CS348n: Neural Representations and Generative Models for 3D Geometry

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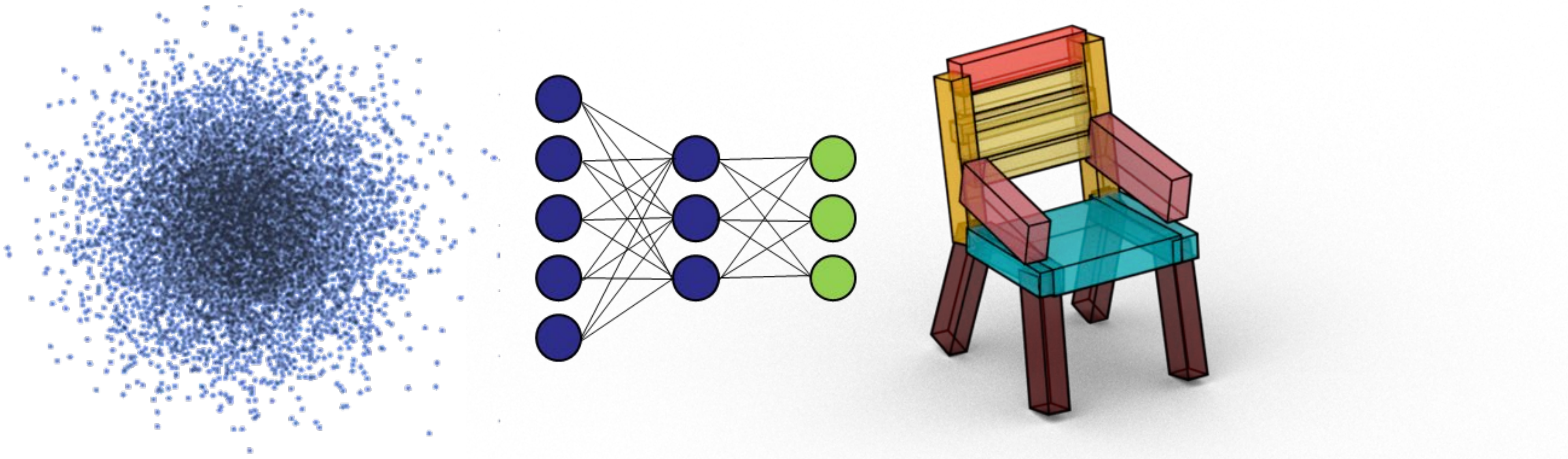
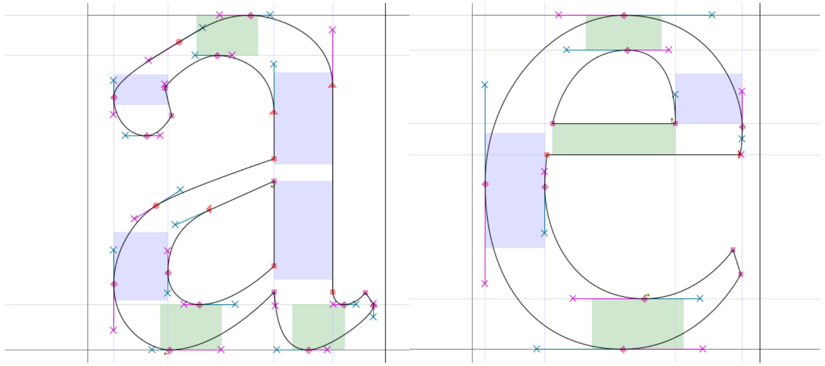


Leonidas Guibas Laboratory

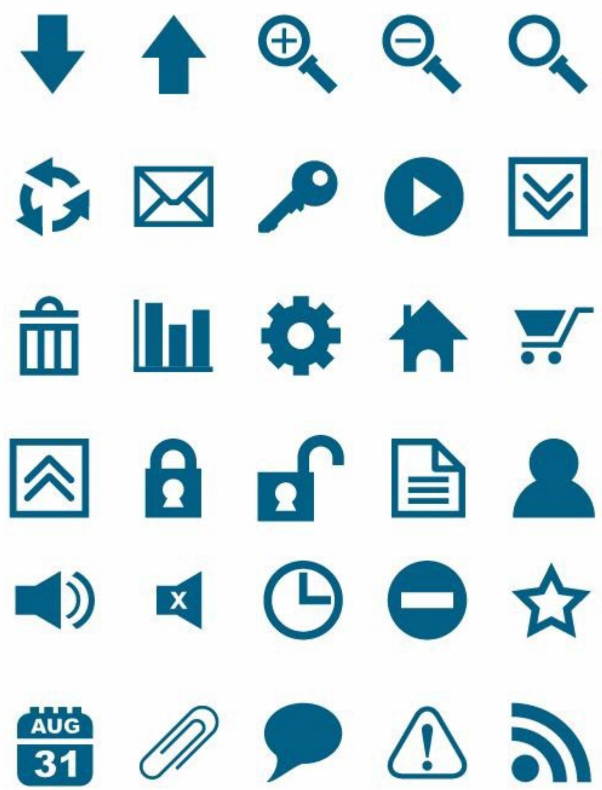
Geometric Computing

Last Time: Learning Vector and Mesh/CAD Representations

Vector (2D) and Mesh/CAD (3) Shape Models



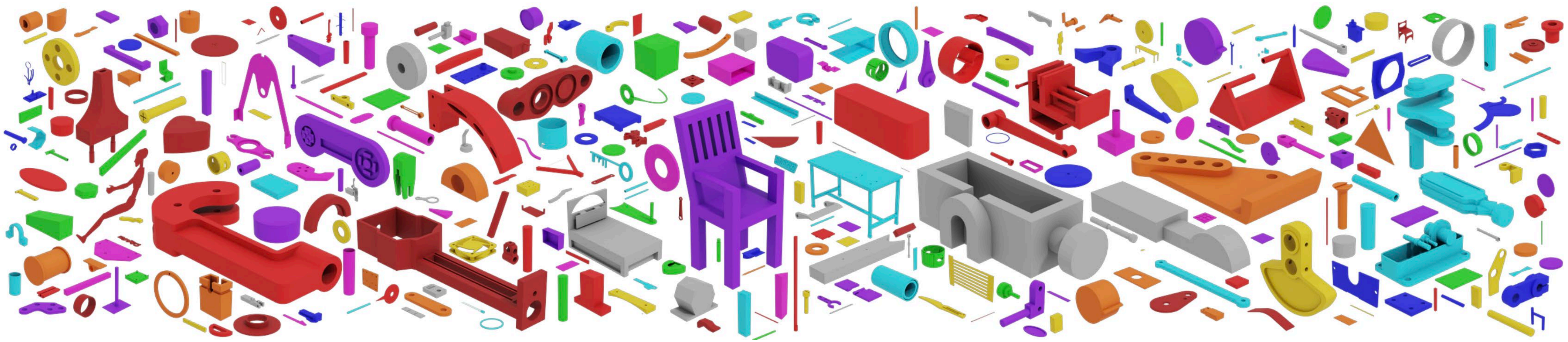
Fonts



Clip Art

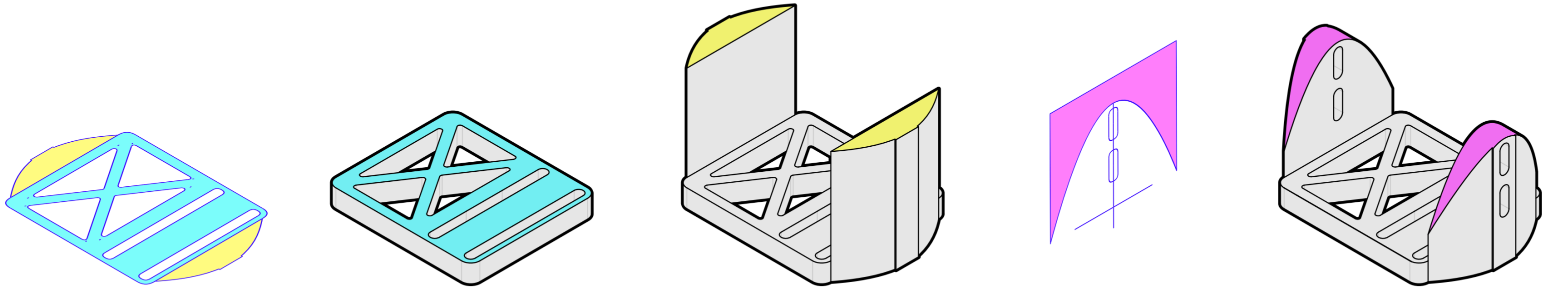
2D

Vector (2D) and Mesh/CAD (3D) Shape Models



Autodesk Fusion 360 Dataset

3D



Sketch 1 → Extrude 1 → Extrude 2 → Sketch 2 → Extrude 3

Sketch 1

```
s1 = add_sketch('XZ')
add_line(.6, .8, .6, 2.4)
add_arc(.6, 2.4, .8, 2.4, 90)
add_line(.8, 2.6, 1, 2.6)
...
p1 = add_line(-1, 2, -1, .8)
```

Extrude 1

```
add_extrude(
    sketch=s1,
    profile=p1[1],
    distance=.8,
    operation='NewBody')
```

Extrude 2

```
add_extrude(
    sketch=s1,
    profile=p1[0::2],
    distance=5,
    operation='Join')
```

Sketch 2

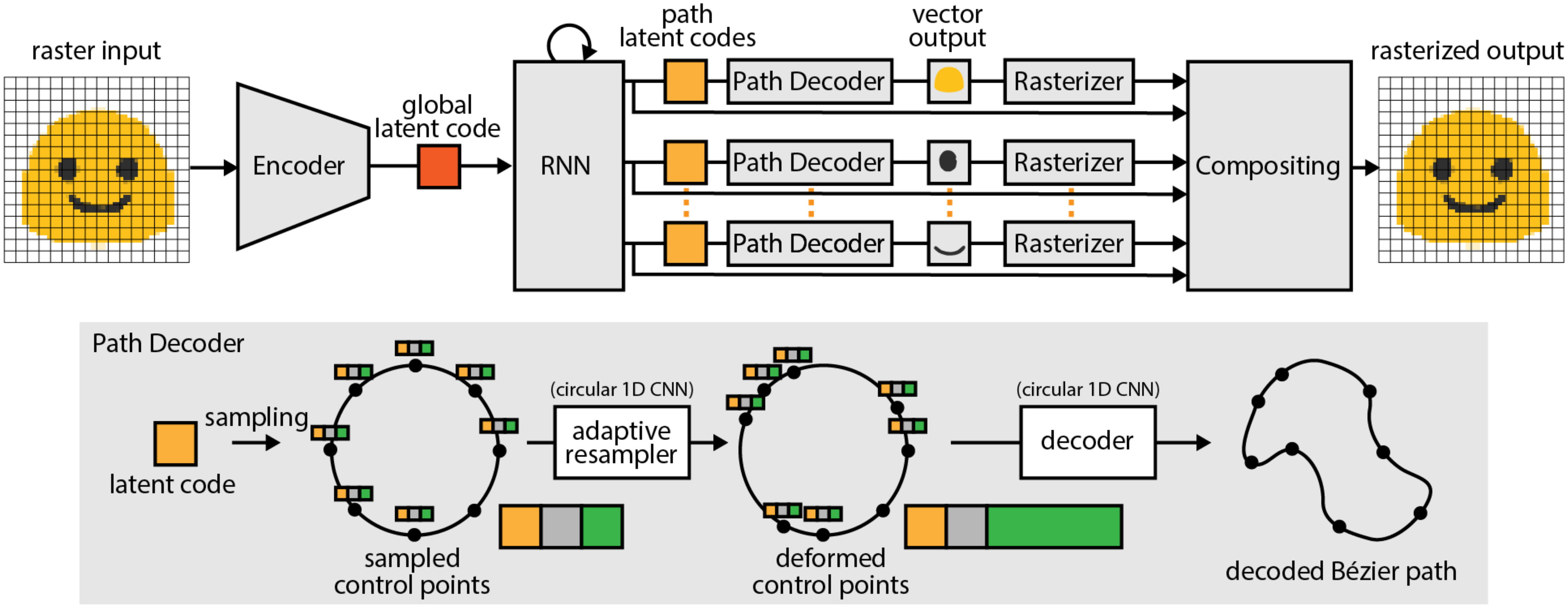
```
s2 = add_sketch('YZ')
add_line(.8, 2.8, 5.8, 2.8)
add_line(5.8, 2.8, 5.8, -2.8)
add_line(5.8, -2.8, .8, -2.8)
...
p2 = add_line(2, .2, 2.8, .2)
```

