Sampling and Reconstruction

The sampling and reconstruction process

- Real world: continuous
- Digital world: discrete

Basic signal processing

- Fourier transforms
- The convolution theorem
- The sampling theorem

Aliasing and antialiasing

- Uniform supersampling
- Stochastic sampling

Imagers = Signal Sampling

All imagers convert a continuous image to a discrete sampled image by integrating over the active “area” of a sensor.

$$R = \int_T \int_\Omega \int \int_A L(x,\omega,t)P(x)S(t)\cos\theta \, dA \, d\omega \, dt$$

Examples:

- Retina: photoreceptors
- CCD array

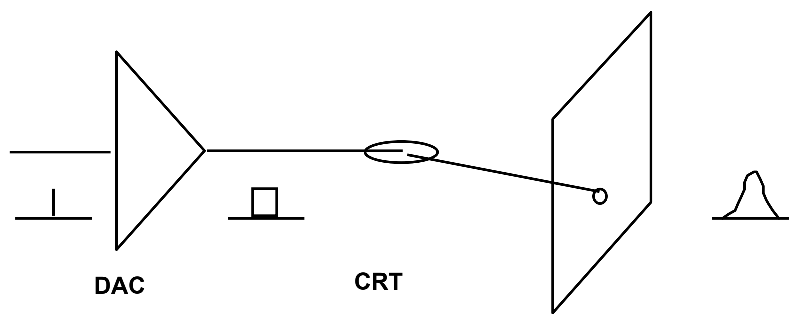
We propose to do this integral using Monte Carlo Integration

Displays = Signal Reconstruction

All physical displays recreate a continuous image from a discrete sampled image by using a finite sized source of light for each pixel.

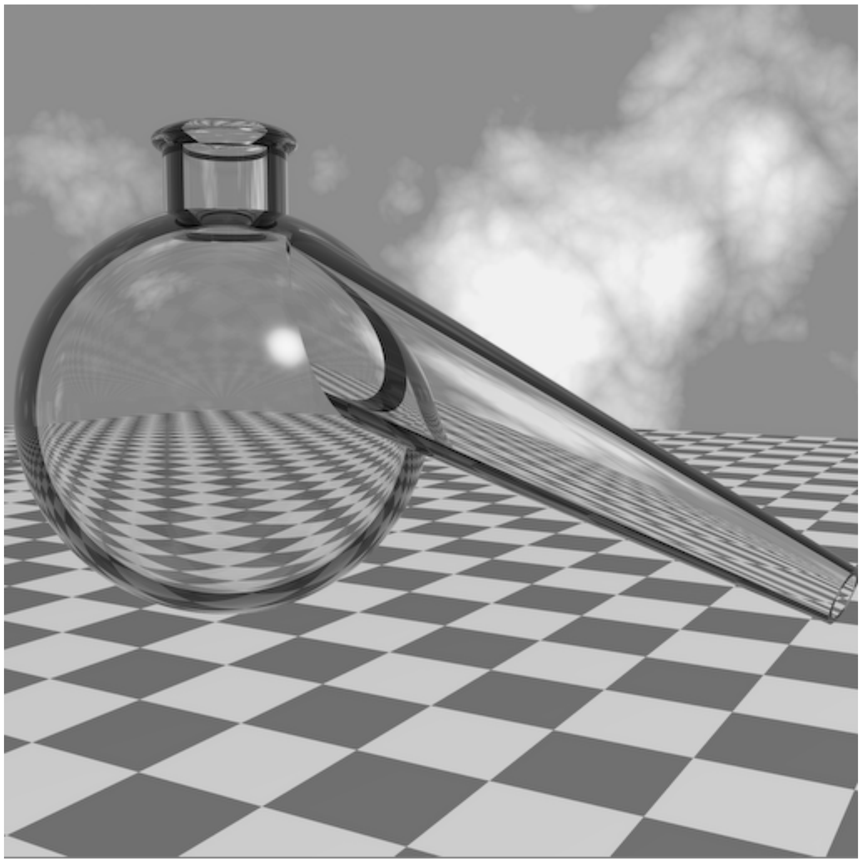
Examples:

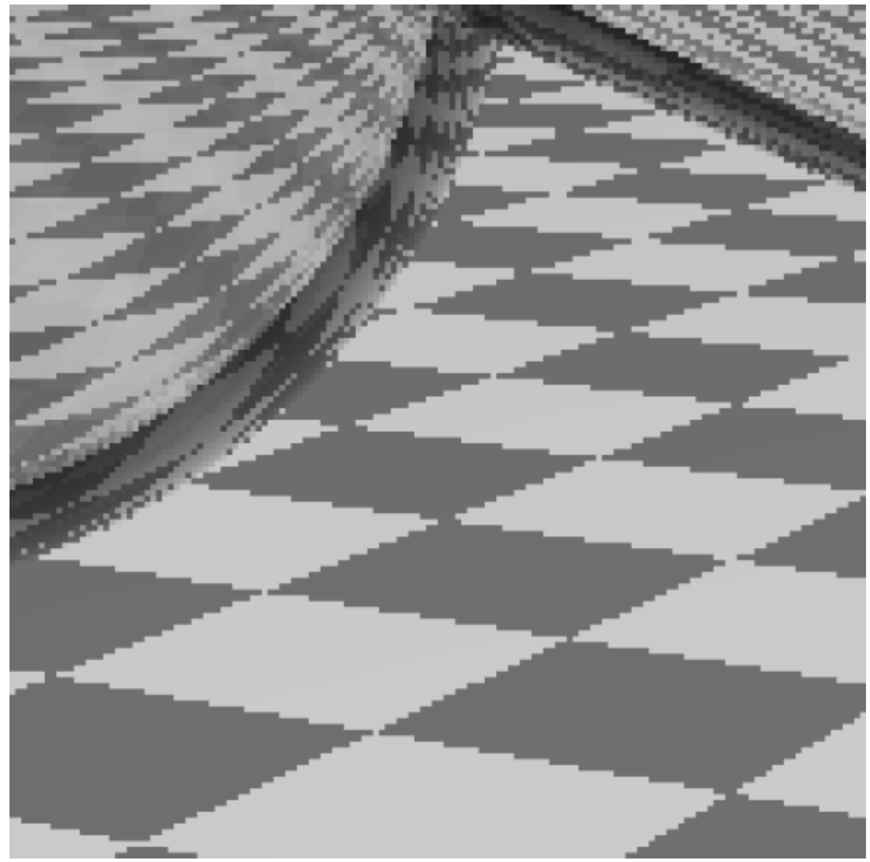
- DACs: sample and hold
- Cathode ray tube: phosphor spot and grid



Jaggies

Retort sequence by Don Mitchell





Staircase pattern or jaggies

Sampling in Computer Graphics

Artifacts due to sampling - Aliasing

- Jaggies
- Moire
- Flickering small objects
- Sparkling highlights
- Temporal strobing

Preventing these artifacts - Antialiasing

Basic Signal Processing

Fourier Transforms

Spectral representation treats the function as a weighted sum of sines and cosines

Each function has two representations

- Spatial domain - normal representation
- Frequency domain - spectral representation

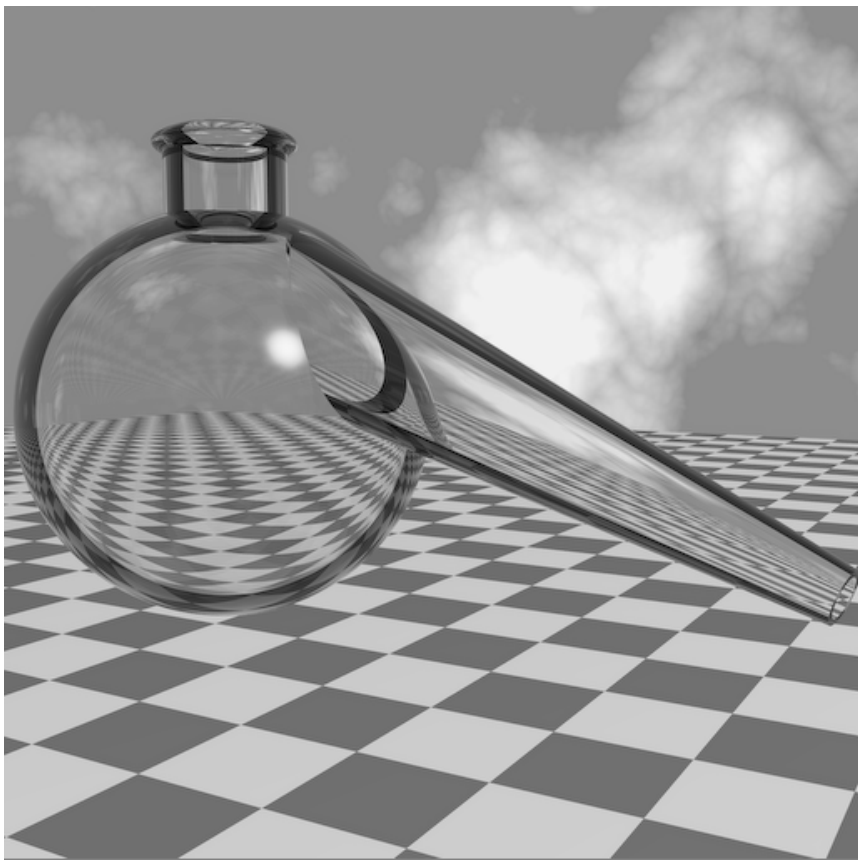
The *Fourier transform* converts between the spatial and frequency domain

Spatial Domain ⇒ $$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$ ⇒ Frequency Domain

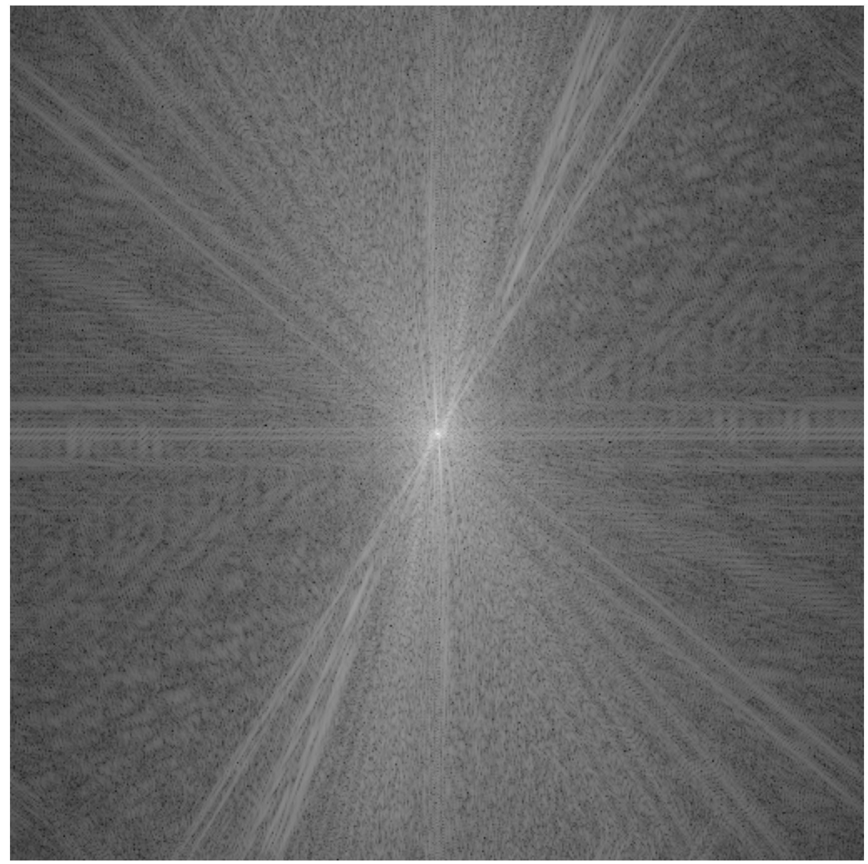
Spatial Domain ⇐ $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega$$ ⇐ Frequency Domain

