

# Monte Carlo I

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## Previous lecture

- Analytical illumination formula

## This lecture

- Review random variables and probability
- Monte Carlo integration
- Sampling from distributions
- Sampling from shapes
- Variance and efficiency

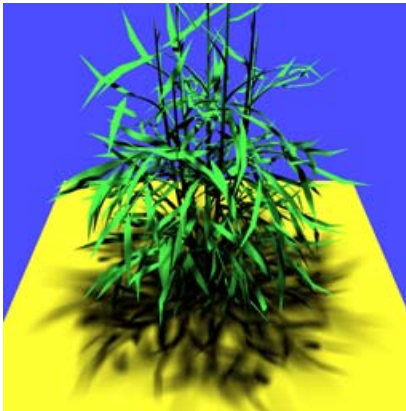
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# Lighting and Soft Shadows

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$$E(x) = \int_{H^2} L_i(x, \omega) \cos \theta d\omega$$



## Challenges

- Visibility and blockers
- Varying light distribution
- Complex source geometry

Source: Agrawala, Ramamoorthi, Heirich, Moll, 2000

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# Monte Carlo Algorithms

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## Advantages

- Easy to implement
- Easy to think about (but be careful of biasing!)
- Robust when used with complex integrands and domains (shapes, lights, ...)
- Efficient for high dimensional integrals
- Efficient solution method for a few selected points

## Disadvantages

- Noisy
- Slow (many samples needed for convergence)

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# Random Variables

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## Random variables

$X$  is chosen by some random process

$X \sim p(x)$  probability distribution function (PDF)

## Expected values

$$E[f] \equiv \int_0^1 f(x)p(x) dx$$

## Properties

$$E[\sum_i f_i] = \sum_i E[f_i]$$

$$E[af] = aE[f]$$

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# Monte Carlo Integration

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**Definite integral**  $I(f) = \int_0^1 f(x) dx$

**Expectation of  $f$**   $E[f] = \int_0^1 f(x)p(x) dx$

**Random variables**  $X_i \sim p(x)$   
 $Y_i = f(X_i)$

**Estimator**  $F_N = \frac{1}{N} \sum_{i=1}^N Y_i$

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# Unbiased Estimator

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$E[F_N] = I(f)$

**Properties**

$$E[\sum_i Y_i] = \sum_i E[Y_i]$$

$$E[aY] = aE[Y]$$

$$\begin{aligned} E[F_N] &= E\left[\frac{1}{N} \sum_{i=1}^N Y_i\right] \\ &= \frac{1}{N} \sum_{i=1}^N E[Y_i] = \frac{1}{N} \sum_{i=1}^N E[f(X_i)] \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^1 f(x)p(x) dx \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^1 f(x) dx \\ &= \int_0^1 f(x) dx \end{aligned}$$

**Assume uniform probability distribution for now**

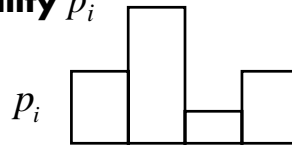
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## Discrete Probability Distributions

Discrete events  $X_i$  with probability  $p_i$

$$p_i \geq 0 \quad \sum_{i=1}^n p_i = 1$$



Cumulative PDF

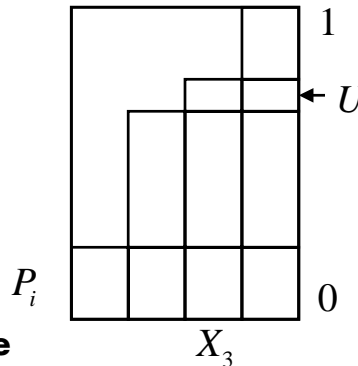
$$P_j = \sum_{i=1}^j p_i$$

Construction of samples

To randomly select an event,

Select  $X_i$  if  $P_{i-1} < U \leq P_i$

Uniform random variable



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## Continuous Probability Distributions

PDF  $p(x)$

$$p(x) \geq 0$$

CDF  $P(x)$

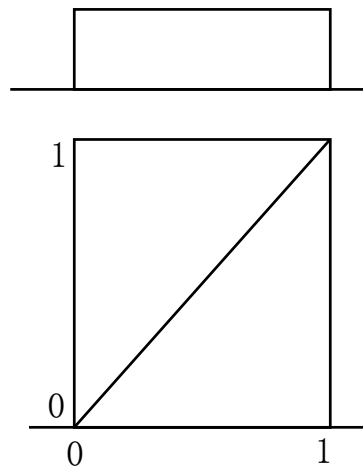
$$P(x) = \int_0^x p(x) dx$$

$$P(x) = \Pr(X < x) \quad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} p(x) dx$$

$$= P(\beta) - P(\alpha)$$

Uniform



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# Sampling Continuous Distributions