Extrude 3

```
add_extrude(
    sketch=s2,
    profile=p2[0],
    distance=14,
    operation='Cut')
```

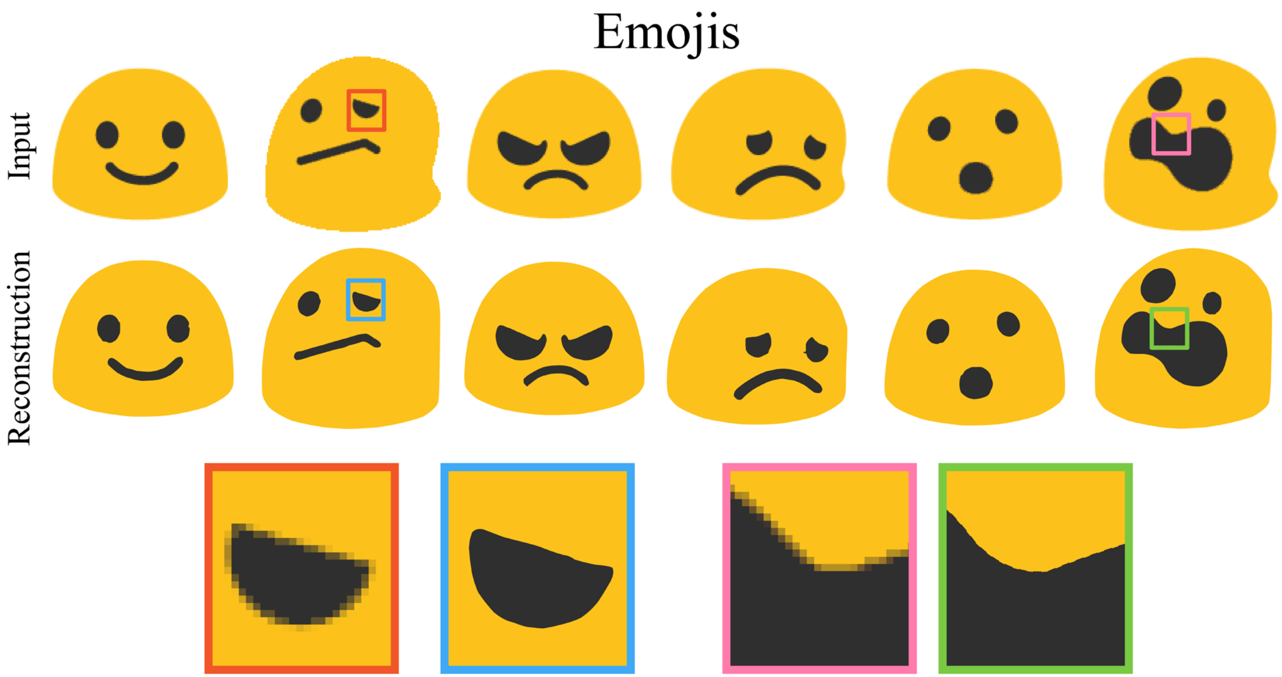
Im2Vec Inference Architecture

4 step process.