Spatial and Frequency Domain

Spatial Domain

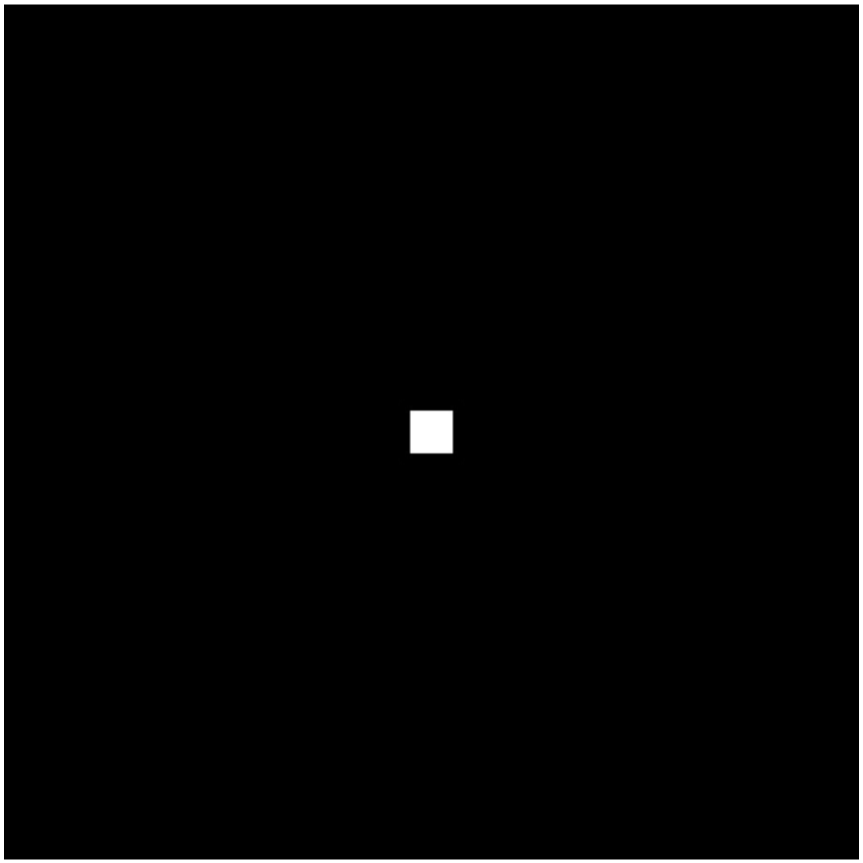


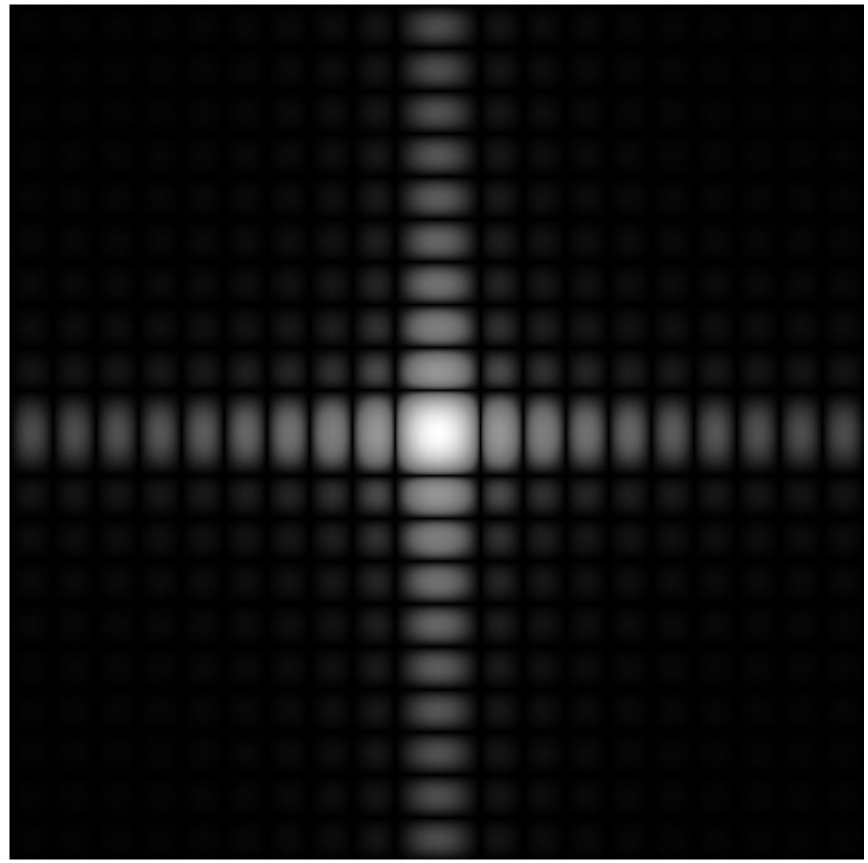
Frequency Domain



More Examples

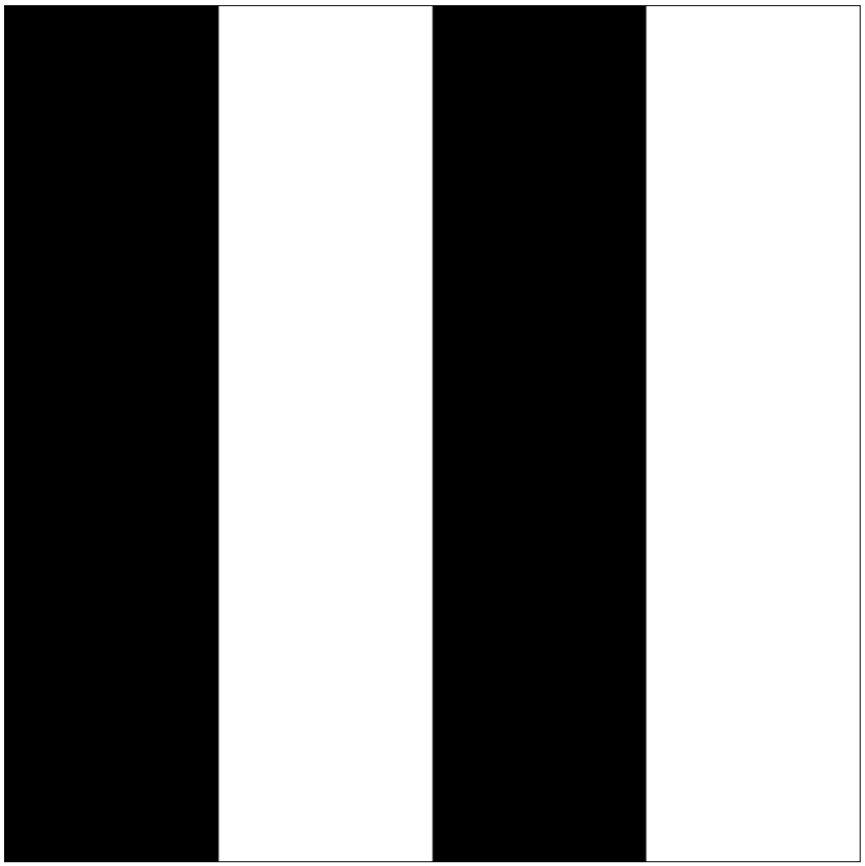
Spatial Domain | **Frequency Domain**



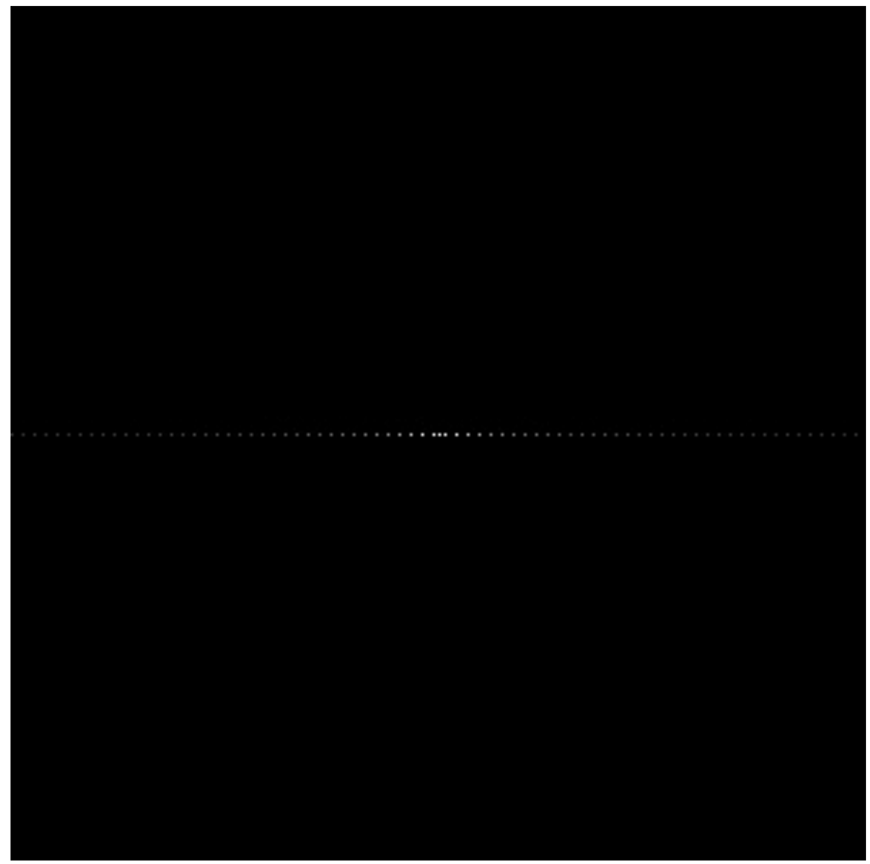


More Examples

Spatial Domain

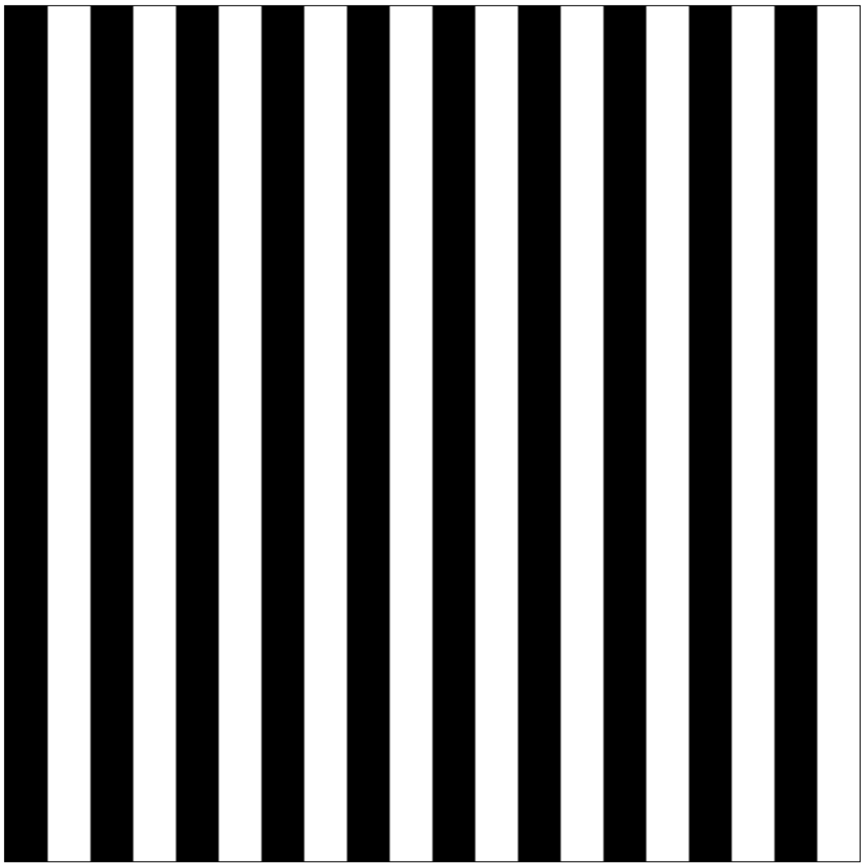


Frequency Domain

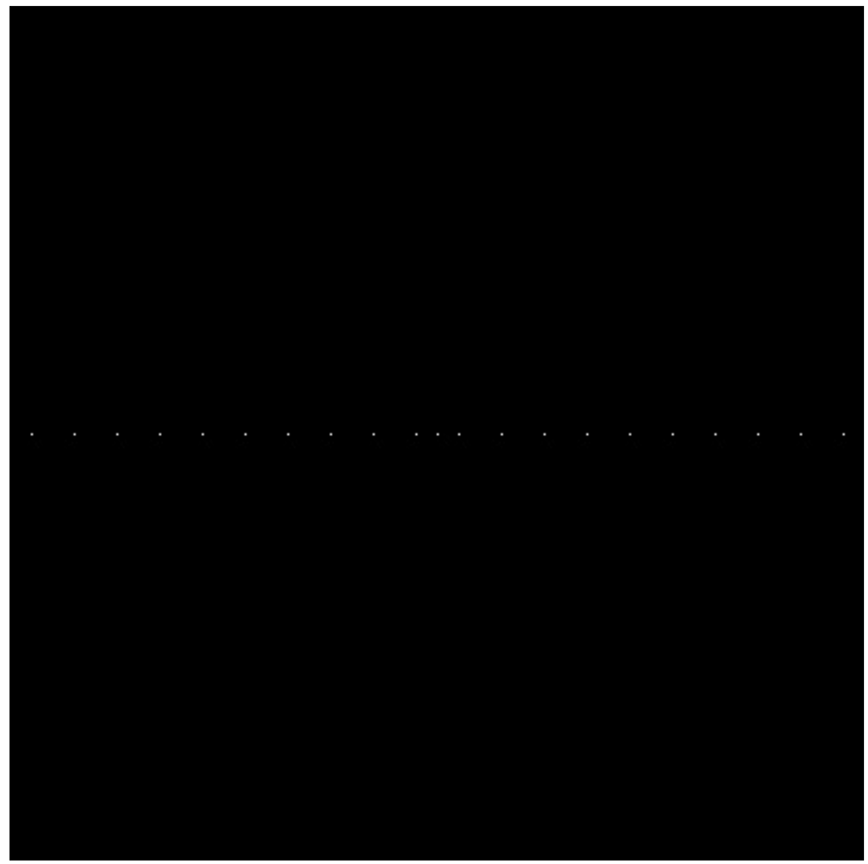


More Examples

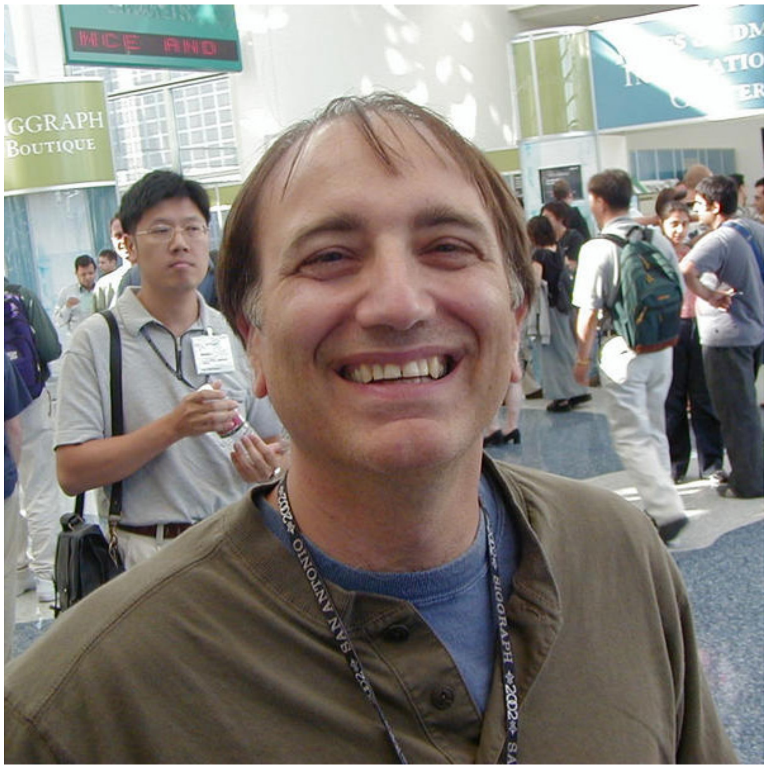
Spatial Domain

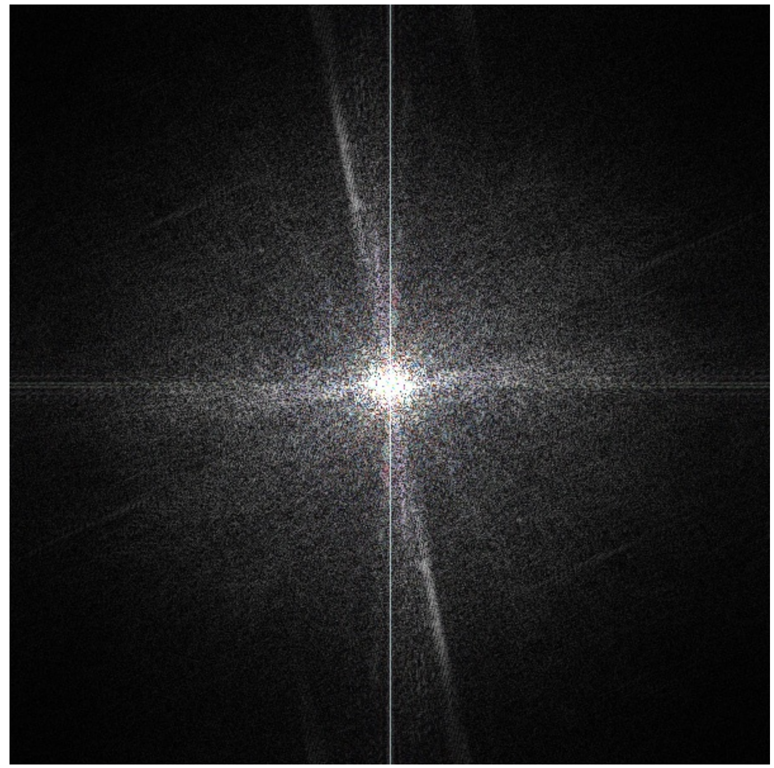


Frequency Domain

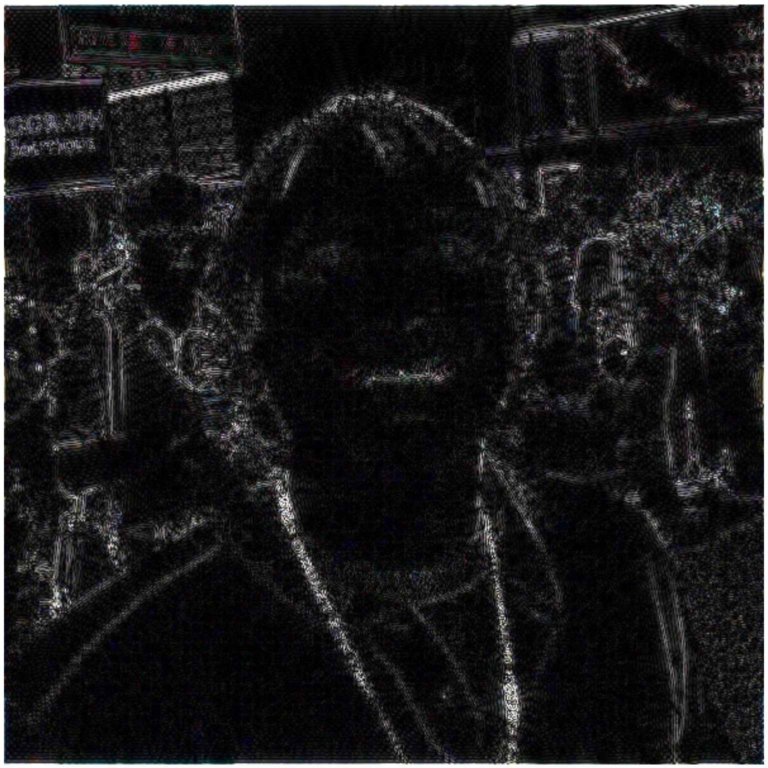