## Cumulative probability distribution function

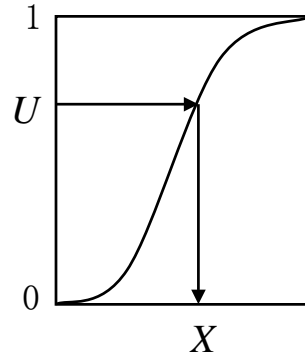
$$P(x) = \Pr(X < x)$$

## Construction of samples

$$\text{Solve for } X = P^{-1}(U)$$

## Must know:

1. The integral of  $p(x)$
2. The inverse function  $P^{-1}(x)$



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## Example: Power Function

### Assume

$$p(x) = (n+1)x^n$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

$$\int_0^1 x^n dx = \frac{x^{n+1}}{n+1} \Big|_0^1 = \frac{1}{n+1}$$

### Trick

$$Y = \max(U_1, U_2, \dots, U_n, U_{n+1})$$

$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U_i < x) = x^{n+1}$$

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## Sampling a Circle

$$A = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta = \int_0^1 r \, dr \int_0^{2\pi} d\theta = \left( \frac{r^2}{2} \right) \Big|_0^1 \theta \Big|_0^{2\pi} = \pi$$

$$p(r, \theta) \, dr \, d\theta = \frac{1}{\pi} r \, dr \, d\theta \Rightarrow p(r, \theta) = \frac{r}{\pi}$$

$$p(r, \theta) = p(r)p(\theta)$$

$$p(\theta) = \frac{1}{2\pi}$$

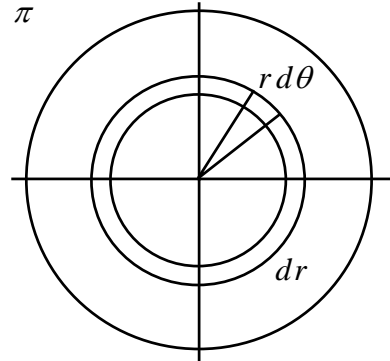
$$P(\theta) = \frac{1}{2\pi} \theta$$

$$p(r) = 2r$$

$$P(r) = r^2$$

$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$



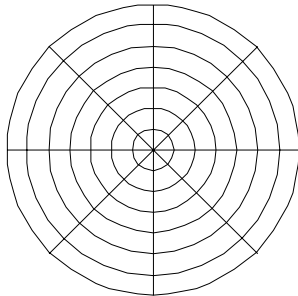
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## Sampling a Circle

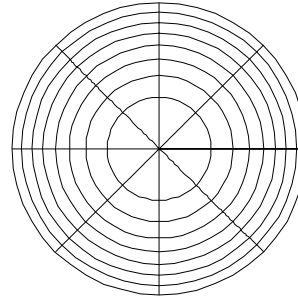
**WRONG**  $\neq$  Equi-Areal

**RIGHT** = Equi-Areal



$$\theta = 2\pi U_1$$

$$r = U_2$$



$$\theta = 2\pi U_1$$

$$r = \sqrt{U_2}$$

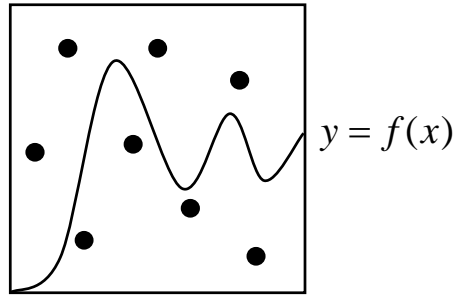
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## Rejection Methods

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$$I = \int_0^1 f(x) dx$$
$$= \iint_{y < f(x)} dx dy$$



### Algorithm

**Pick**  $U_1$  and  $U_2$

**Accept**  $U_1$  if  $U_2 < f(U_1)$

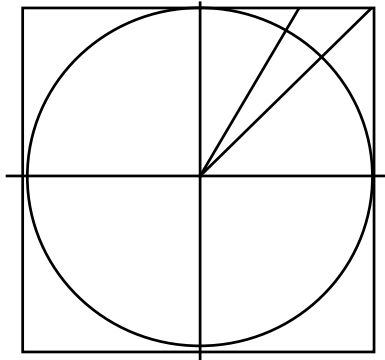
**Wasteful? Efficiency = Area / Area of rectangle**

