [5]. The following object (on the left) is quite difficult to reconstruct when it is sampled with low density:



Low density requires a rather large  $\alpha$ -ball to accept the triangles on the surface. But a large  $\alpha$ -ball will unfortunately connect the interstice on top of the image, and connect the two (actually separate) objects, the triangle and the sphere. In such a situation, the best thing we could achieve with classical  $\alpha$ -shapes is (assuming low density in the same points) what we see on the right. The large value of  $\alpha$  results in (among possibly other things)

- *interstices* being closed,
- *neighbourhood* objects being connected, and
- *joints* (eg. sharp turns) being destroyed.

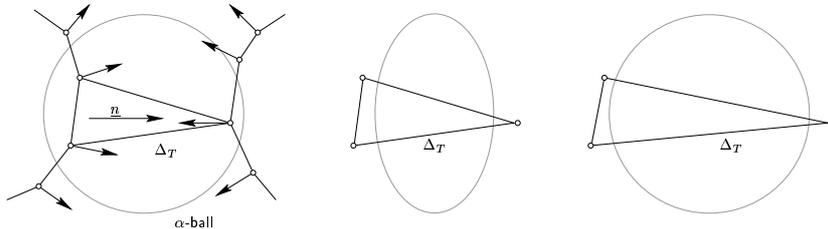
## 4.2 Extensions

**Weighted  $\alpha$ -shapes.** In this system (described in [1]) each point in  $S$  is assigned a *weight* with the interpretation that a large (small) weight favors (discourages) connections to neighboring points. The resulting *weighted  $\alpha$ -shapes* are a generalization of "classical"  $\alpha$ -shapes in the sense that latter are weighted  $\alpha$ -shapes with all weights equal to zero. Obviously, the problem is how to assign the weights in order to solve the above problems (but more I don't know.)

**Density-scaling.** In [5], “density-scaled  $\alpha$ -shapes” are presented. Briefly, one computes the point-density of each point and uses this to get an approximation of the point-density of a triangle (for instance by avering the point-densities of the triangle’s points or by taking the maximum of these densities and so on). Then, the  $\alpha$ -ball is reduced in size in areas where the triangle’s point-density is higher than average. By computing the density of the triangle in different ways (avering, taking the maximum, and so on), one can achieve a finer level of detail in higher density areas, or detect (that is, separate) neighboured objects if they have different point densities.

**Anisotropic Scaling.** In the same paper [5], Teichmann and Capps present “anisotropic  $\alpha$ -shapes” to handle the interstice-problem mentioned above. The idea is the following. Since “we would like for triangles spanning the interstice to fail the alpha-test and be deleted from the  $\alpha$ -shape,” we compress the  $\alpha$ -ball used for the alpha-test of the triangle in question along an axis perpendicular to a local plane separating the interstice. To find this axis, Teichmann and Capps assume that (approximations to) normal vectors are available, eg. that we have (an approximation) of the normal vectors of the real surface (the one to be reconstructed) in every point  $p \in S$ .

Here’s an example (taken from [5], too) of how the heuristic works. The leftmost figure below shows an interstice where the real surfaces are shown as polygons. You can see a triangle  $\Delta_T$  which is being inspecting in order to determine whether it adheres to the  $\alpha$ -shape or not. In the system of classical  $\alpha$ -shapes, the triangle belongs to the shape since its circumsphere has a smaller radius than  $\alpha$  (as illustrated in the figure).

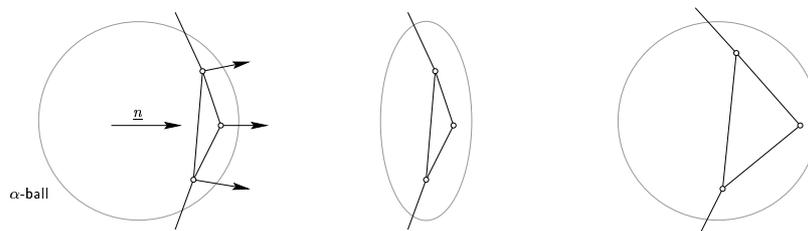


However in the system of anisotropic  $\alpha$ -shapes, we proceed as follows for this triangle, which we assume to have normals  $\underline{n}_1, \underline{n}_2, \underline{n}_3$  say. First, we choose among the vectors

$$\{s_1 \underline{n}_1 + s_2 \underline{n}_2 + s_3 \underline{n}_3 \mid s_1, s_2, s_3 \in \{\pm 1\}\}$$

the vector  $\underline{n}$  which has maximal length. This gives us some kind of “average” over the involved normals  $\underline{n}_i$ . The second step is to normalize  $\underline{n}$  and scale it according to some user parameter  $\tau$  which controls the anisotropy. (For instance,  $\tau = 0$  would mean no anisotropy that is “classical”  $\alpha$ -shapes.) Finally, we compress the  $\alpha$ -ball used for the alpha-test along the axis  $\underline{n}$  as shown in middle of the three figures. Equivalently, we could say that we expanded the points as shown in the rightmost figure. Looking at this latter situation, we see that if we now perform the  $\alpha$ -test, the triangle is not accepted any more. That’s just what we wanted.

For triangles not spanning interstices however, the compression alters the  $\alpha$ -ball in a direction which doesn’t change the outcome of the  $\alpha$ -test, roughly. This is illustrated below.



## References

- [1] H. Edelsbrunner. Weighted alpha shapes. Technical Report UIUCDCS-R-92-1760, Dept. Comput. Sci., Univ. Illinois, Urbana, IL, 1992.
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- [4] H. Edelsbrunner and E. P. Mücke. Three-dimensional alpha shapes. *ACM Trans. Graph.*, 13(1):43–72, January 1994.
- [5] M. Teichmann and M. Capps. Surface reconstruction with anisotropic density-scaled alpha-shapes. In *IEEE Visualization '98 Proceedings*, pages 67–72, San Francisco, CA, October 1998. ACM/SIGGRAPH Press.

The pictures on page 2 and 5 have been taken from [3] and from Walter Luh's homepage at <http://www.stanford.edu/~wluh/cs448b/alphashapes.html>. The images in section 4 have been taken from [5].

Kaspar Fischer, [kfischer@iic.ethz.ch](mailto:kfischer@iic.ethz.ch)—26. V. 2000