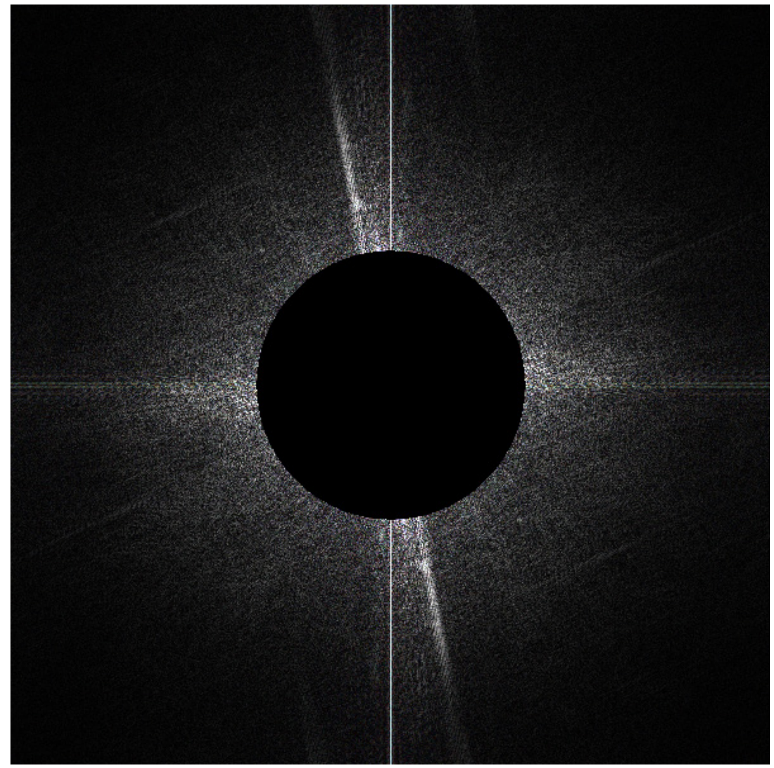
Pat’s Frequencies





Pat's Frequencies





Convolution

Definition

$$h(x) = f \otimes g = \int f(x') g(x - x')\, dx'$$

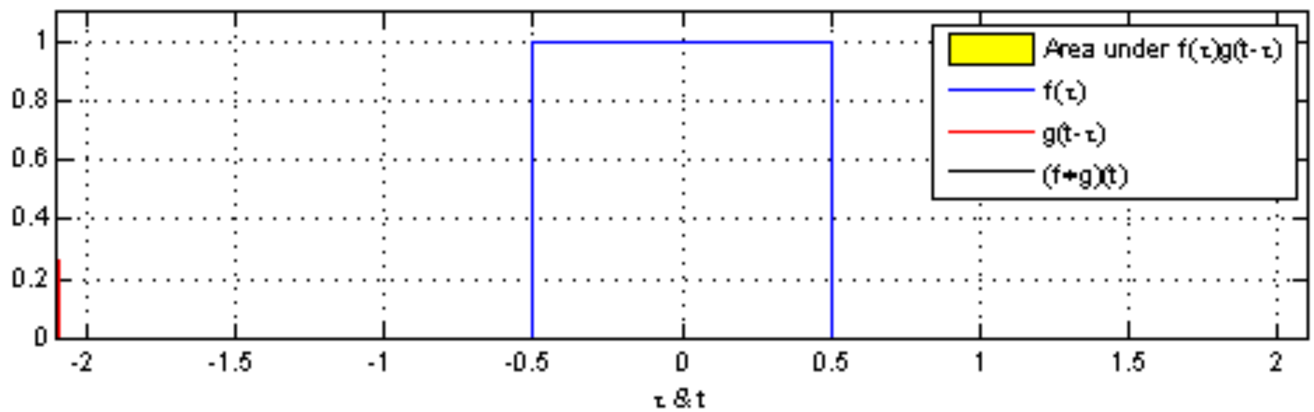
***Convolution Theorem*: Multiplication in the frequency domain is equivalent to convolution in the space domain.**

$$f \otimes g \leftrightarrow F \times G$$

***Symmetric Theorem*: Multiplication in the space domain is equivalent to convolution in the frequency domain.**

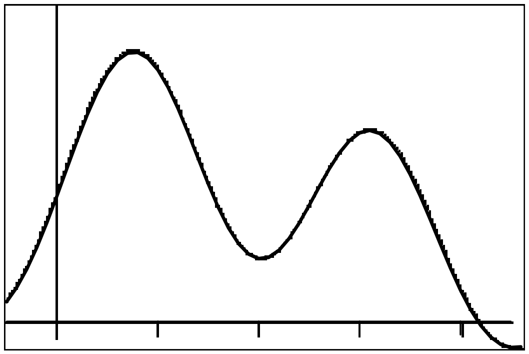
$$f \times g \leftrightarrow F \otimes G$$