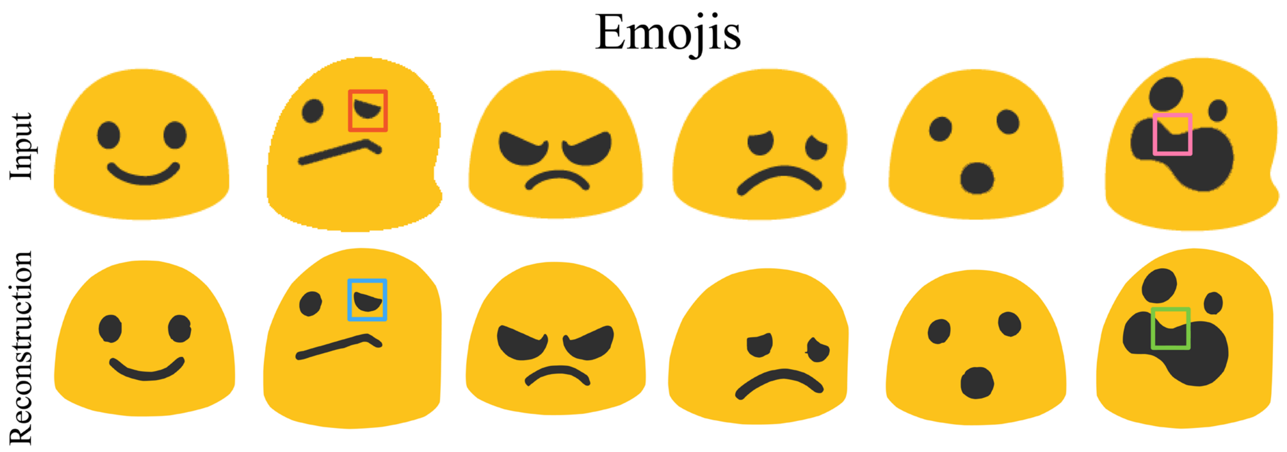
Reconstruction

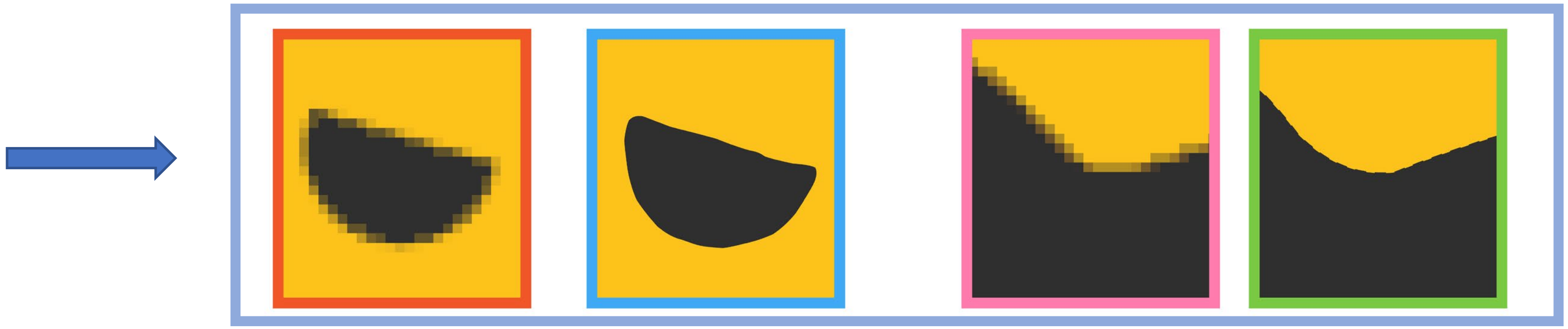
Input images are 128x128



Reconstruction

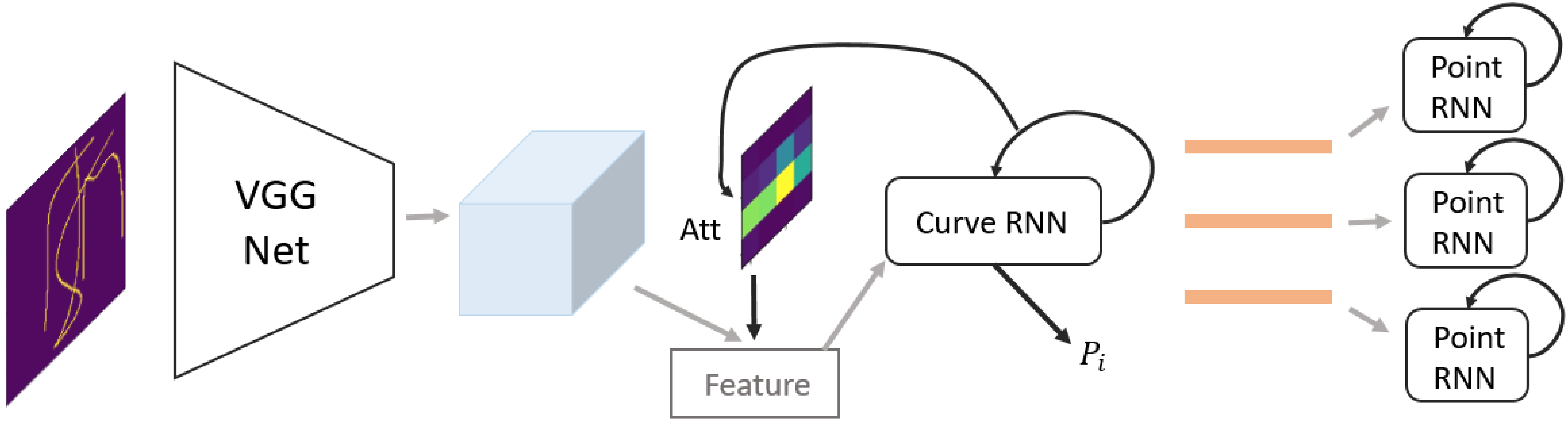
Input images are 128x128



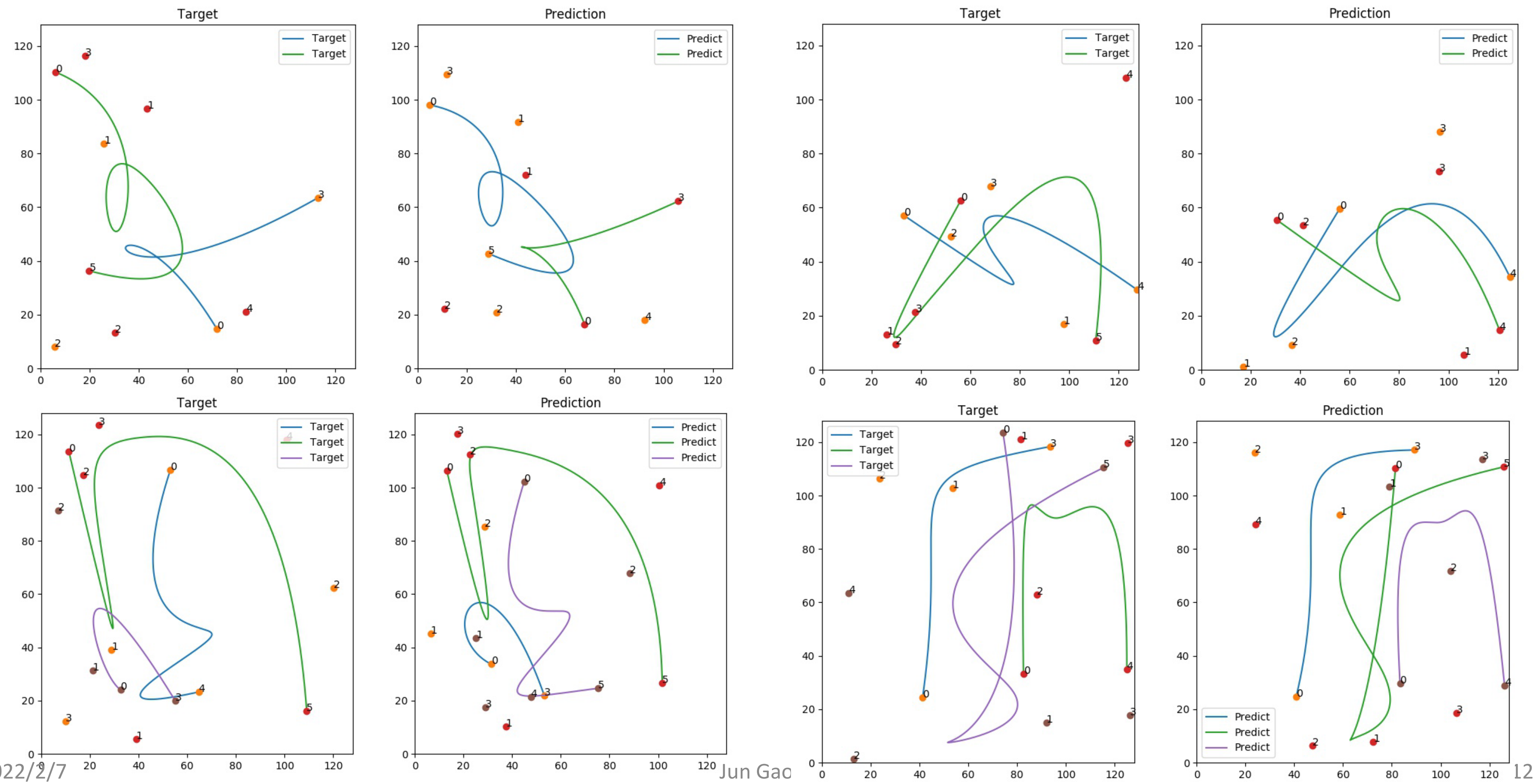


Multiple spline curves with variable numbers of control points

- Whole model



Results


Target
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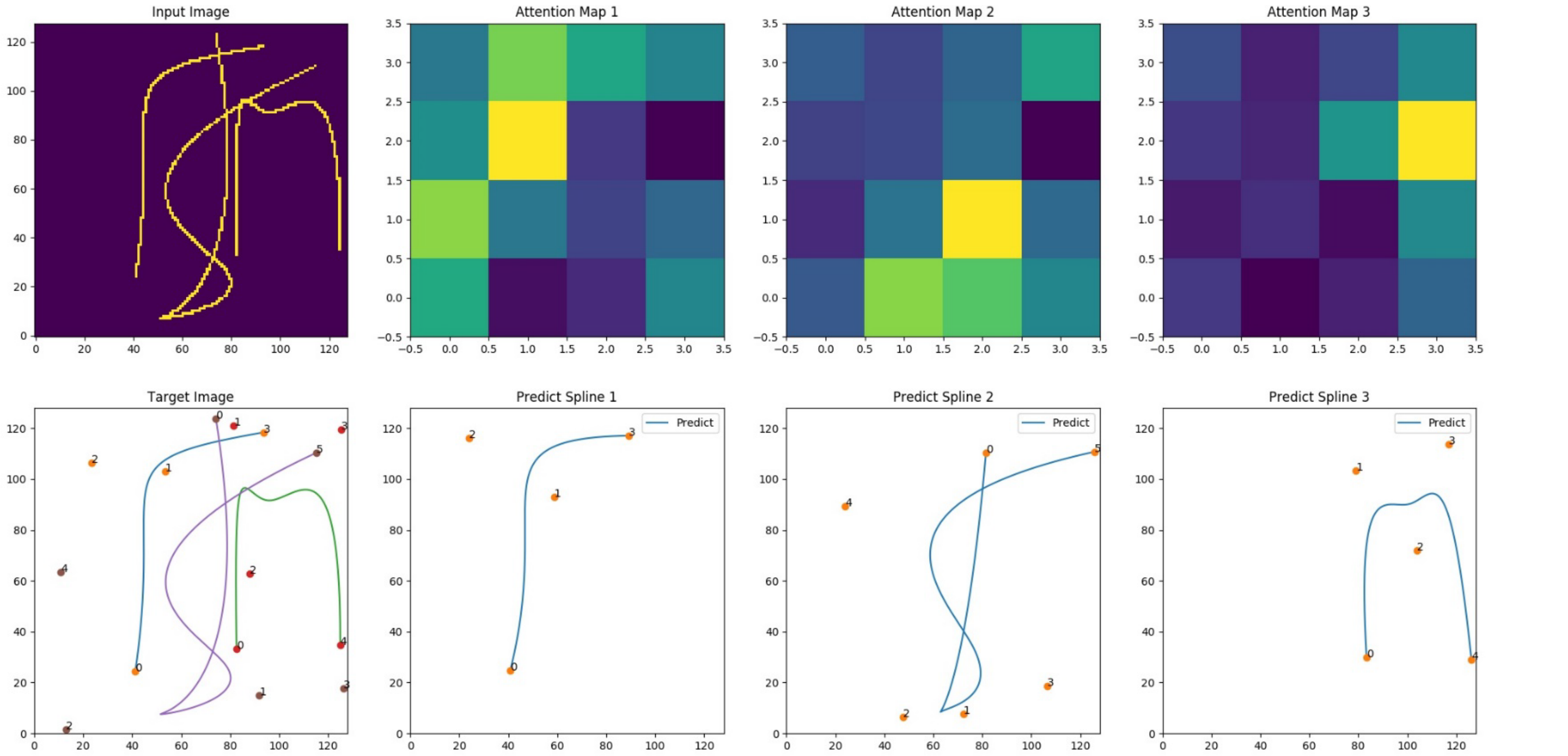
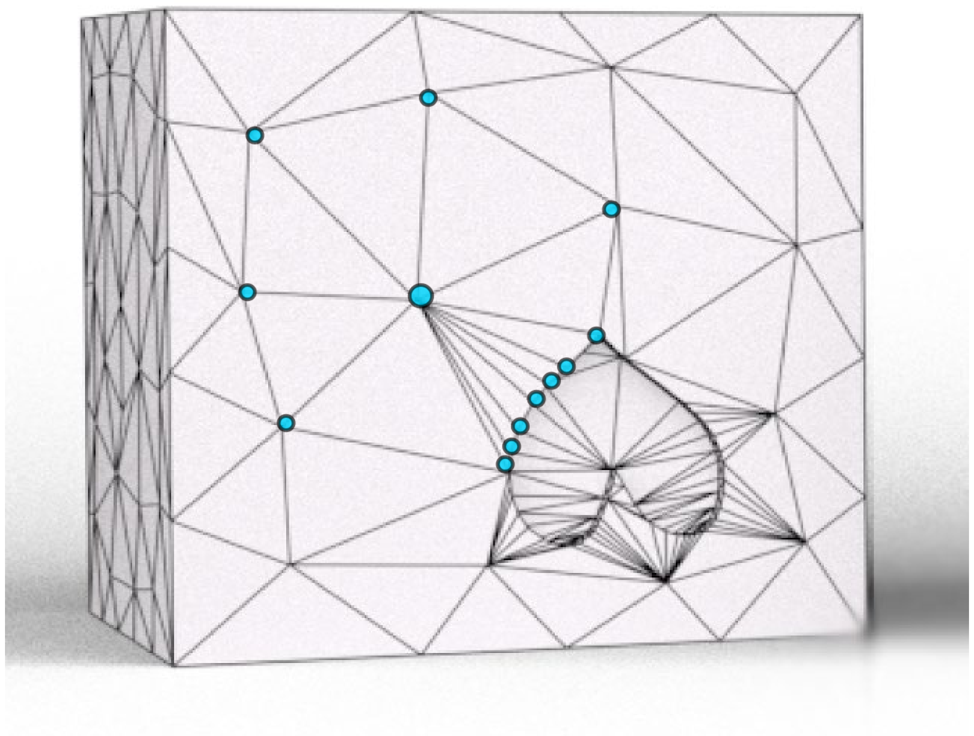
Attention Results