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## Sampling a Circle: Rejection

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```
do {  
    X=1-2*U1  
    Y=1-2*U2  
while( X2+ Y2 >1 )
```

**May be used to pick random 2D directions**

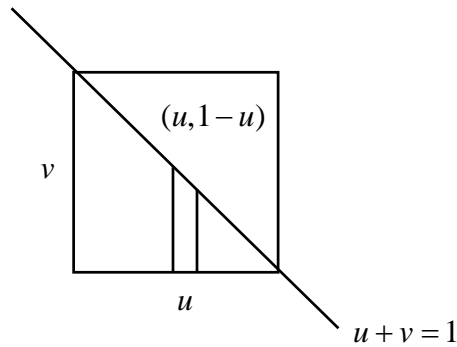
**Circle techniques may also be applied to the sphere**

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## Sampling a Triangle

$$\begin{aligned} u &\geq 0 \\ v &\geq 0 \\ u + v &\leq 1 \end{aligned}$$



$$A = \int_0^1 \int_0^{1-u} dv du = \int_0^1 (1-u) du = -\frac{(1-u)^2}{2} \Big|_0^1 = \frac{1}{2}$$

$$p(u, v) = 2$$

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## Sampling a Triangle

**Here  $u$  and  $v$  are not independent!**  $p(u, v) = 2$

**Conditional probability**

$$p(u) \equiv \int p(u, v) dv \quad p(u | v) \equiv \frac{p(u, v)}{p(u)}$$

$$p(u) = 2 \int_0^{1-u} dv = 2(1-u)$$

$$P(u_0) = \int_0^{u_0} 2(1-u) du = (1-u_0)^2$$

$$u_0 = 1 - \sqrt{U_1}$$

$$p(v | u) = \frac{1}{(1-u)}$$

$$v_0 = \sqrt{U_1} U_2$$

$$P(v_0 | u_0) = \int_0^{v_0} p(v | u_0) dv = \int_0^{v_0} \frac{1}{(1-u_0)} dv = \frac{v_0}{(1-u_0)}$$

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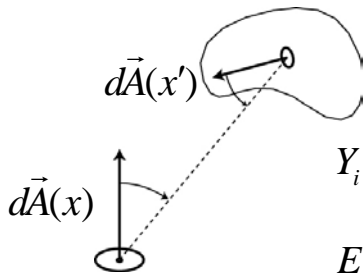
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## Direct Lighting - Area Sampling

$$E(x) = \int_{\Omega} L(x, \omega) \cos \theta d\omega$$

$$= \int_A L(x, \omega) \frac{\cos \theta \cos \theta'}{|x - x'|^2} V(x, x') dA(x')$$



Sample  $A$  uniformly by area

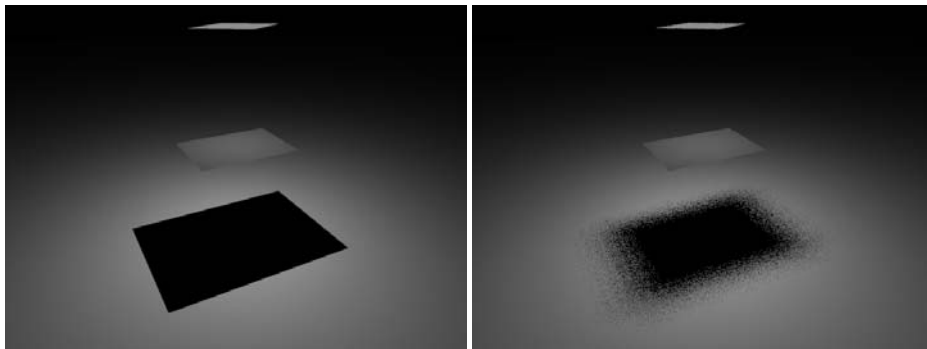