Convolution

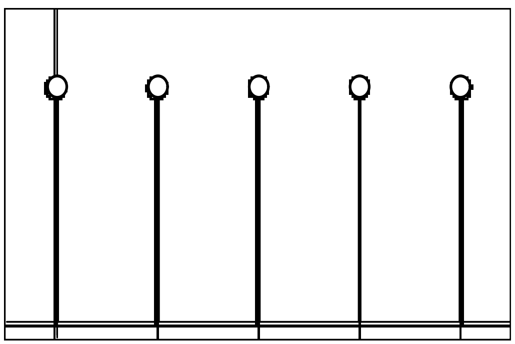


The Sampling Theorem

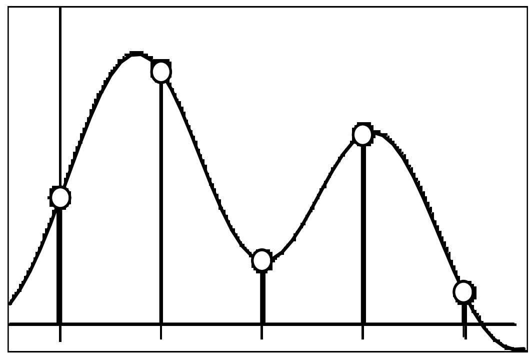
Sampling: Spatial Domain



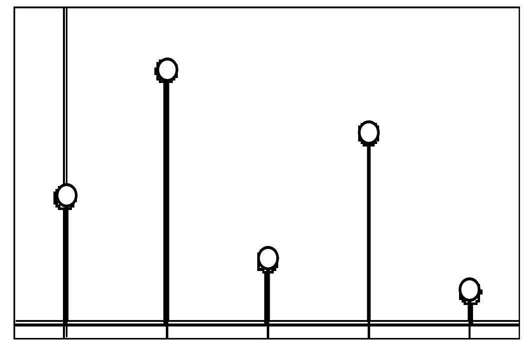
$f(x)$



$\times$



$=$

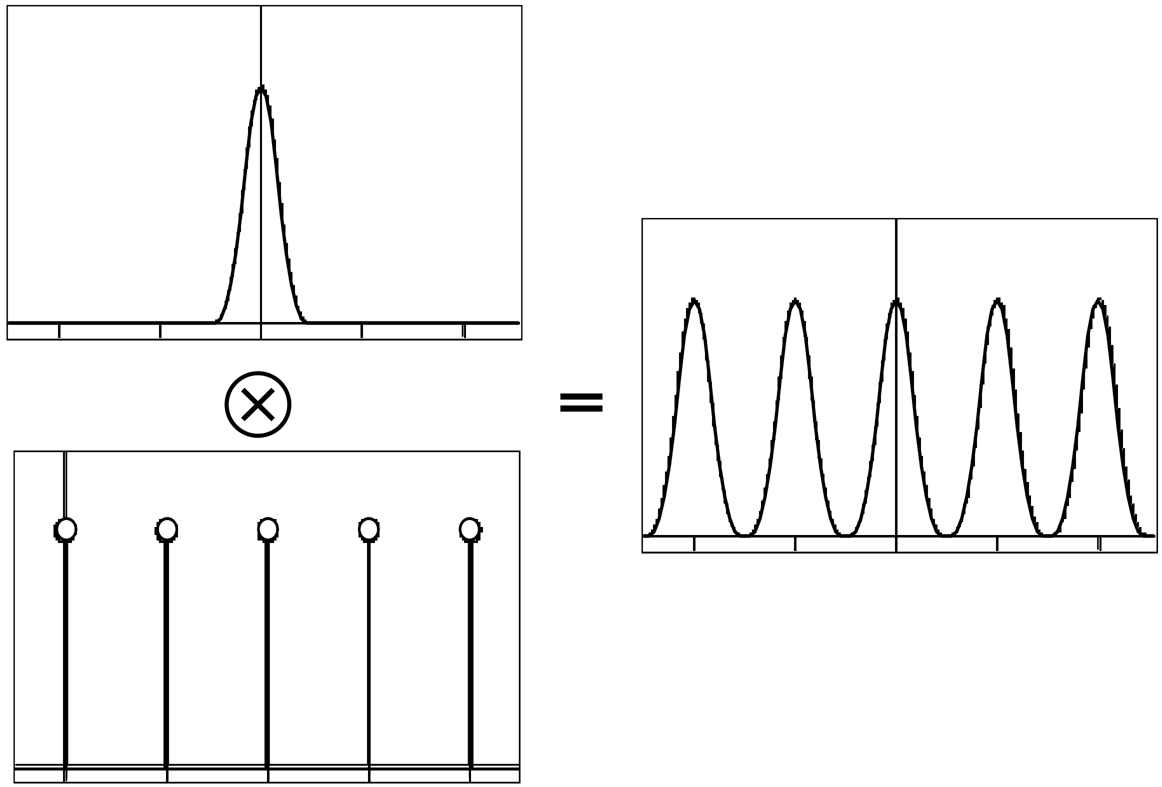


$$\sum_{n=-\infty}^{\infty} \delta(x - nT) f(nT)$$

$$\text{III}(x) = \sum_{n=-\infty}^{n=\infty} \delta(x - nT)$$

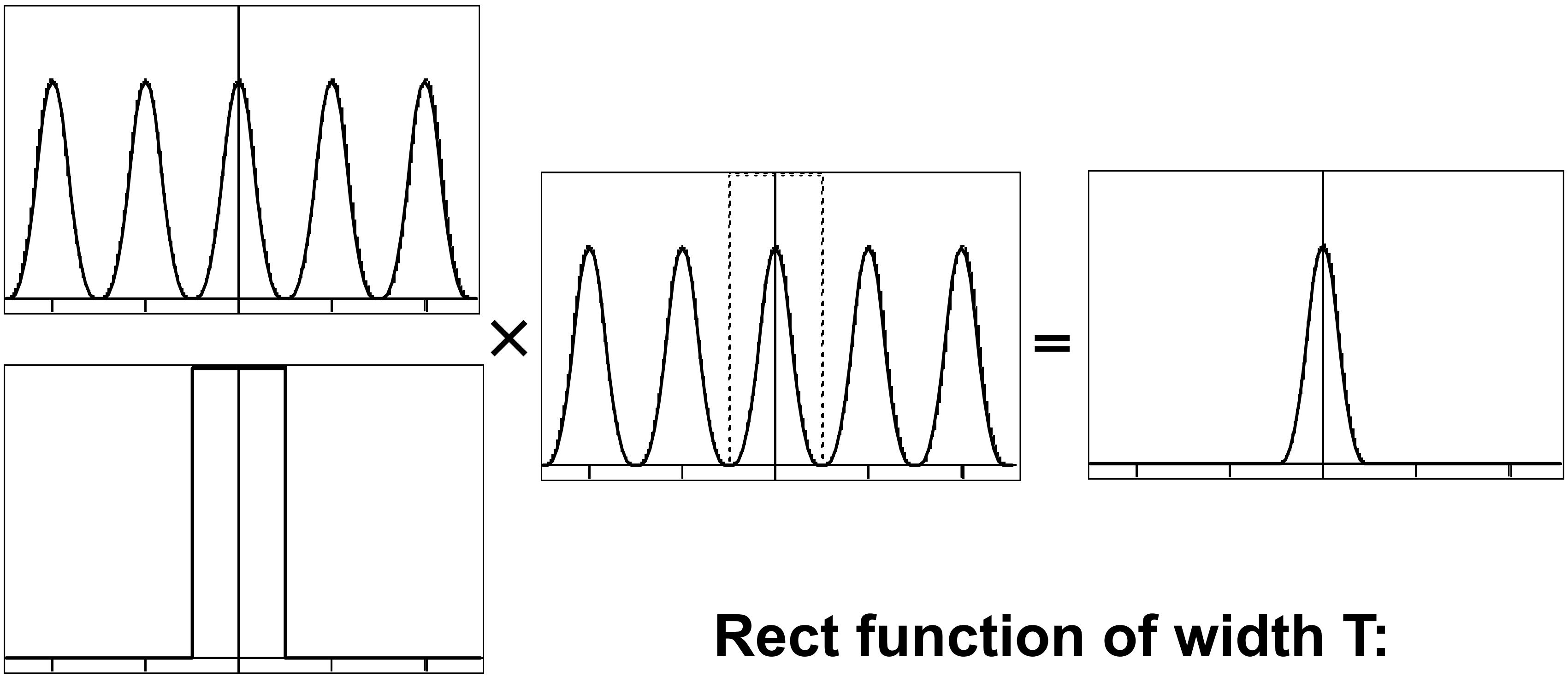
Sampling: Frequency Domain

$$F(\omega)$$



$$III_{1/T}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - n/T)$$

Reconstruction: Frequency Domain

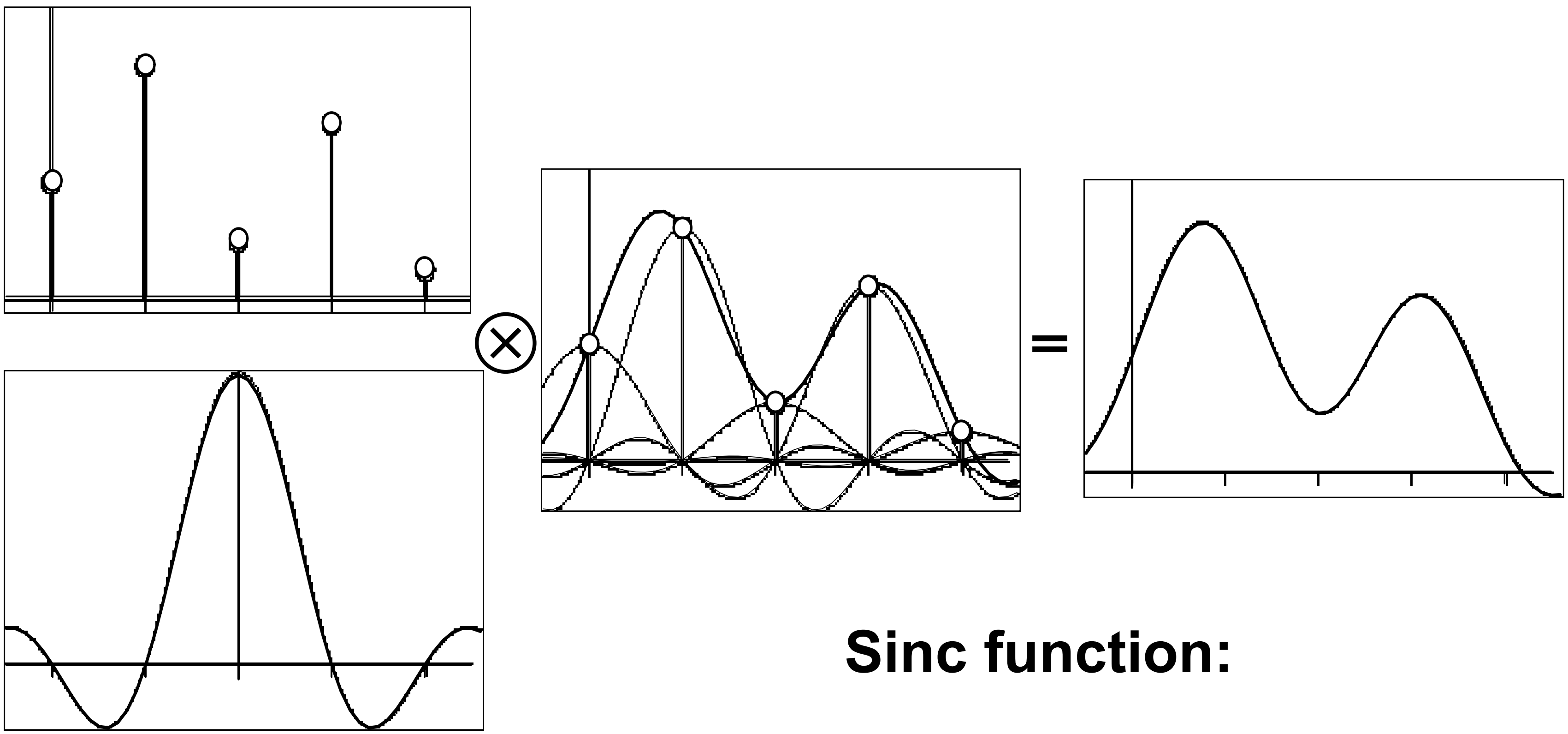


$$\prod_{1/T}(x)$$

Rect function of width T:

$$\prod_T(x) = \begin{cases} 1 & |x| \le \dfrac{T}{2} \\ 0 & |x| > \dfrac{T}{2} \end{cases}$$