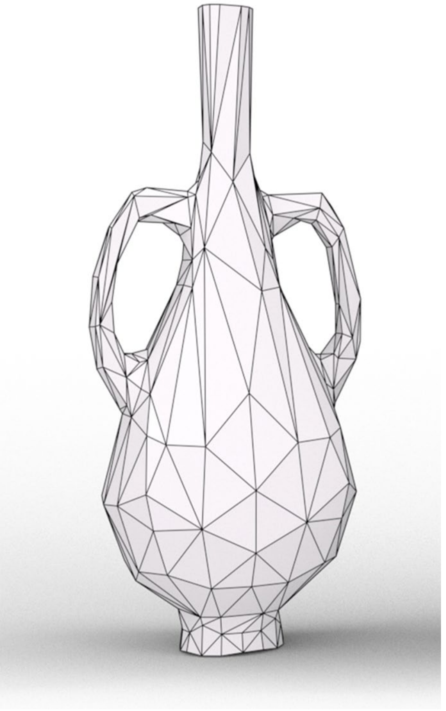


Irregular
Unordered
&
Unoriented

Goal: CNN directly on the irregular mesh elements

Classification

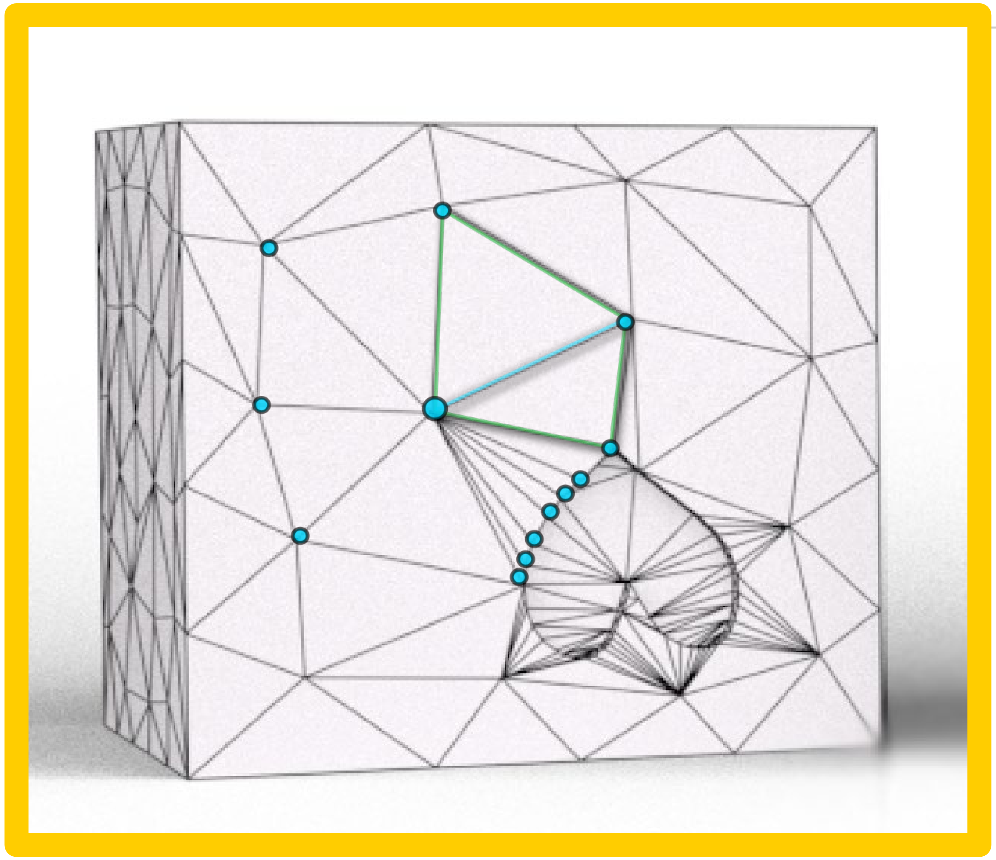
Segmentation



Vase

Fixed Size

Neighborhood



Vertices

<x, y, z>

Edges

<vi, vj>

Faces

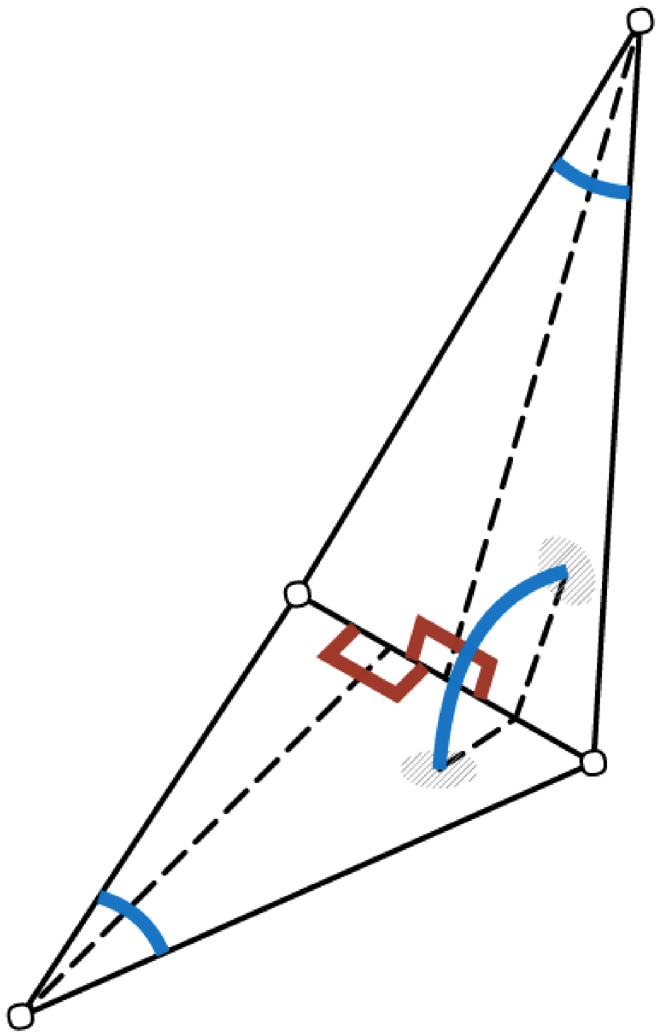
<vi, vj, vk>

Input Edge Features

Relative Geometric Features

→ Invariant to *similarity* transformations

5-dimensional vector



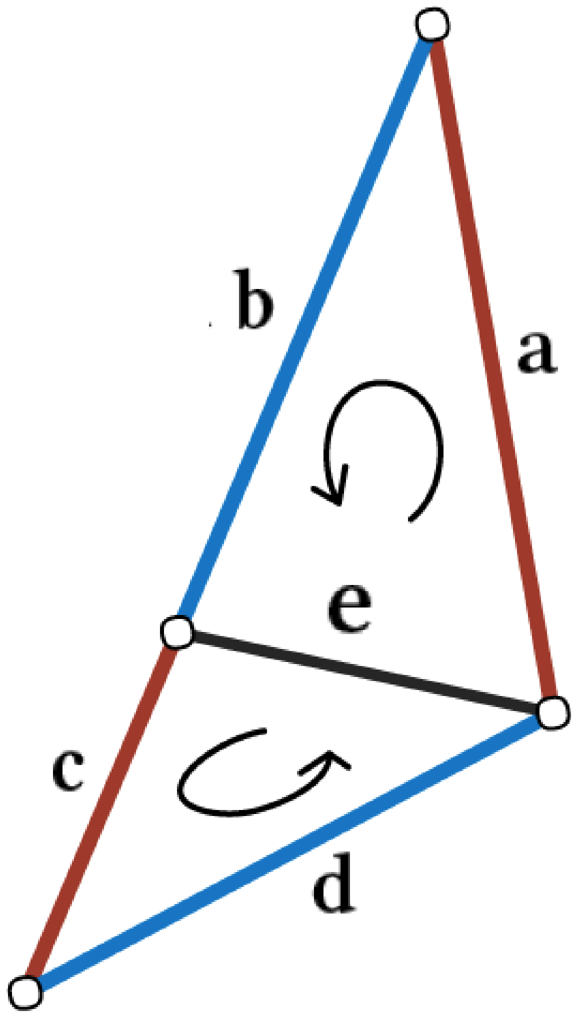
Mesh Convolution Order

Face normal

- ➔ Consistent ordering in each face
- ➔ Two *valid* orderings

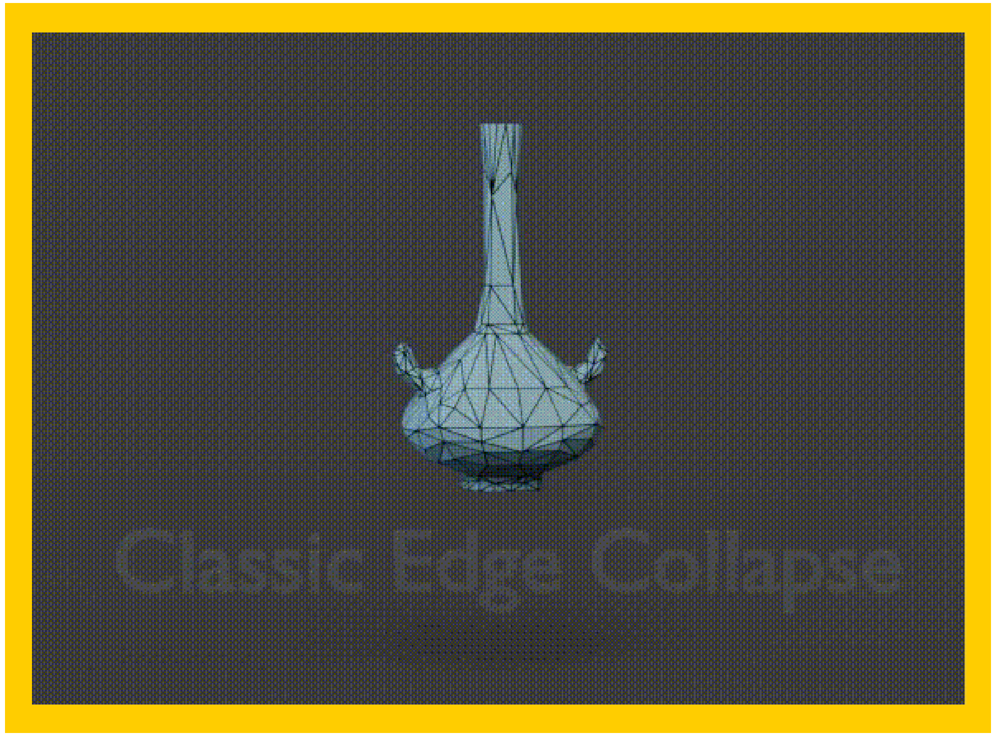
Solution: build symmetric features

- ➔ e-> (a+c, |a-c|, b+d, |b-d|)



Learned Edge Collapse

- Network decides collapse
- Strengthens the learned representation
- Visual insights from network

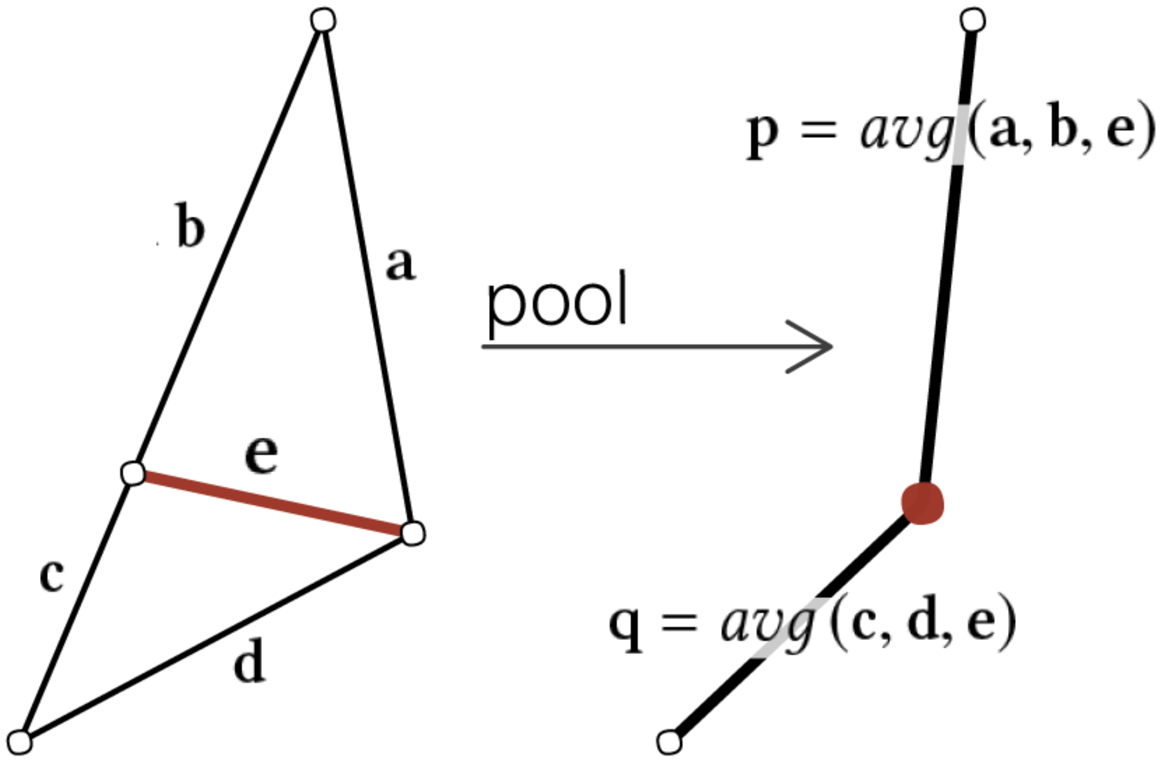


Mesh Pooling

Delete edge with smallest feature activations

→ Aggregate features

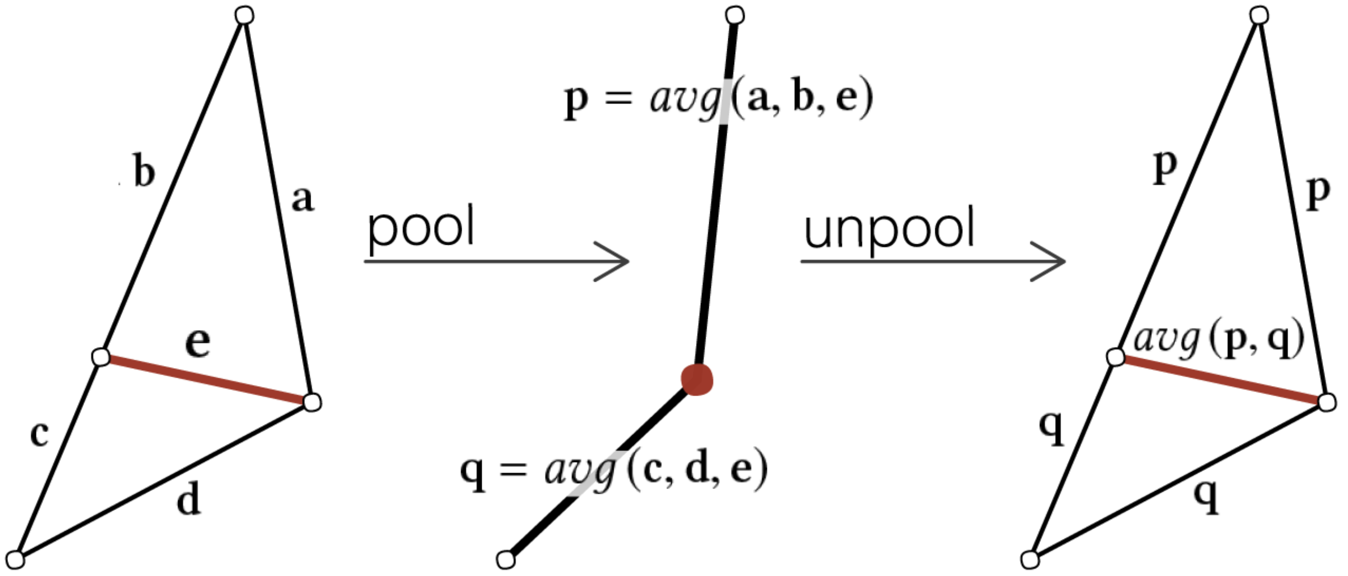
→ Update topology



Mesh Unpooling

Partial Inverse of Pooling

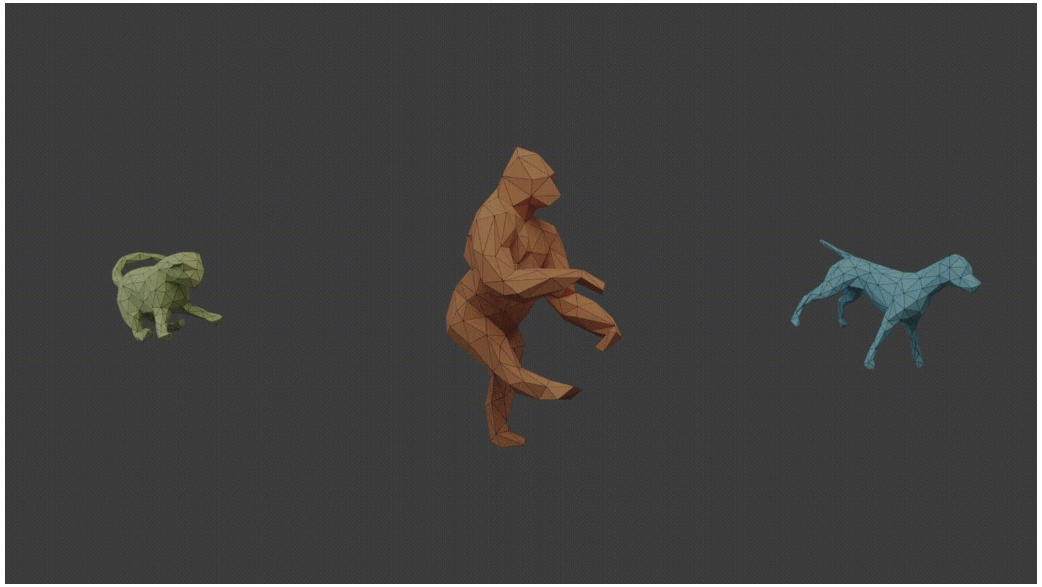
- Restores upsampled topology (reversible)
- Unpooled features weighted combination of pooled features