$$Y_i = L(x, x_i) \frac{\cos \theta \cos \theta'}{|x - x_i|^2} A V(x, x_i)$$

$$E(x) = \frac{1}{N} \sum_{i=1}^N Y_i$$

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## Examples



4 eye rays per pixel  
1 shadow ray per eye ray

Fixed

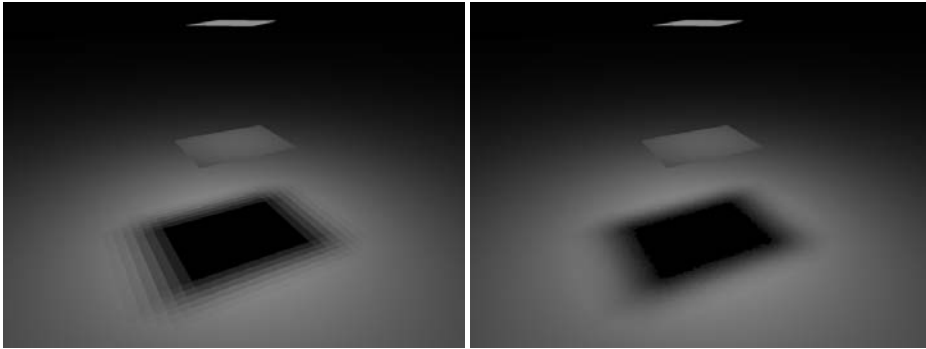
Random

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## Examples

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**4 eye rays per pixel  
16 shadow rays per eye ray**

**Uniform grid**

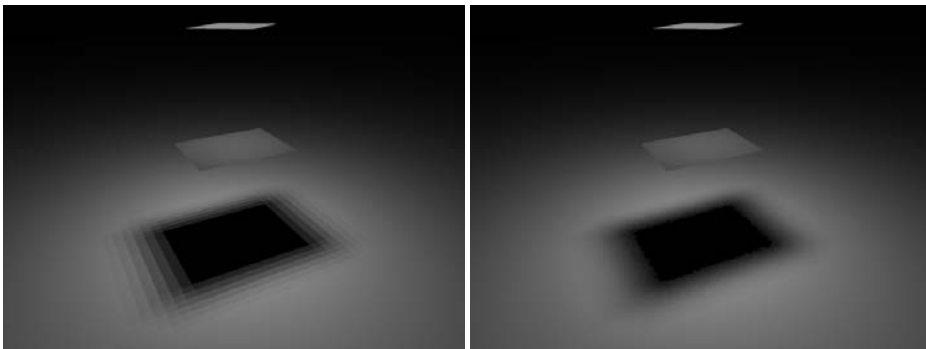
**Stratified random**

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## Examples

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**4 eye rays per pixel  
64 shadow rays per eye ray**

**Uniform grid**

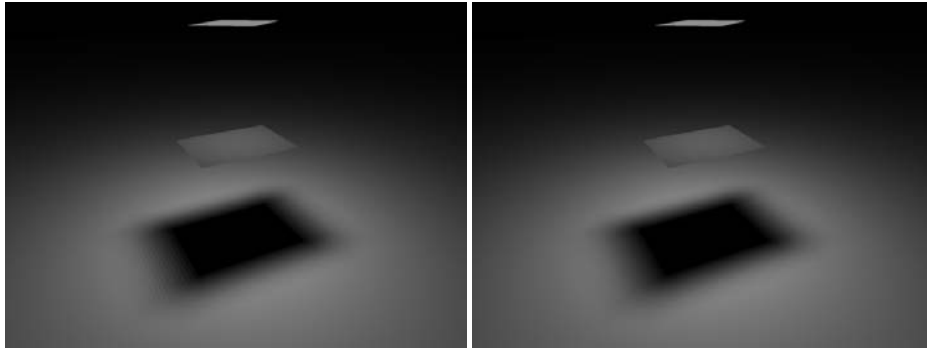
**Stratified random**

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## Examples

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**4 eye rays per pixel  
100 shadow rays per eye ray**

**Uniform grid**

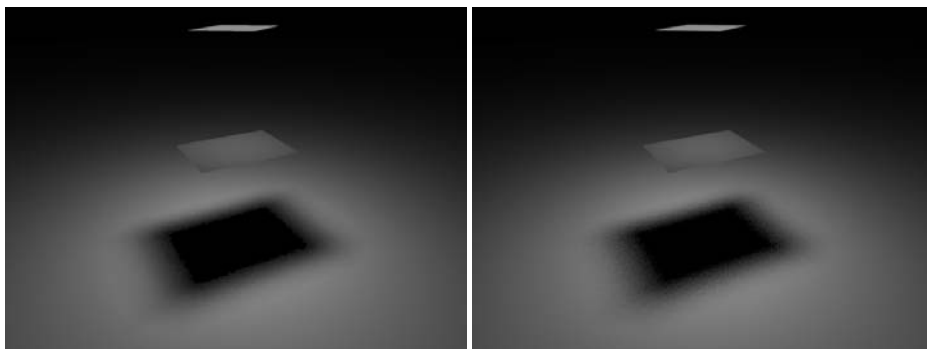
**Stratified random**

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## Examples

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**4 eye rays per pixel  
16 shadow rays per eye ray**

**64 eye rays per pixel  
1 shadow ray per eye ray**

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# Variance

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## Definition

$$\begin{aligned}V[Y] &\equiv E[(Y - E[Y])^2] \\ &= E[Y^2 - 2YE[Y] + E[Y]^2] \\ &= E[Y^2] - E[Y]^2\end{aligned}$$