Reconstruction: Spatial Domain

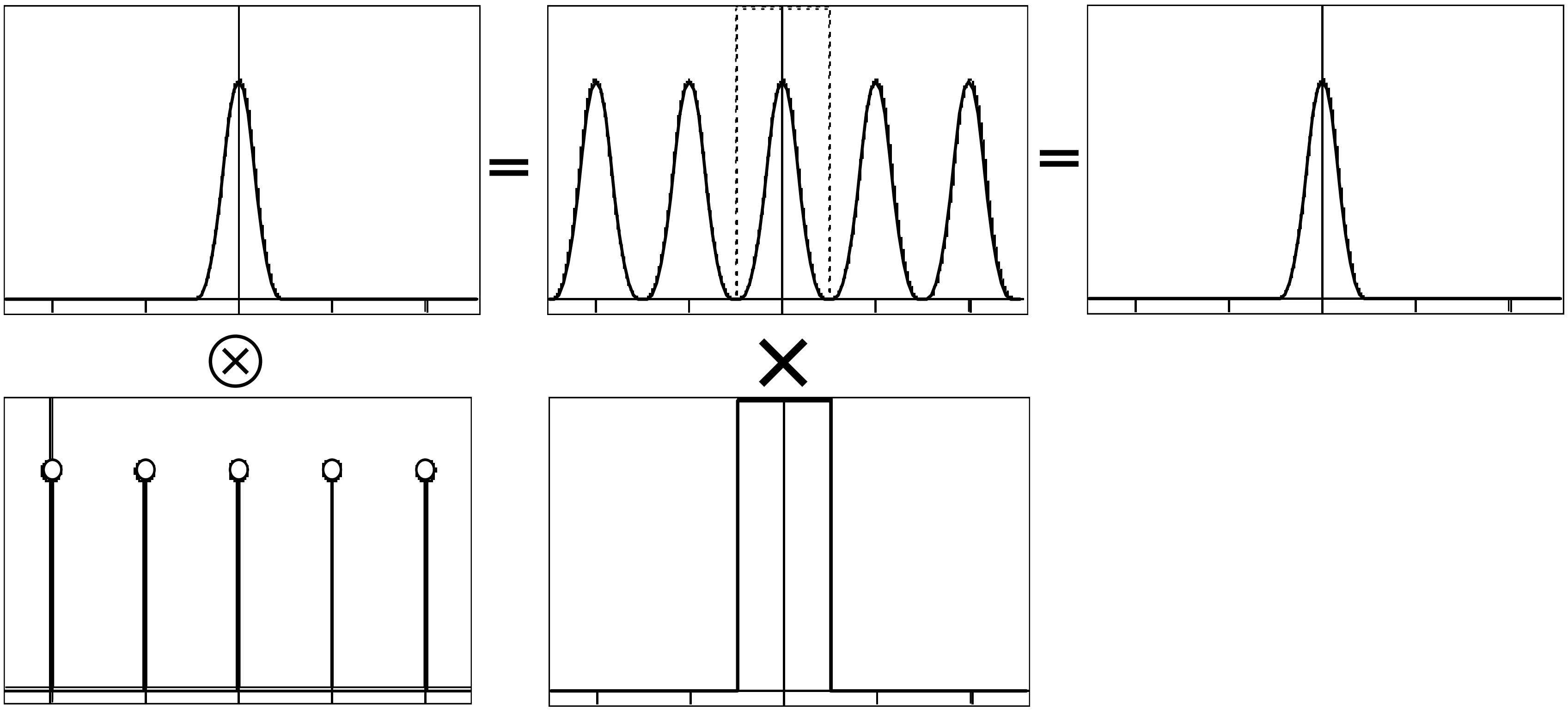


$$\operatorname{sinc}(x/T)$$

Sinc function:

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

Sampling and Reconstruction



Sampling Theorem

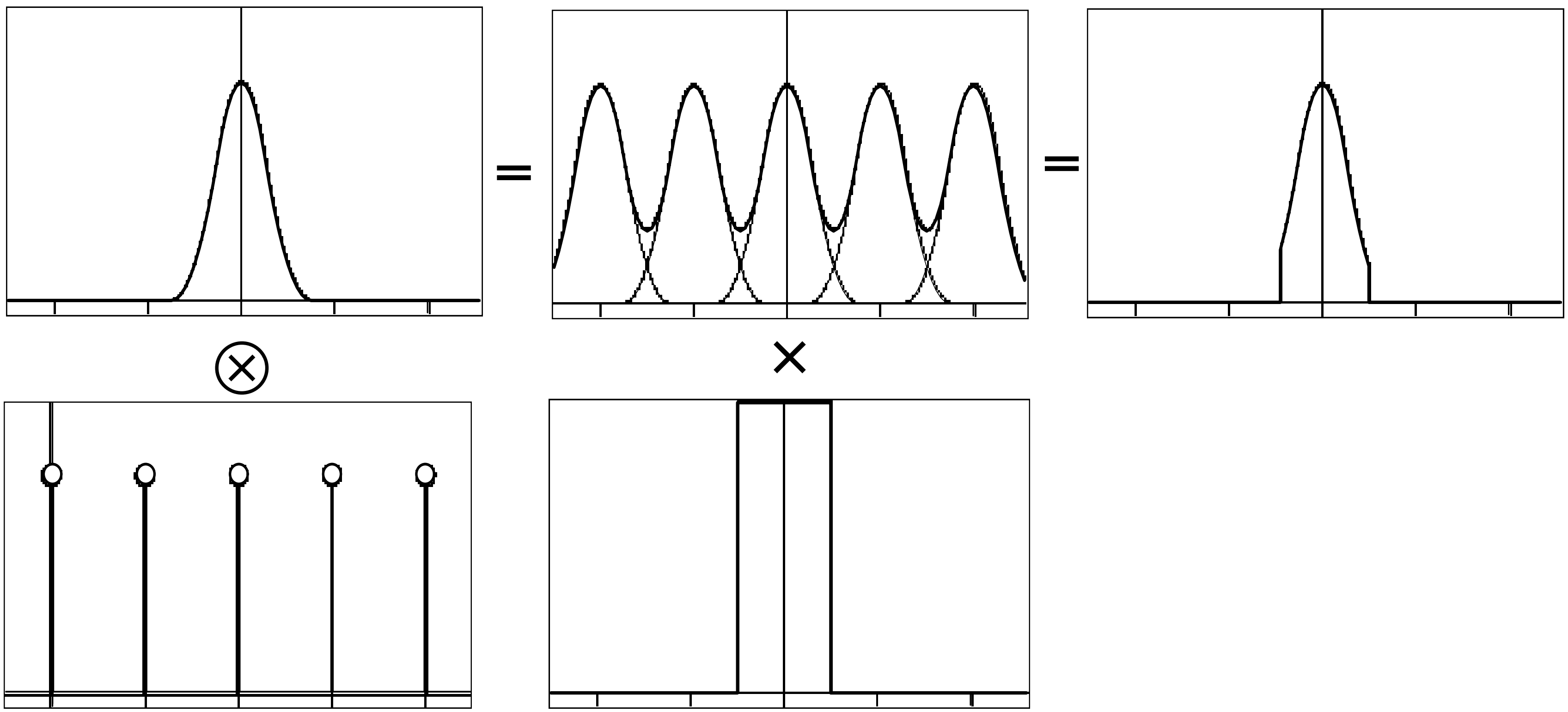
This result if known as the Sampling Theorem and is due to Claude Shannon who first discovered it in 1949

> A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above 1/2 the Sampling frequency

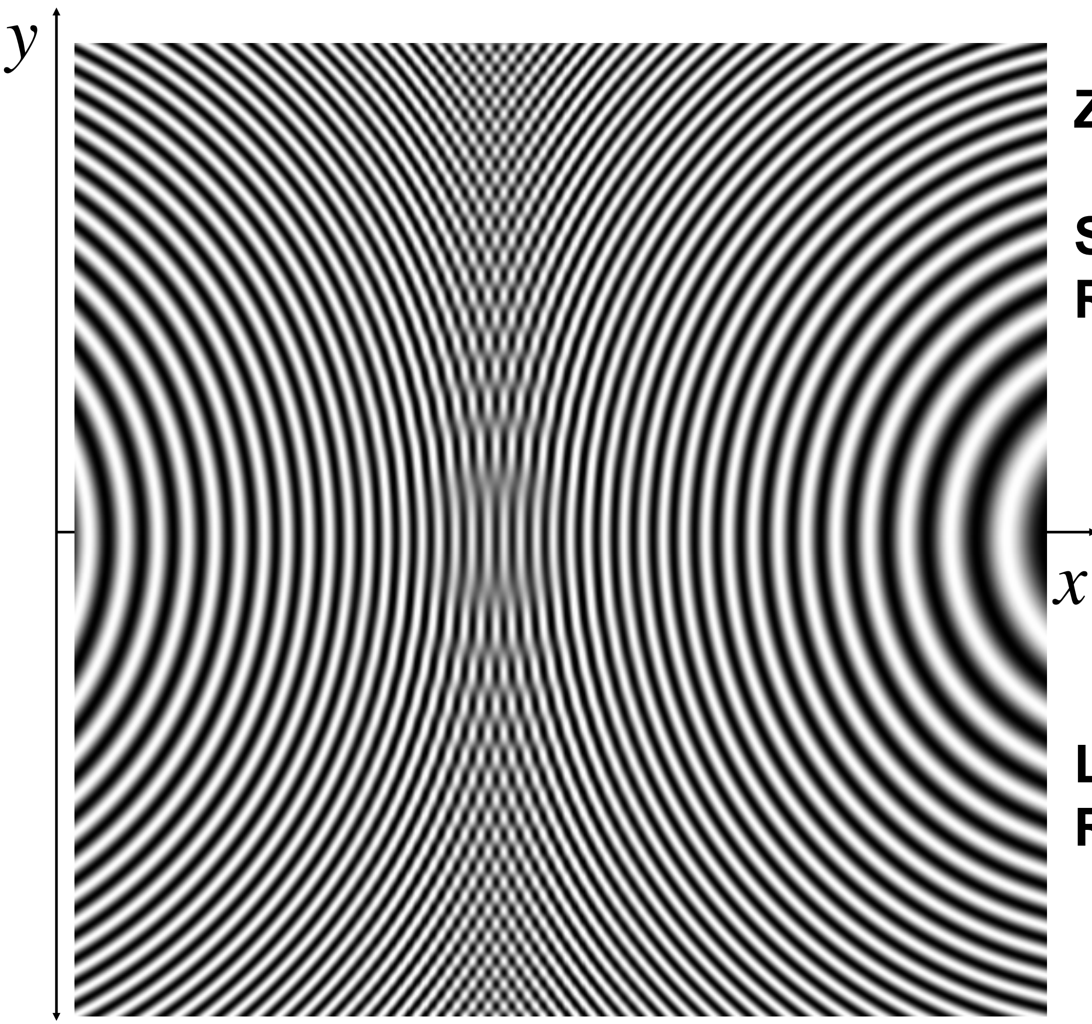
For a given bandlimited function, the rate it must be sampled is called the *Nyquist Frequency*

Aliasing

Undersampling: Aliasing



Sampling a "Zone Plate"



Zone plate: $\sin x^2 + y^2$

Sampled at 128x128
Reconstructed to 512x512
Using a 30-wide
Kaiser windowed sinc

Left rings: part of signal
Right rings: prealiasing

Ideal Reconstruction

Ideally, use a perfect low-pass filter - the sinc function - to bandlimit the sampled signal and thus remove all copies of the spectra introduced by sampling

Unfortunately,

- The sinc has infinite extent and we must use simpler filters with finite extents. Physical processes in particular do not reconstruct with sincs
- The sinc may introduce ringing which are perceptually objectionable

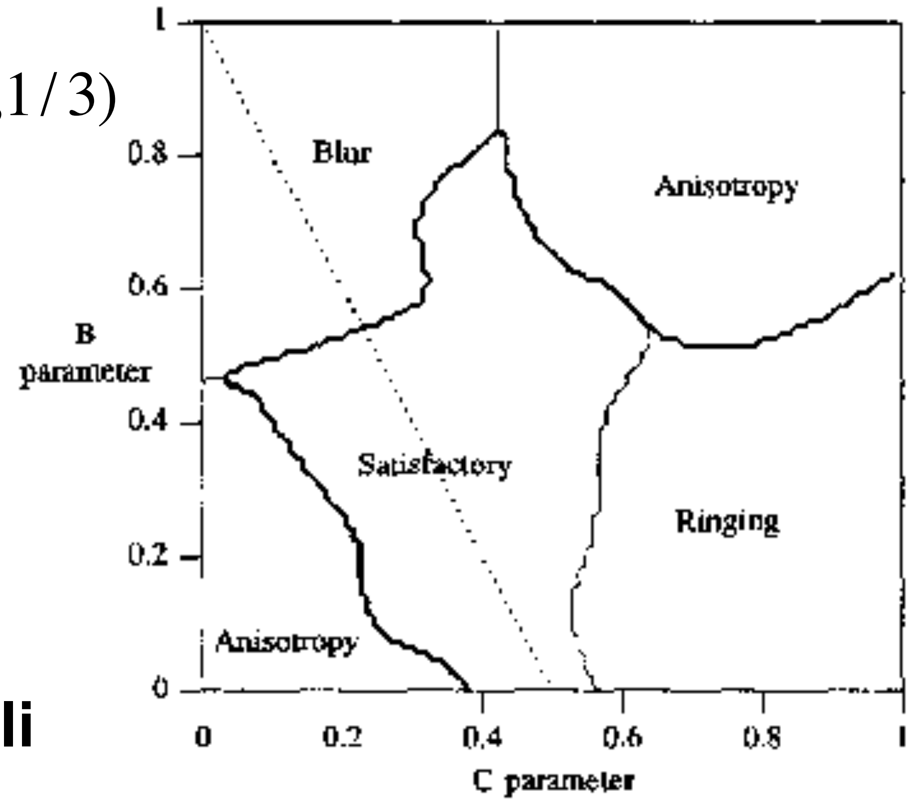
Aliasing

- Prealiasing: due to sampling under Nyquist rate
- Postaliasing: due to use of imperfect reconstruction filter