Applications of MeshCNN

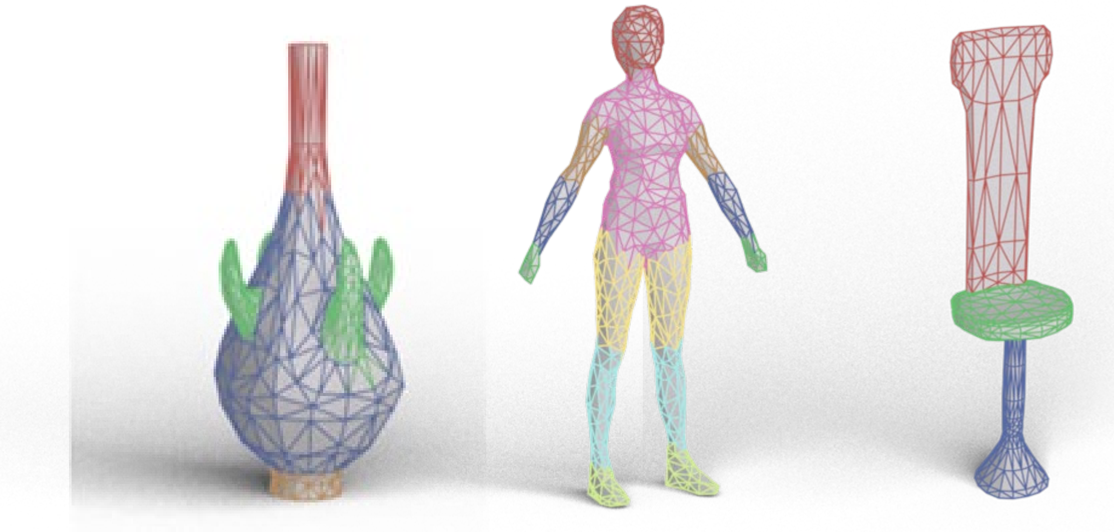
Classification

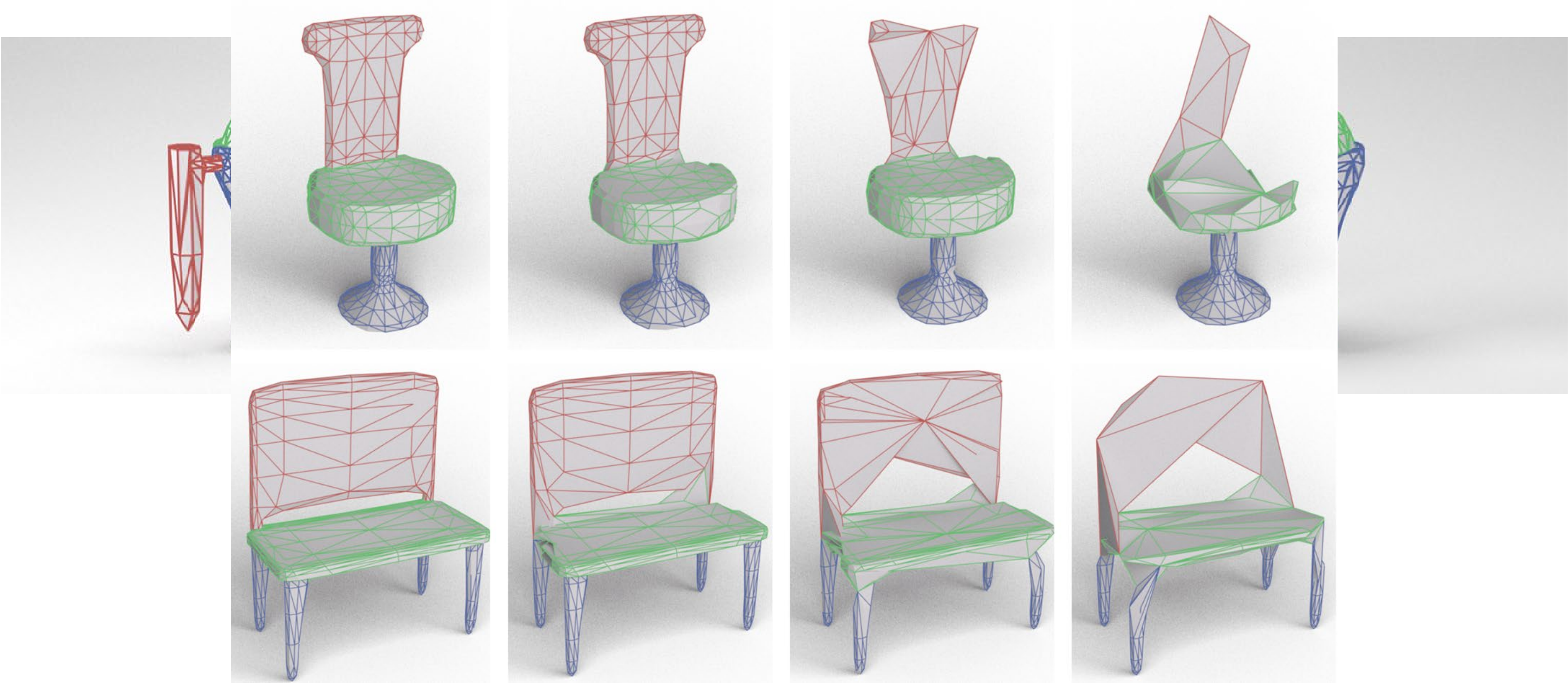
- Conv & Pooling Layers
- Fully-Connected Layers



Segmentation

- Fully convolutional
- Conv & Pooling & Unpooling



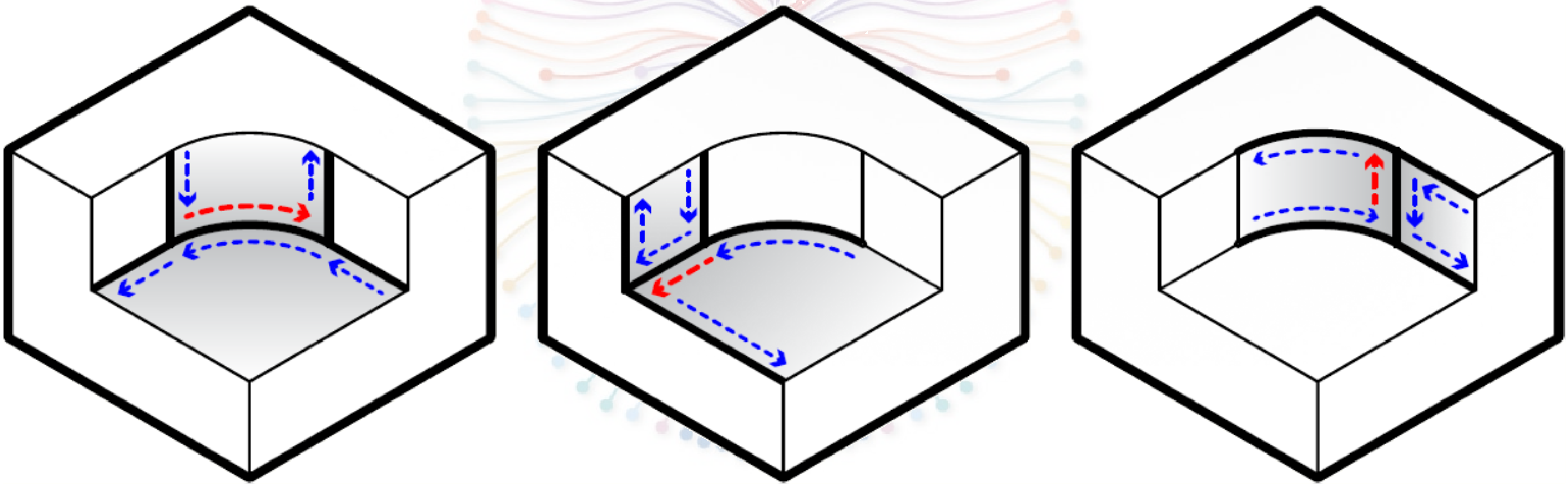


Segmentation

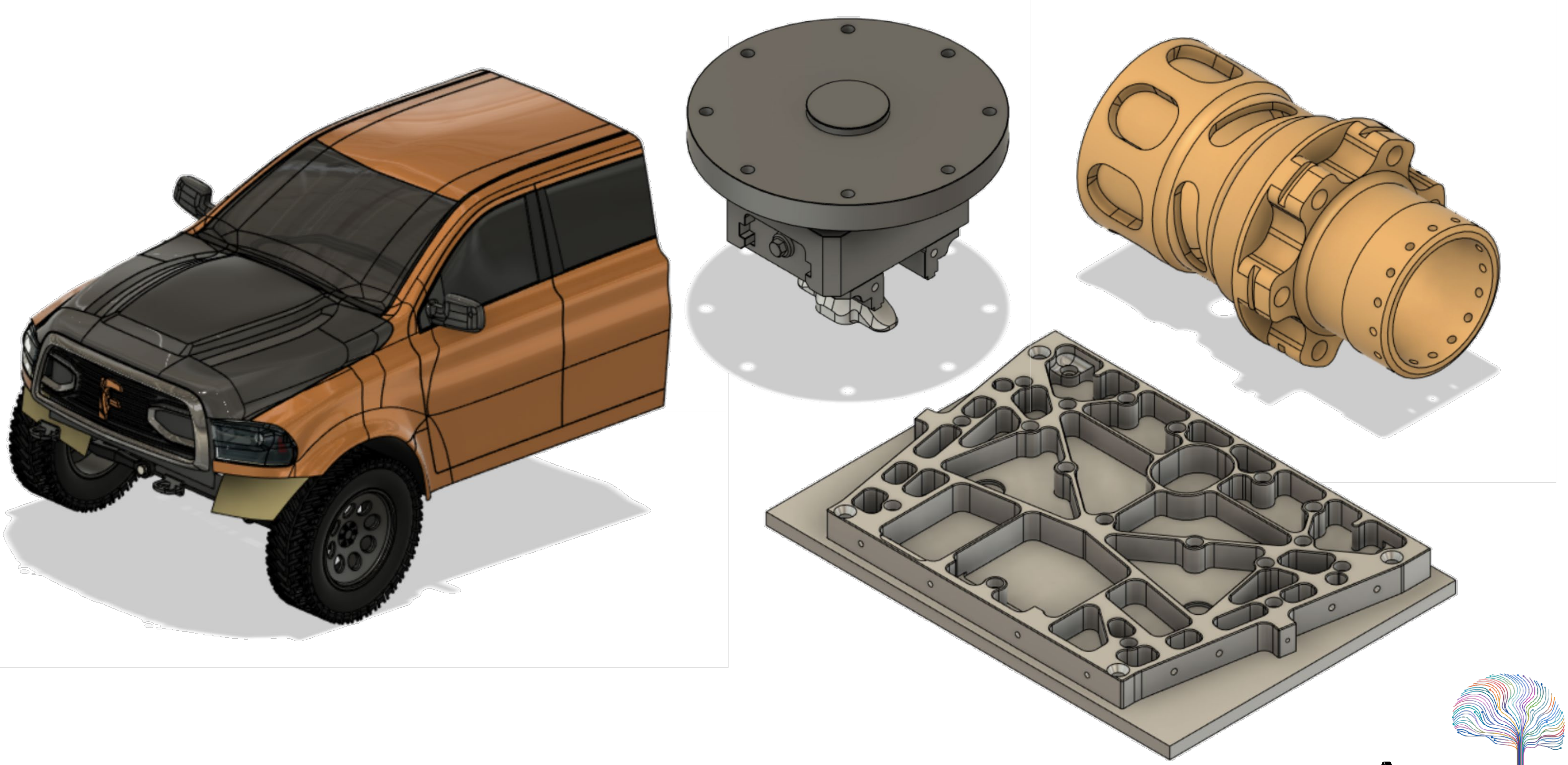
BRepNet: A topological message passing system for solid models