## Properties

$$V[\sum_i Y_i] = \sum_i V[Y_i]$$

$$V[aY] = a^2V[Y]$$

## Variance decreases with sample size

$$V[\frac{1}{N} \sum_{i=1}^N Y_i] = \frac{1}{N^2} \sum_{i=1}^N V[Y_i] = \frac{1}{N} V[Y]$$

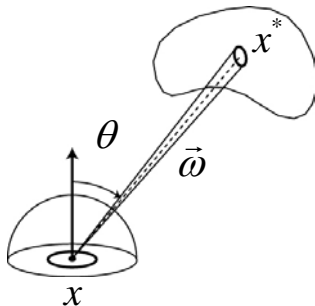
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# Direct Lighting - Directional Sampling

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$$E(x) = \int_{\Omega} L(x, \omega) \cos \theta d\omega$$



**Ray intersection**  $x^*(x, \omega)$

**Sample uniformly by**  $\Omega$

$$Y_i = L(x^*(x, \omega_i), x) \cos \theta \pi$$

**Sample uniformly by**  $\tilde{\Omega}$

$$Y_i = L(x^*(x, \omega_i), x) \pi$$

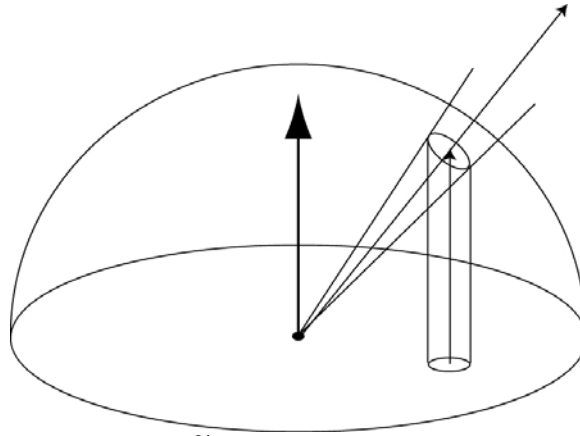
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# Sampling Projected Solid Angle

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Generate cosine weighted distribution



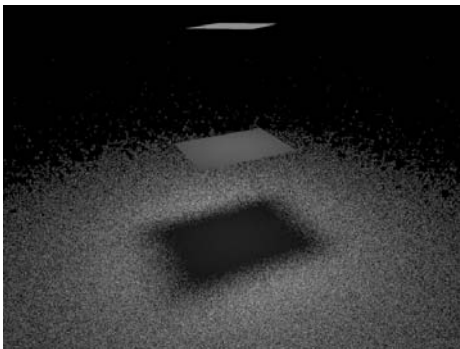
$$d\tilde{\Omega} = \cos \theta d\omega$$

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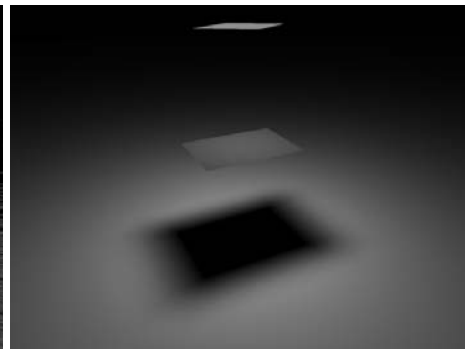
## Examples

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**Projected solid angle**

**4 eye rays per pixel  
100 shadow rays**



**Area**

**4 eye rays per pixel  
100 shadow rays**

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# Variance Reduction

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## Efficiency measure

$$\text{Efficiency} \propto \frac{1}{\text{Variance} \bullet \text{Cost}}$$

## Techniques

- Importance sampling
- Sampling patterns: stratified, ...