Mitchell Cubic Filter

$$h(x) = \frac{1}{6}\begin{cases} (12-9B-6C)x^3 + (-18+12B+6C)x^2 + (6-2B) & |x| < 1 \\ (-B-6C)x^3 + (6B+30C)x^2 + (-12B-48C)x + (8B+24C) & 1 < |x| < 2 \\ 0 & otherwise \end{cases}$$

$$\text{Good: } (1/3, 1/3)$$



Properties:

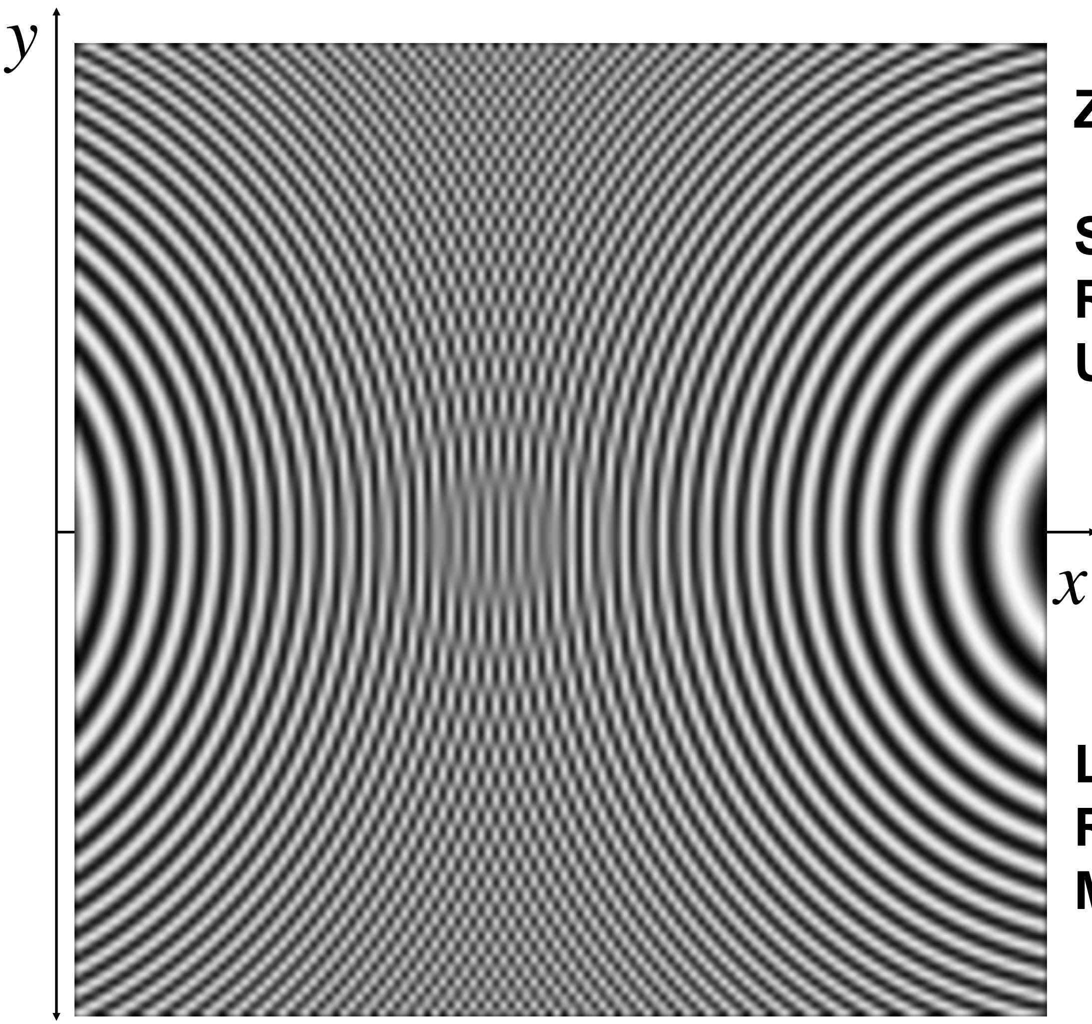
$$\sum_{n=-\infty}^{n=\infty} h(x) = 1$$

$$\text{B-spline: } (1,0)$$

$$\text{Catmull-Rom: } (0,1/2)$$

From Mitchell and Netravali

Sampling a "Zone Plate"



Zone plate: $\sin x^2 + y^2$

Sampled at 128x128
Reconstructed to 512x512
Using optimal cubic

Left rings: part of signal
Right rings: prealiasing
Middle rings: postaliasing

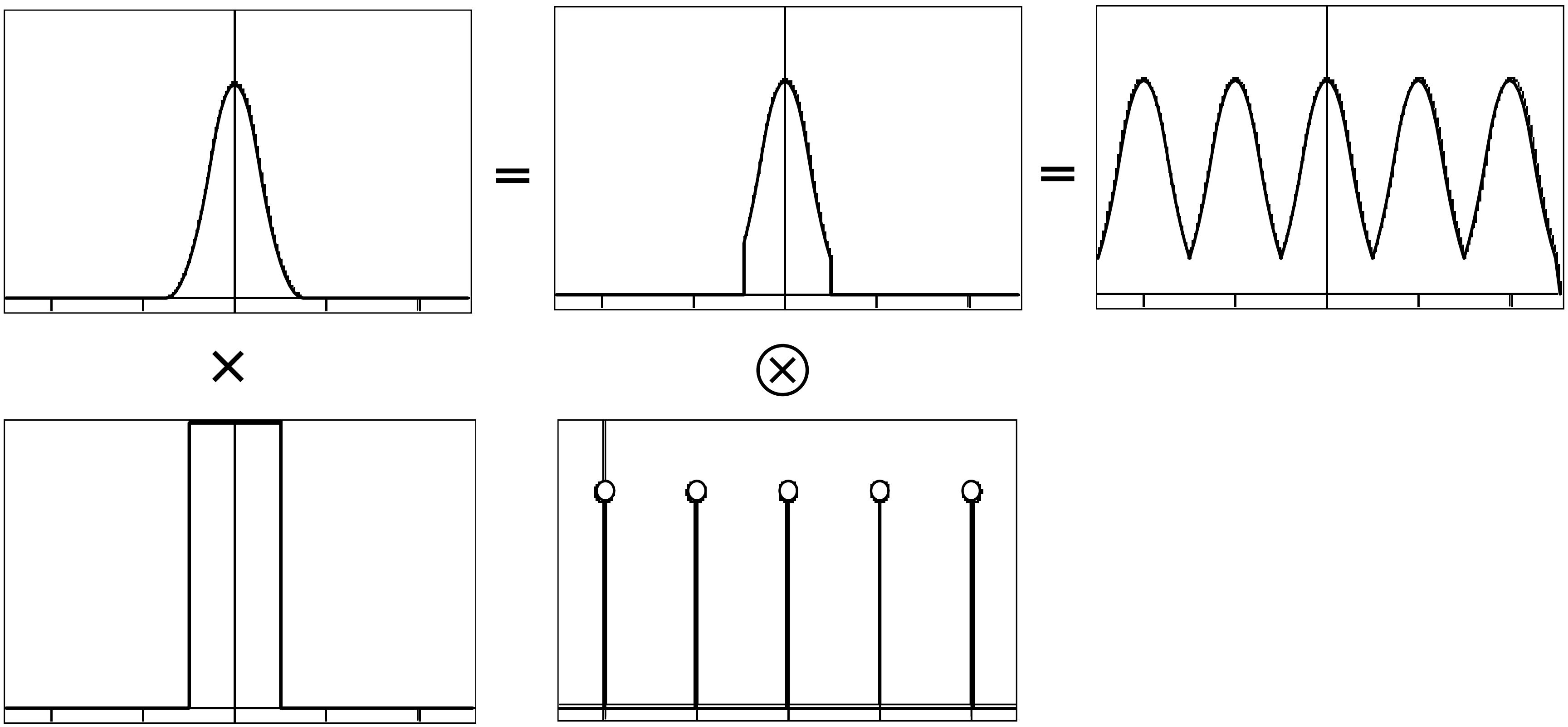
Antialiasing

Antialiasing

Antialiasing = Preventing aliasing

1. **Analytically prefilter the signal**
   - **Solvable for points, lines and polygons**
   - **Not solvable in general**
     e.g. procedurally defined images
2. **Uniform supersampling and resample**
3. **Nonuniform or stochastic sampling**

Antialiasing by Prefiltering

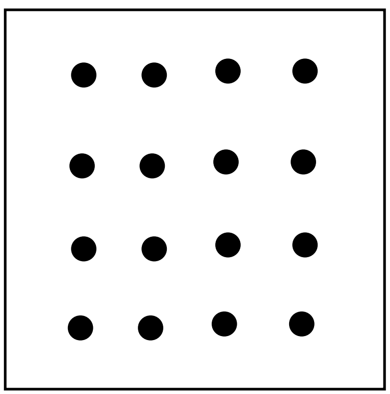


Frequency Space

Uniform Supersampling

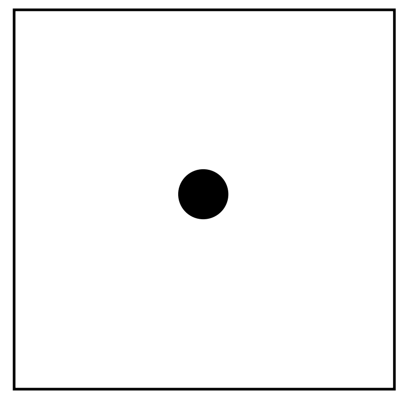
Increasing the sampling rate moves each copy of the spectra further apart, potentially reducing the overlap and thus aliasing