Joseph G. Lambourne[1], Karl D.D. Willis[1], Pradeep Kumar Jayaraman[1], Aditya Sanghi[1], Peter Meltzer[2], Hooman Shayani[1]

Autodesk Research[1], UCL, Computer Science[2]

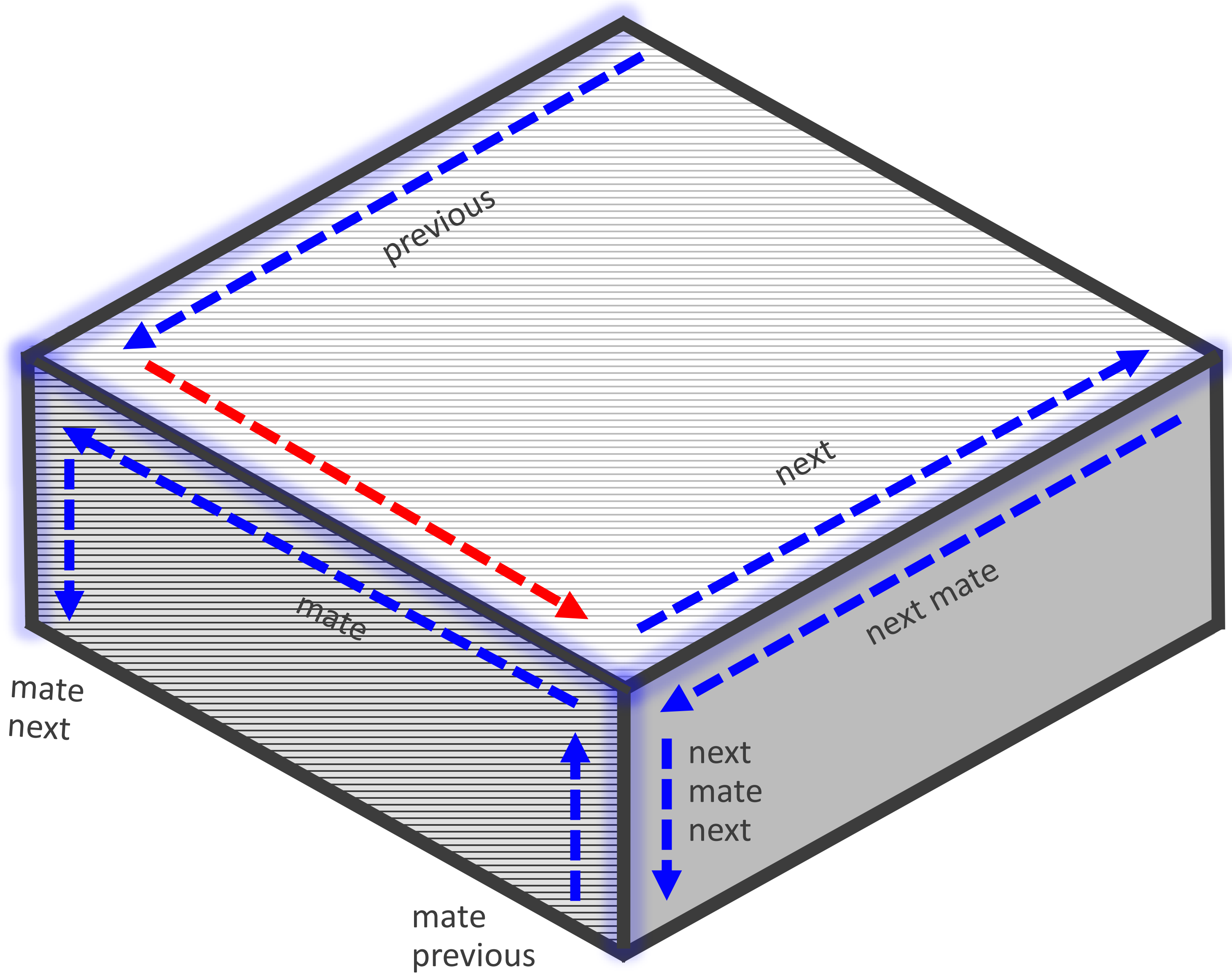


Boundary representation models



AUTODESK AI LAB

Topological walks defining neighboring entities



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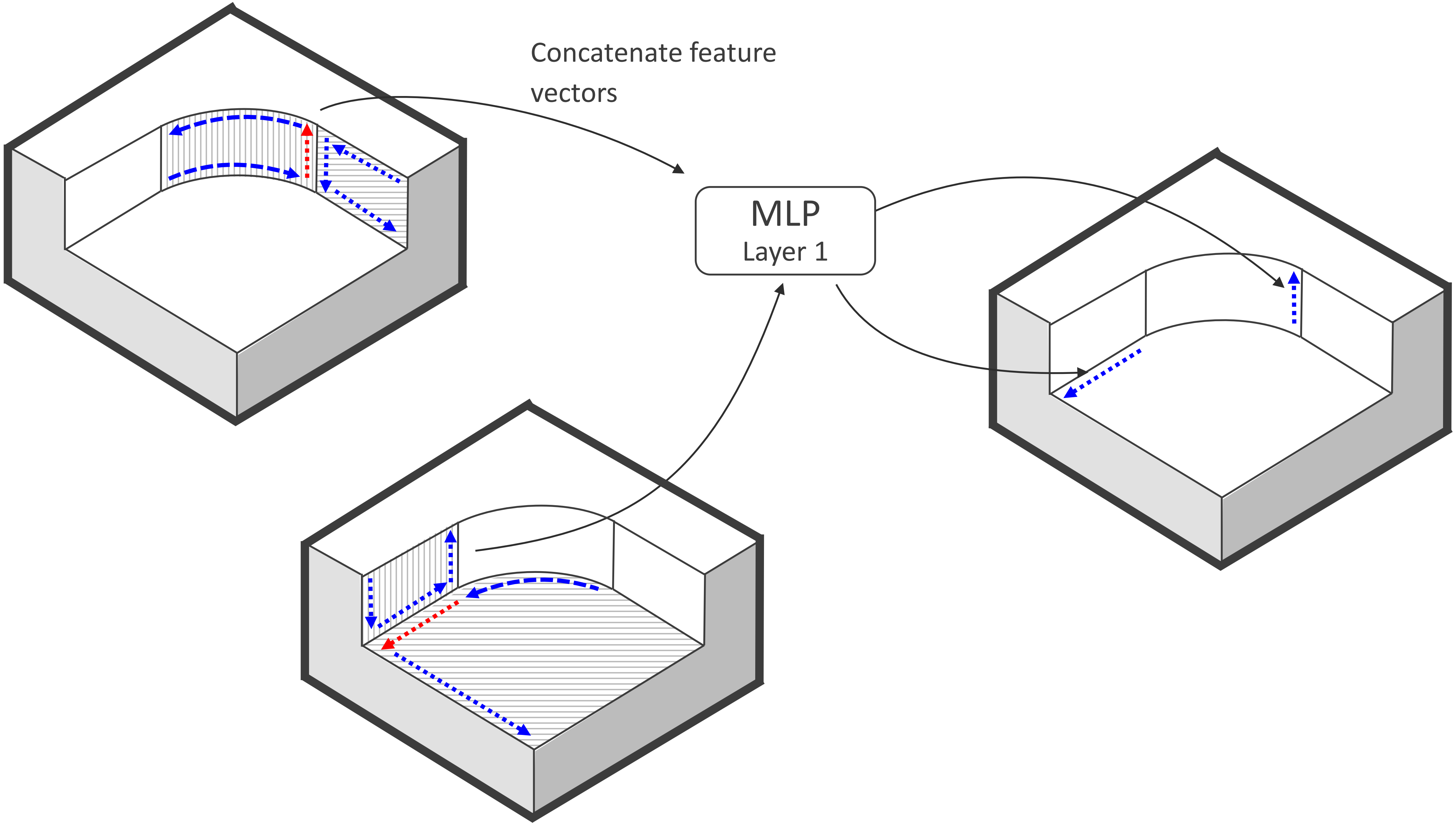
BRepNet convolution



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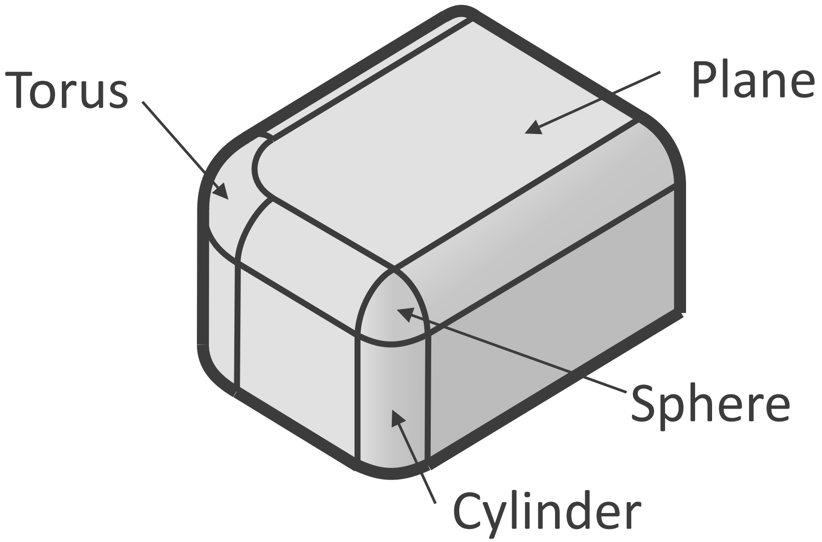
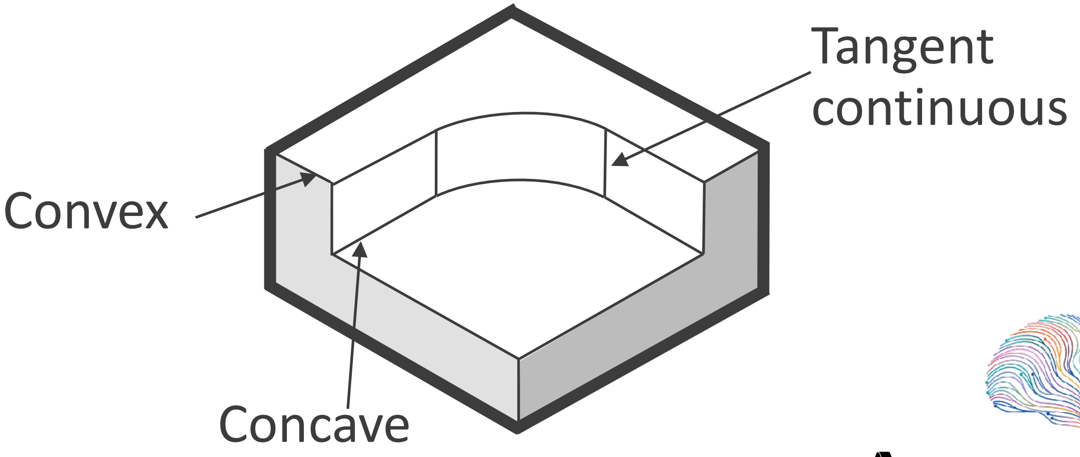
Input features