Resulting samples must be resampled (filtered) to image sampling rate



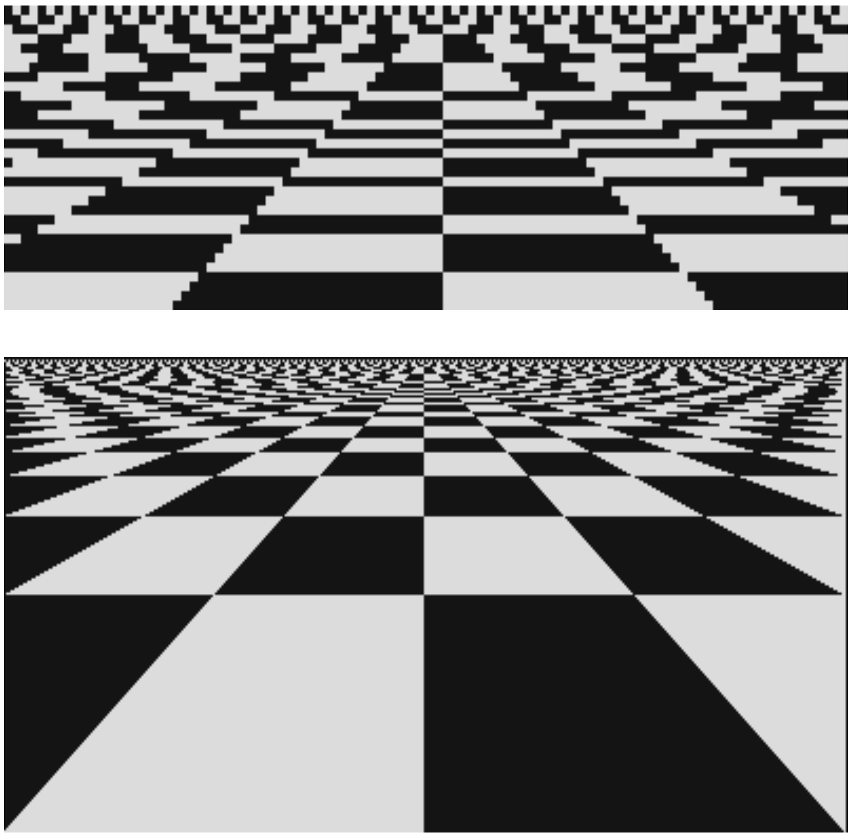
Samples

$$Pixel = \sum_s w_s \cdot Sample_s$$

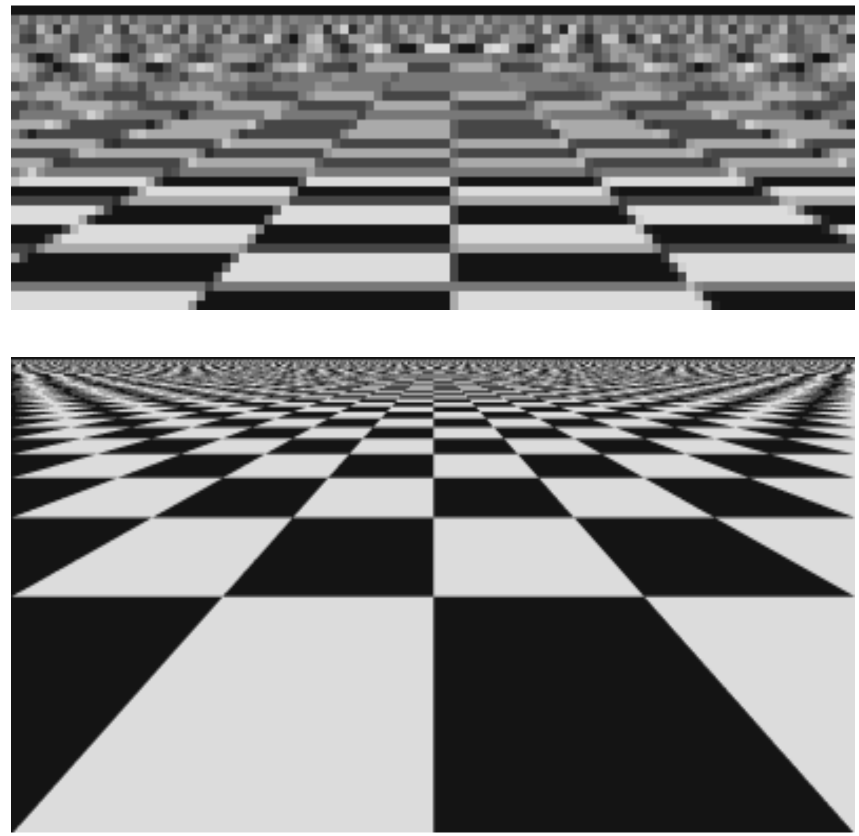


Pixel

Point vs. Supersampled



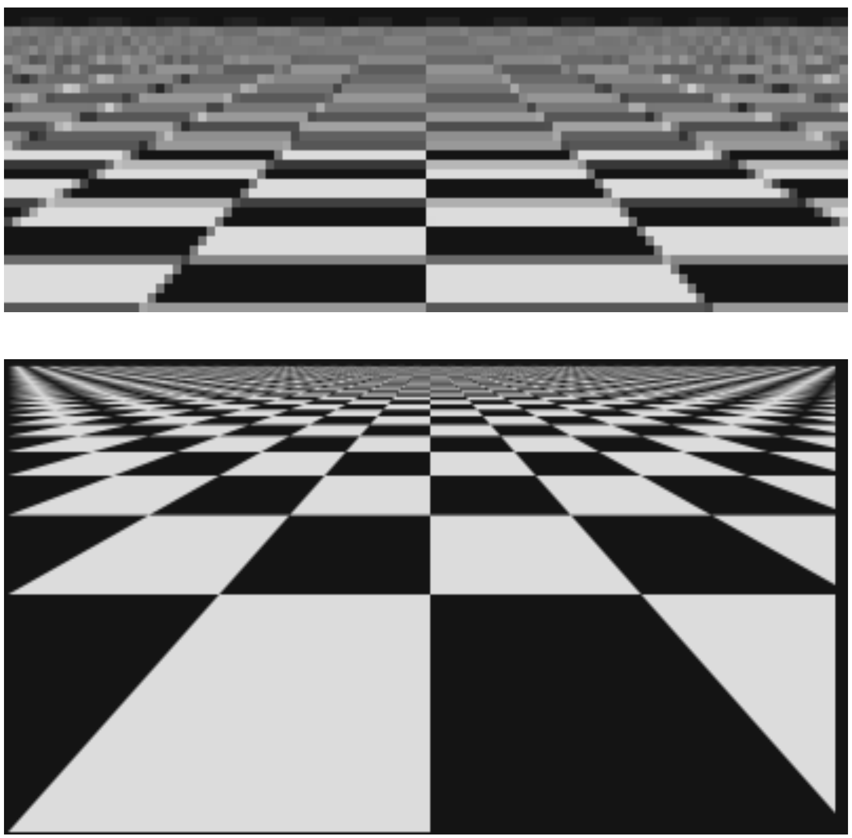
Point



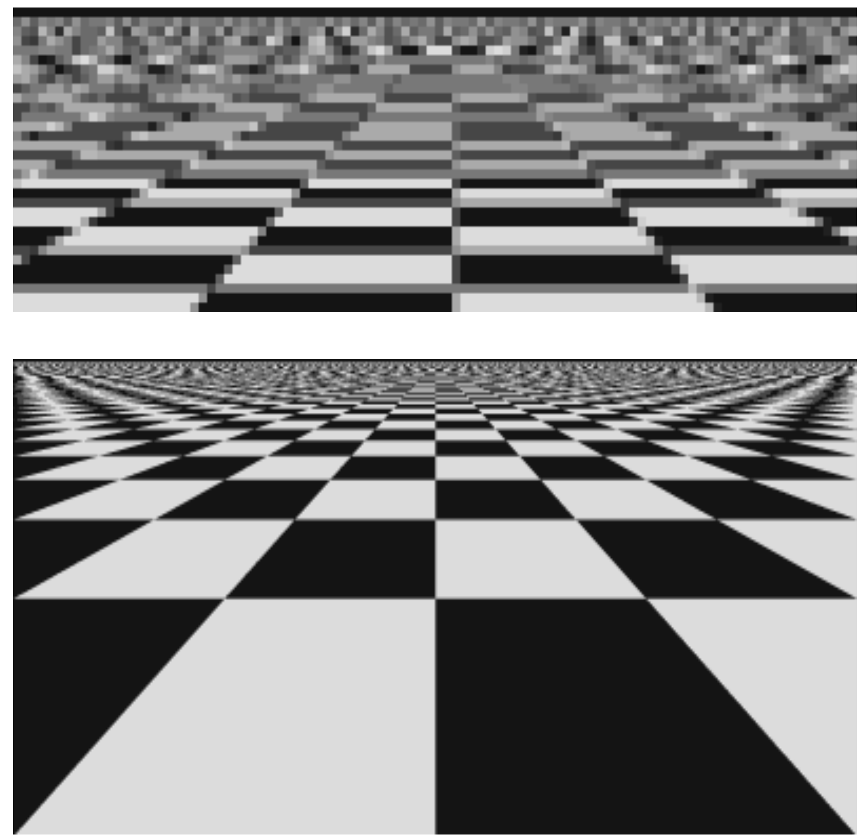
4x4 Supersampled

Checkerboard sequence by Tom Duff

Analytic vs. Supersampled

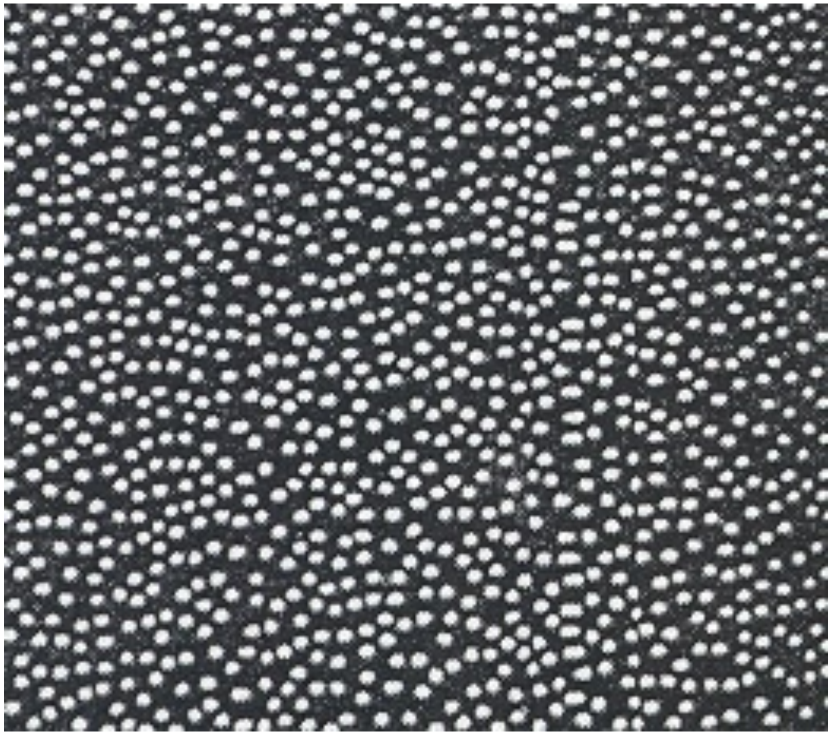


Exact Area



4x4 Supersampled

Distribution of Extrafoveal Cones



Monkey eye
cone distribution

Fourier transform

Yellot theory

- Aliases replaced by noise
- Visual system less sensitive to high freq noise

Non-uniform Sampling

Intuition

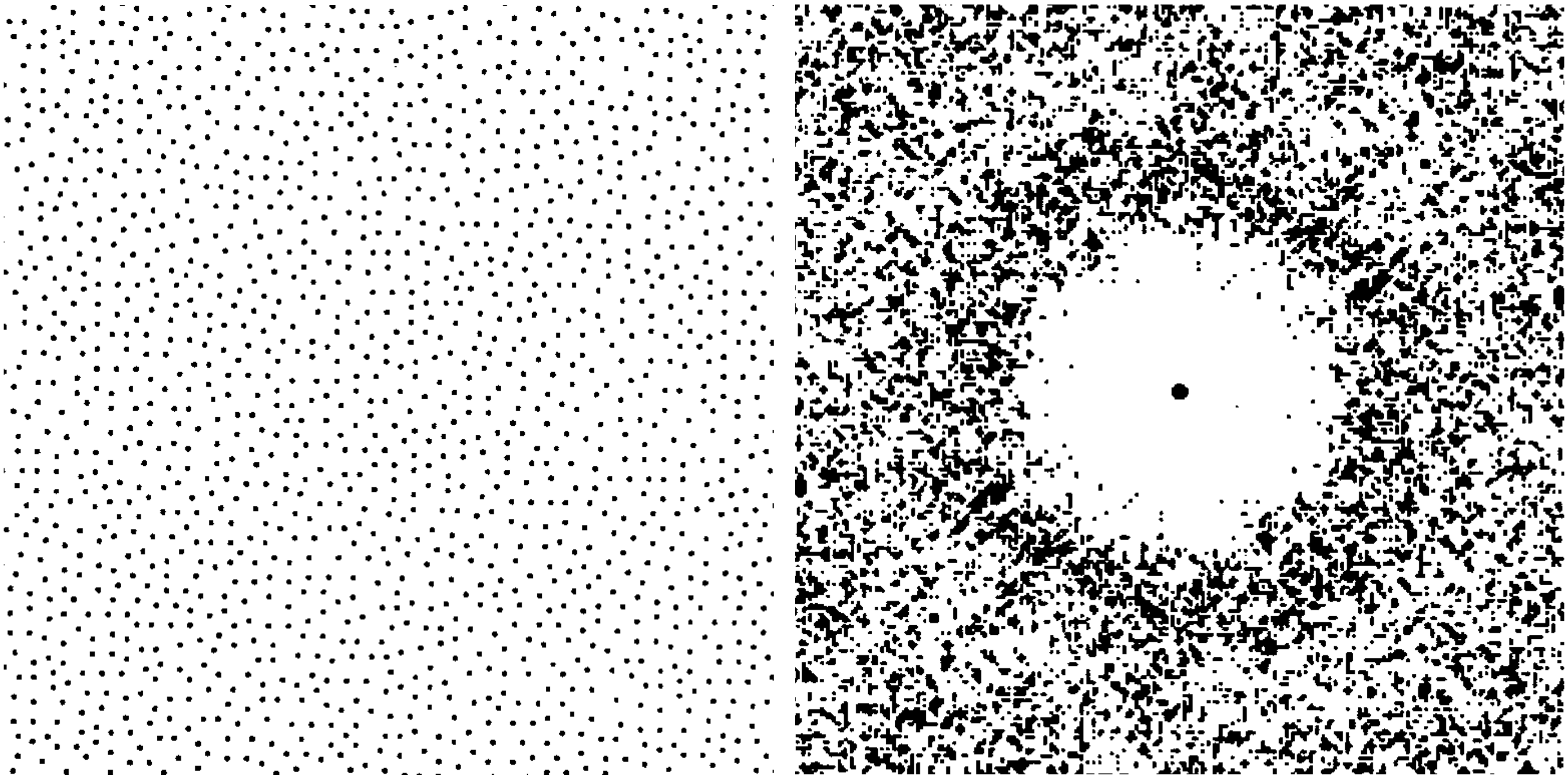
Uniform sampling

- The spectrum of uniformly spaced samples is also a set of uniformly spaced spikes
- Multiplying the signal by the sampling pattern corresponds to placing a copy of the spectrum at each spike (in freq. space)
- Aliases are coherent, and very noticable

Non-uniform sampling

- Samples at non-uniform locations have a different spectrum; a single spike plus noise
- Sampling a signal in this way converts aliases into broadband noise
- Noise is incoherent, and much less objectionable