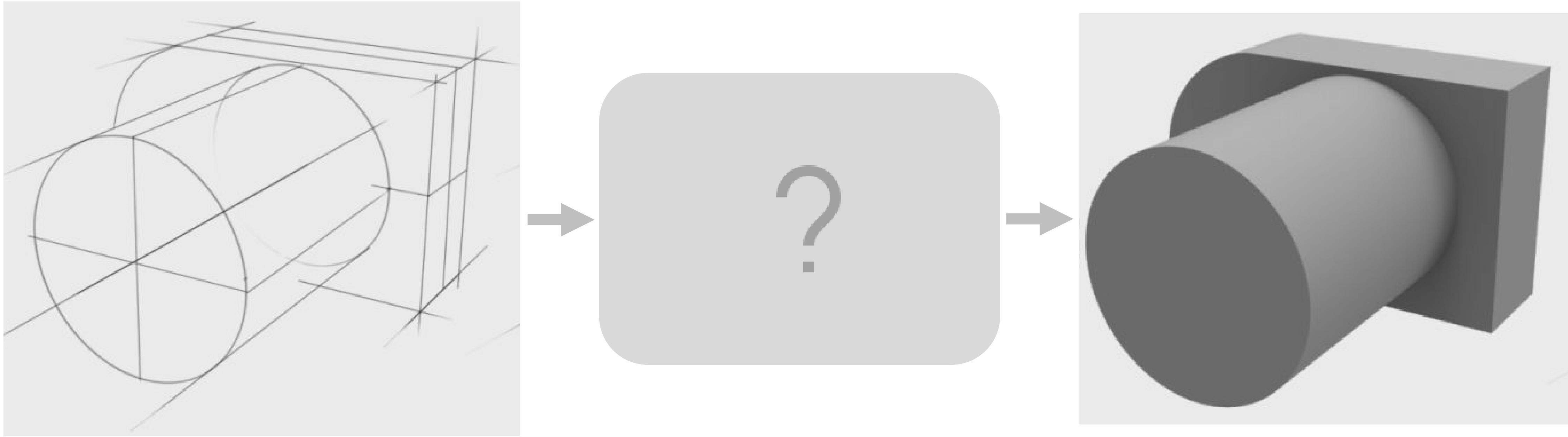
- Face features
  - Surface type (plane, cylinder, cone, sphere, torus, spline)
  - Face area
- Edge features
  - Curve type (line, circle, ellipse, intersection curve, spline)
  - Edge convexity (concave, convex, tangent plane continuous)
  - Edge length





Sketch2CAD Goal

- Gap: converting sketches to a CAD model?

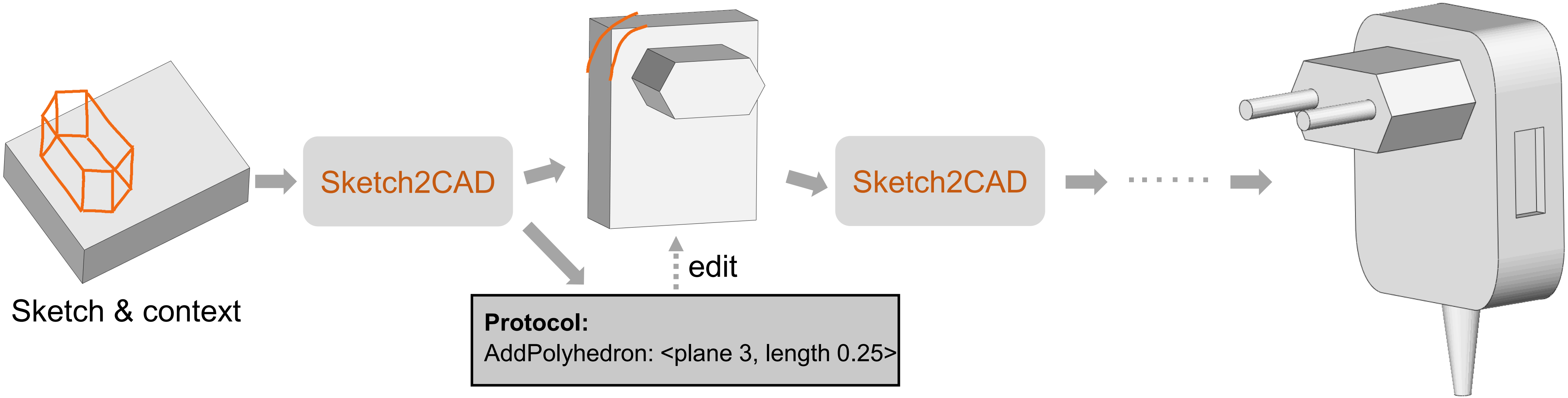


User sketches

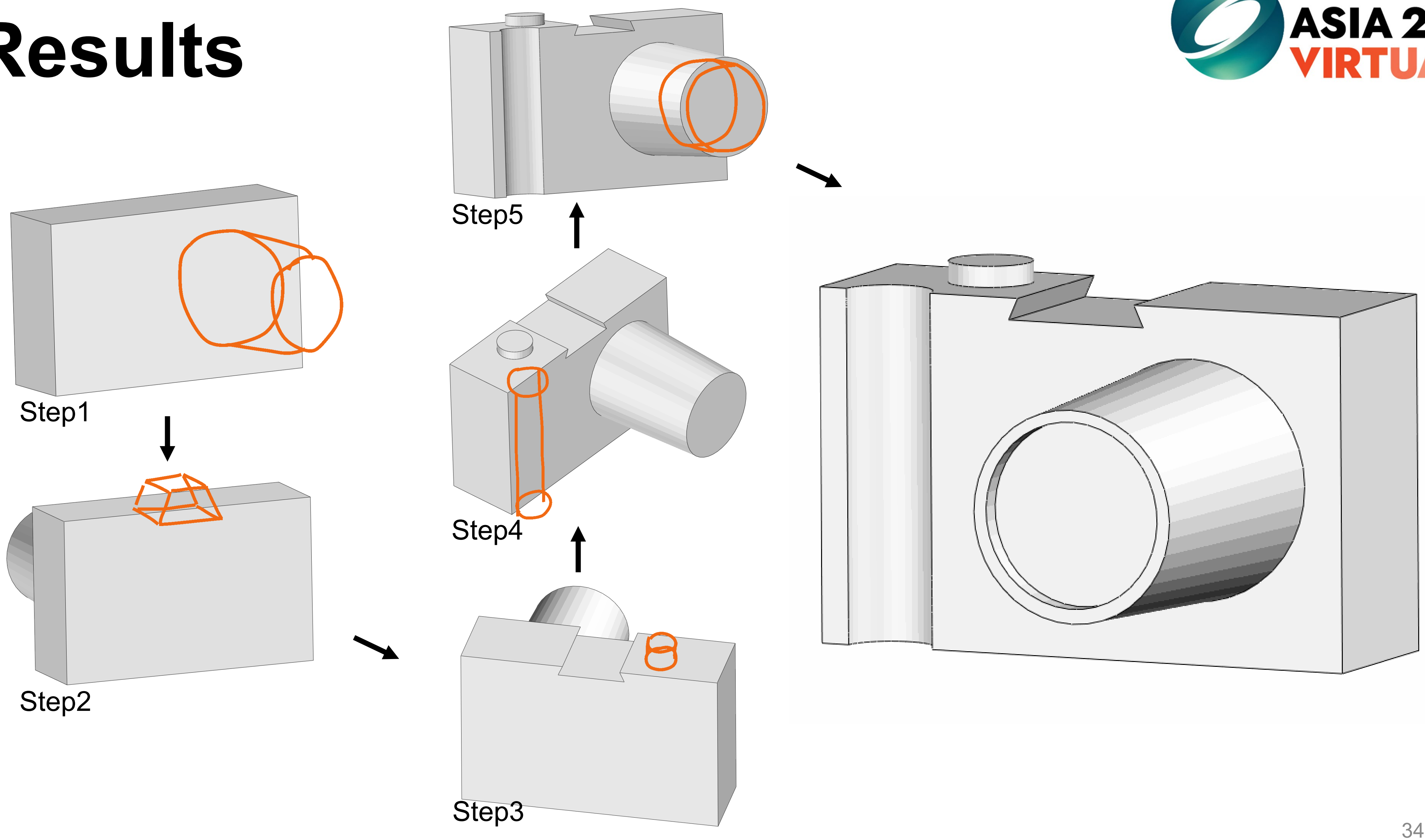
3D model

Sketch2CAD

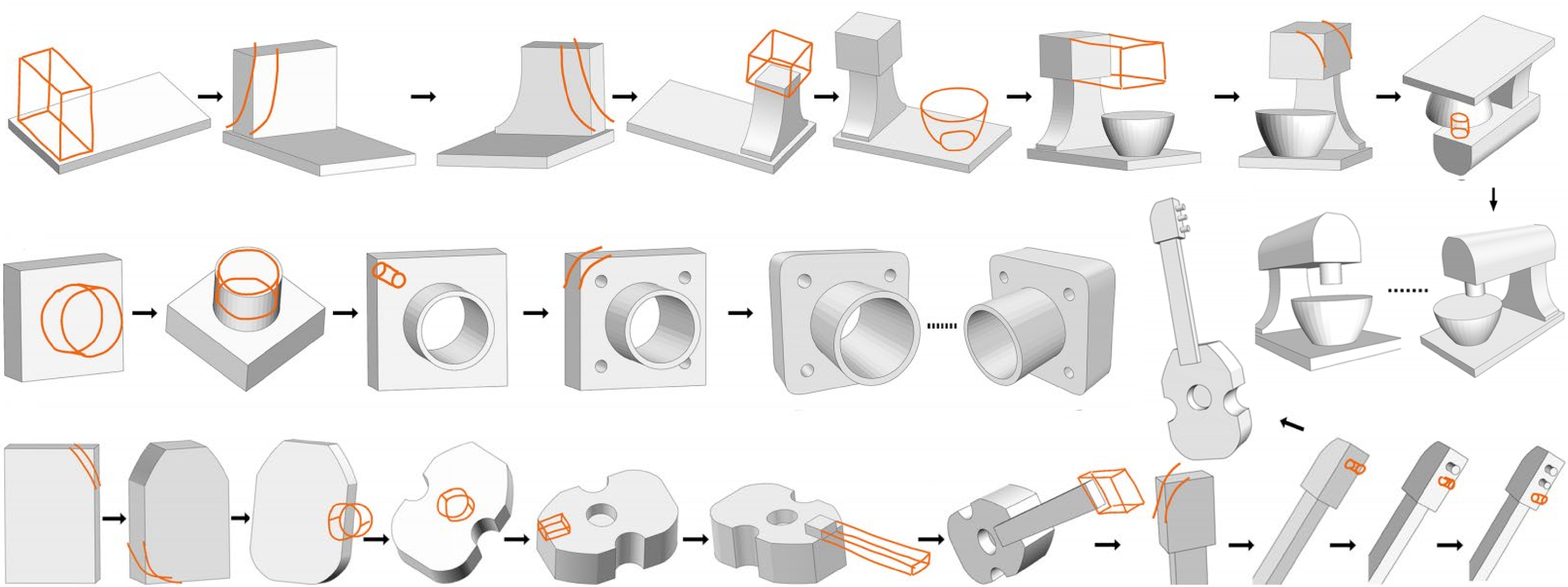
- A learning-based modeling system that translates sketching operations to their corresponding CAD operations, along with the associated parameters.