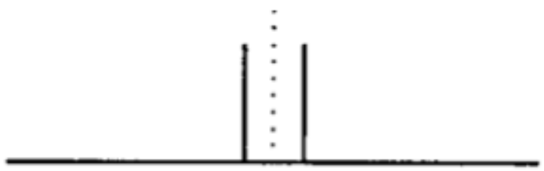
Poisson Disk Sampling



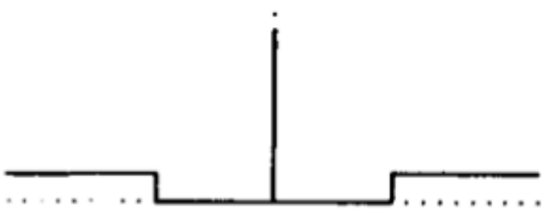
Dart throwing algorithm

Poisson Disk Sampling

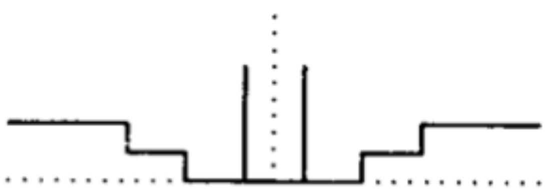
From Cook



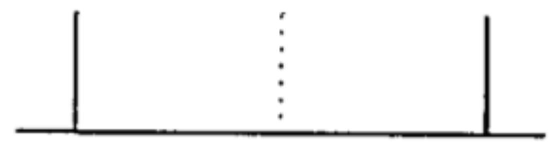
(a) Original signal $F(x)$



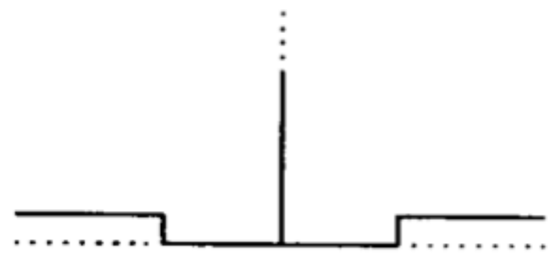
(b) Sampling function $P(x)$



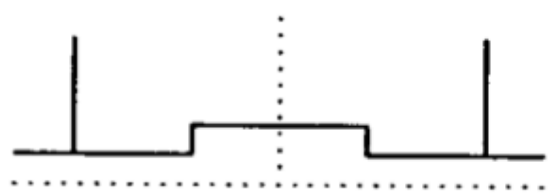
(c) Sampled signal $F(x) * P(x)$



(f) Original signal $F(x)$

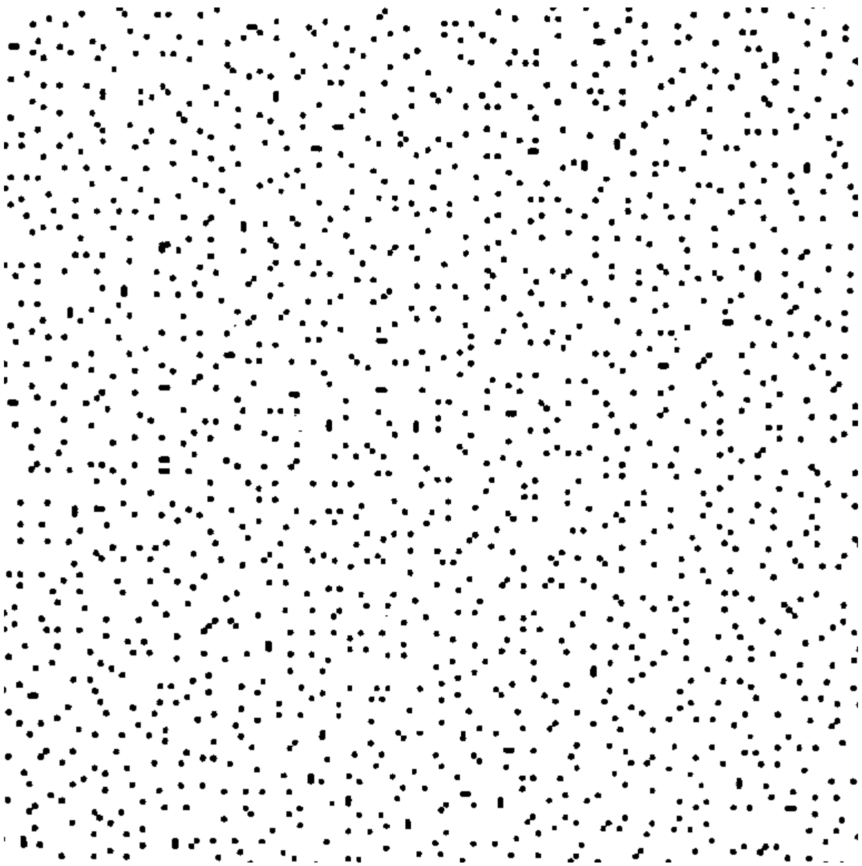


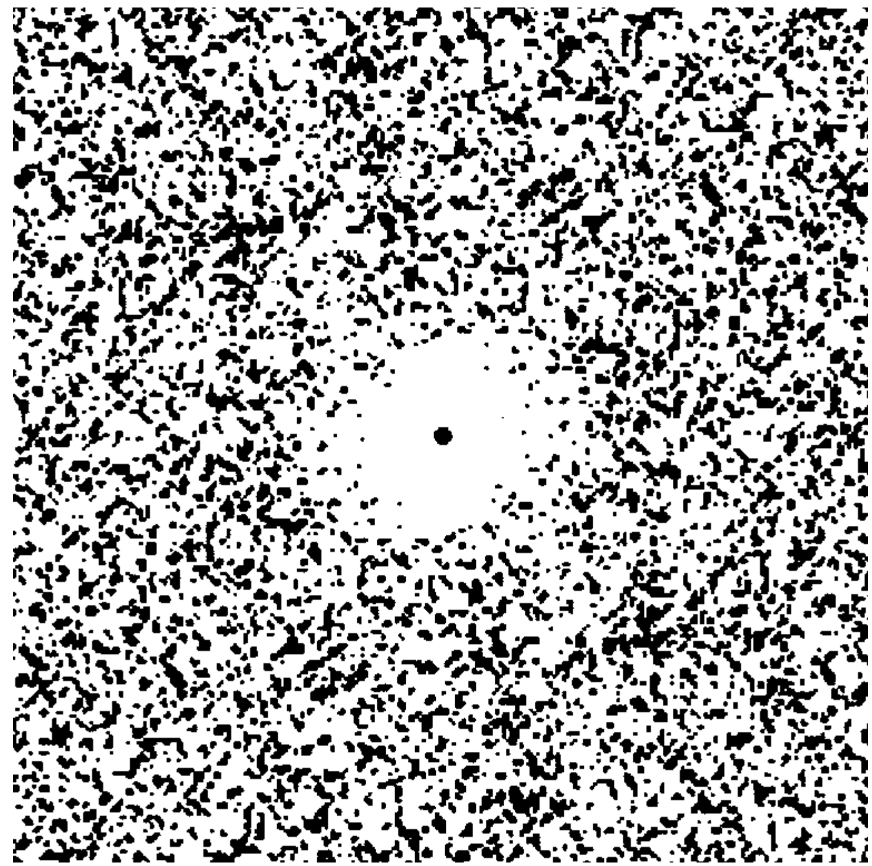
(g) Sampling function $P(x)$



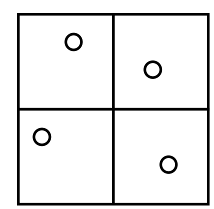
(h) Sampled signal $F(x) * P(x)$

Jittered Sampling

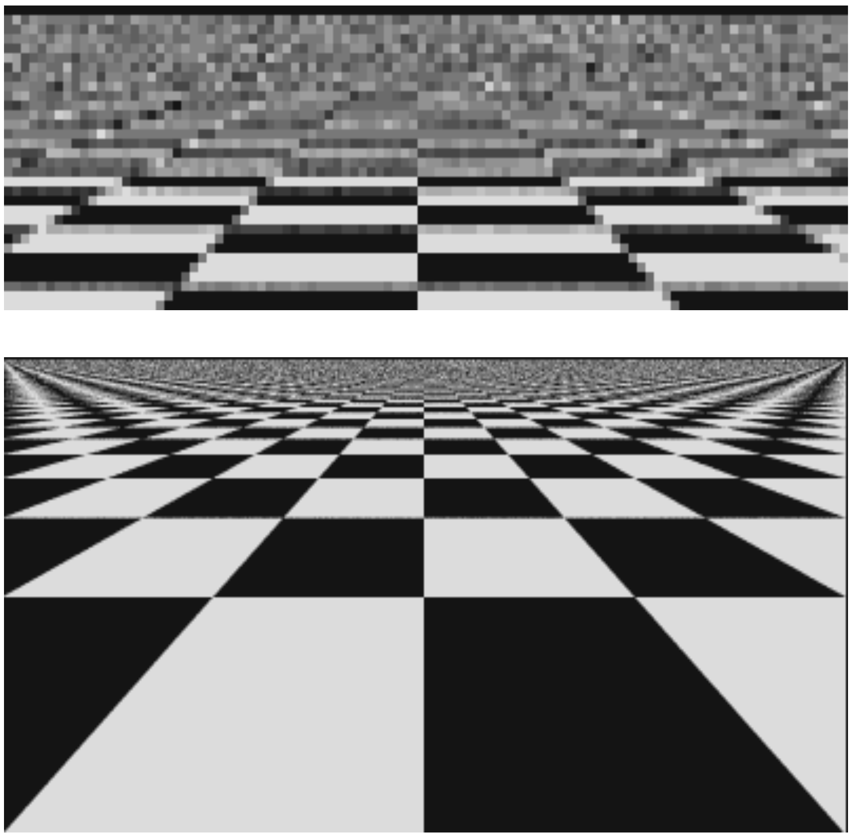




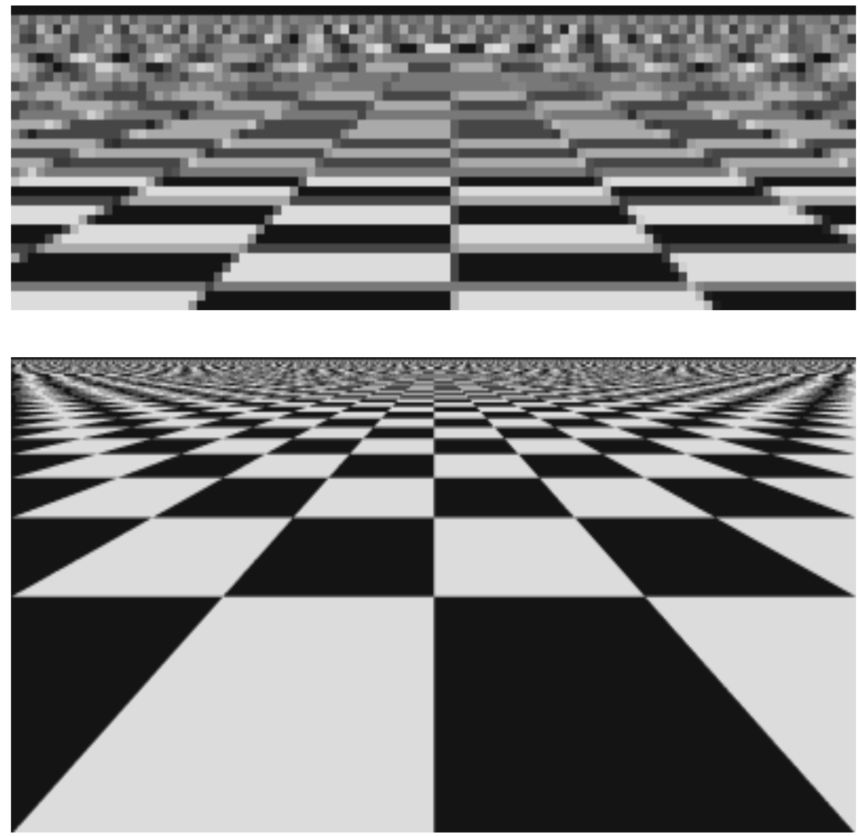
Add uniform random jitter to each sample



Jittered vs. Uniform Supersampling



4x4 Jittered Sampling



4x4 Uniform

Analysis of Jitter

Non-uniform sampling

$$s(x) = \sum_{n=-\infty}^{n=\infty} \delta(x - x_n)$$

$$x_n = nT + j_n$$

Jittered sampling

$$j_n \sim j(x)$$

$$j(x) = \begin{cases} 1 & |x| \le 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

$$J(\omega) = \operatorname{sinc} \omega$$

$$S(\omega) = \frac{1}{T}\left[1 - |J(\omega)|^2\right] + \frac{2\pi}{T^2}|J(\omega)|^2 \sum_{n=-\infty}^{n=-\infty} \delta(\omega - \frac{2\pi n}{T})$$

$$= \frac{1}{T}\left[1 - \operatorname{sinc}^2 \omega\right] + \delta(\omega)$$