Results



Results



3D Equivariance and Invariance

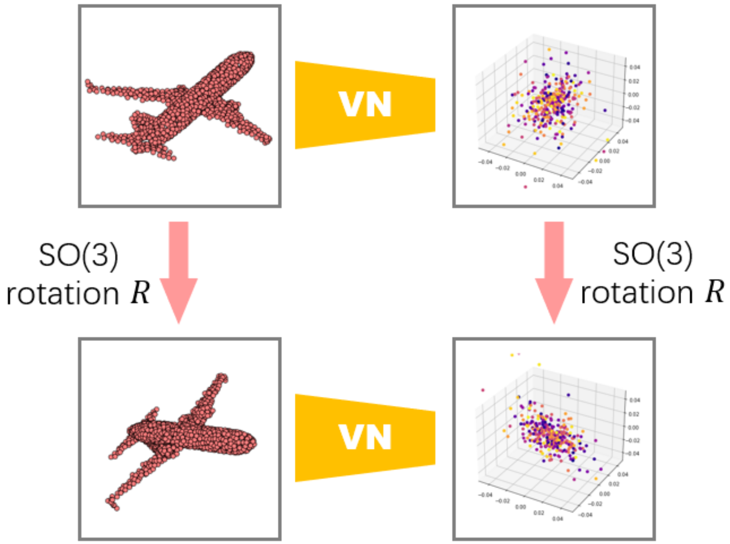
Class Questionnaire

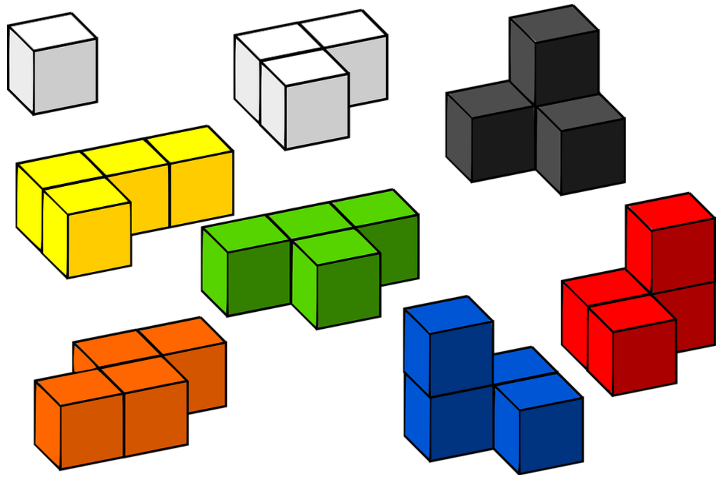
Class Questionnaire

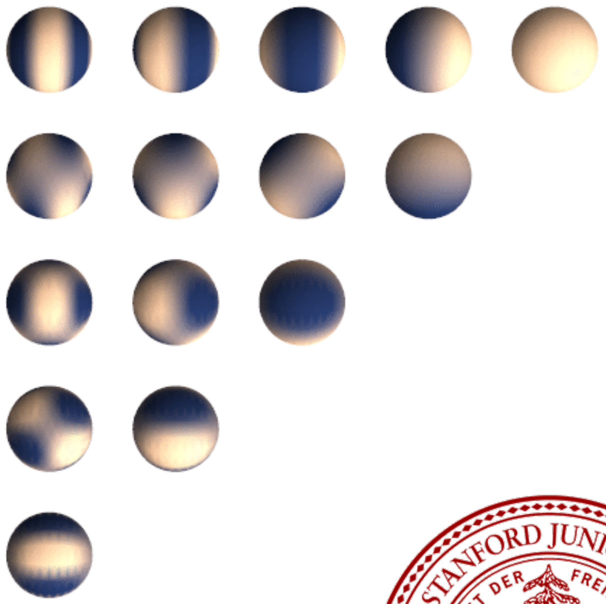
- Please take the questionnaire below to provide us with feedback on the class:

- https://forms.gle/igFFpmnWaWL11Tfw9

CS348N Lecture 10:
3D Rotation Equivariance and Invariance







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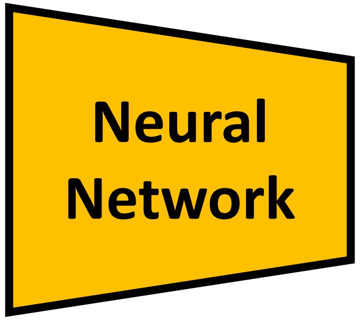
Why Equivariance / Invariance?

- What neural networks can do?



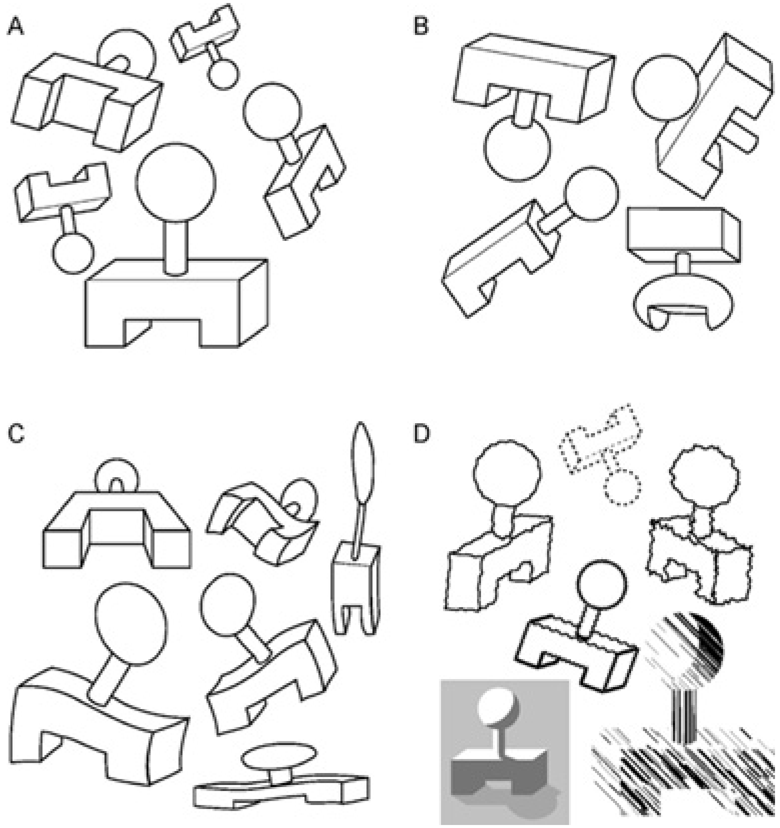
“chair”

- But what is this?

Not too Hard for a Human…

Gestalt theory – how human perceive objects

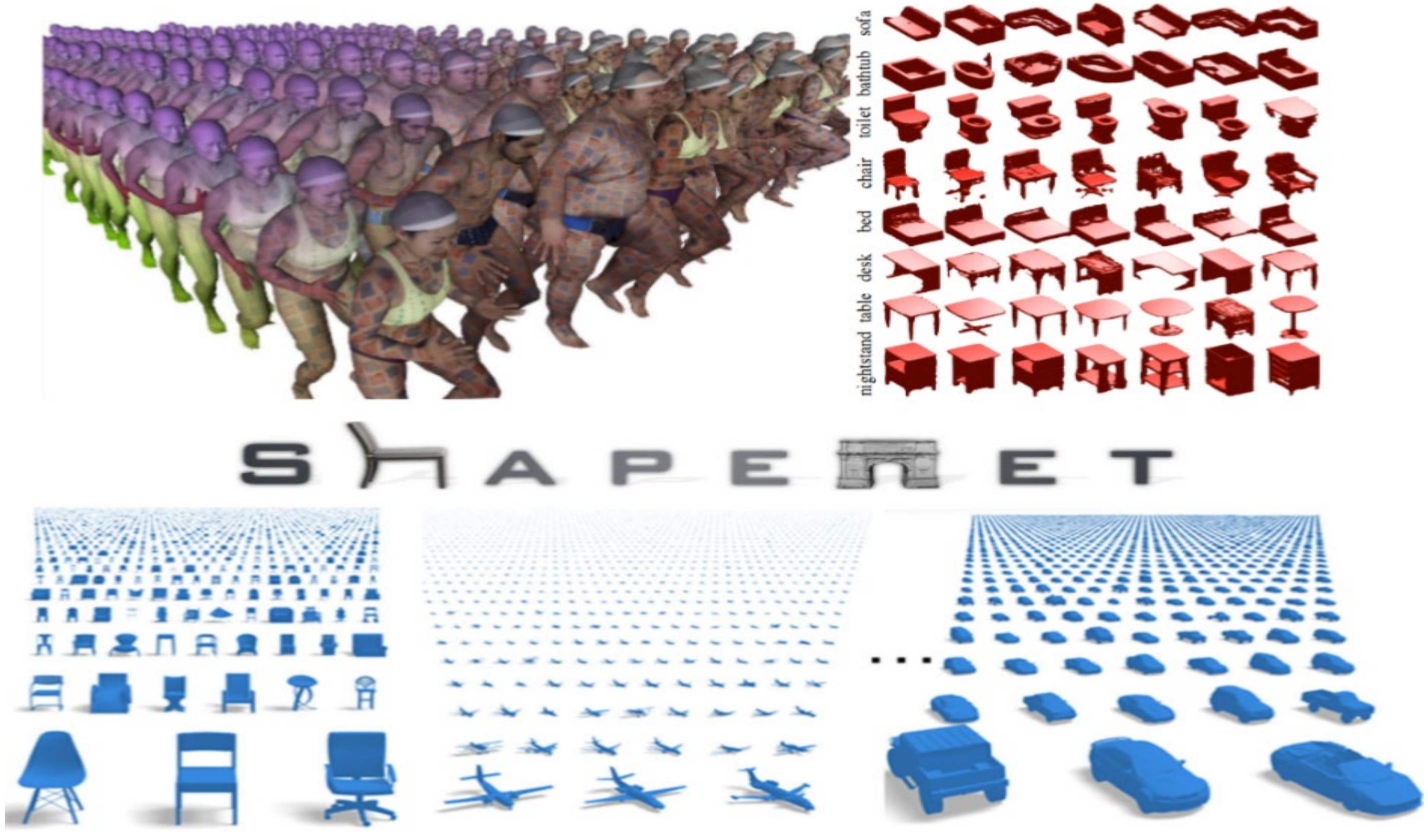
- Emergence
- Reification
- Multistability
- **Invariance**



https://en.wikipedia.org/wiki/Gestalt_psychology#Properties

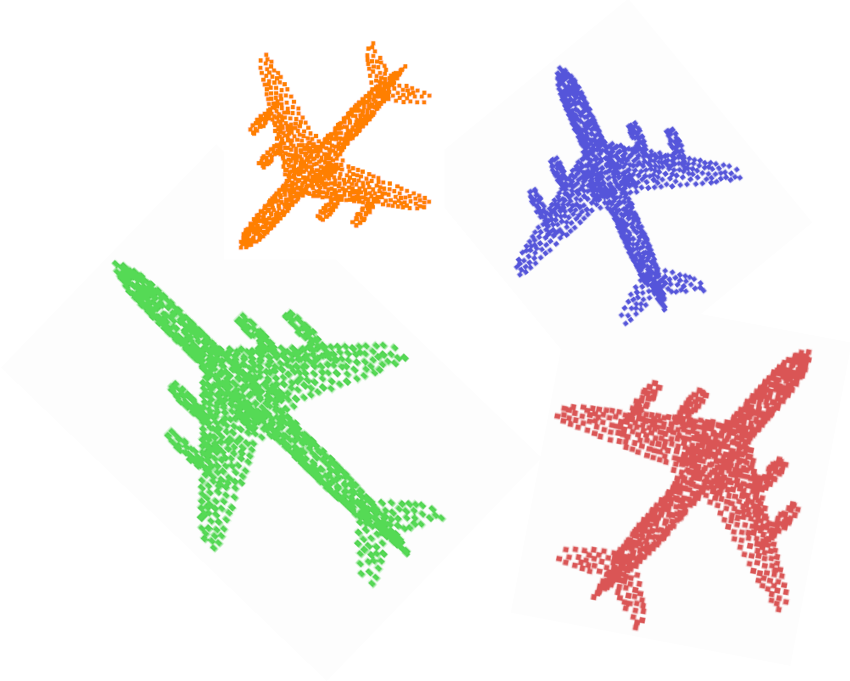
But Neural Networks Struggle…

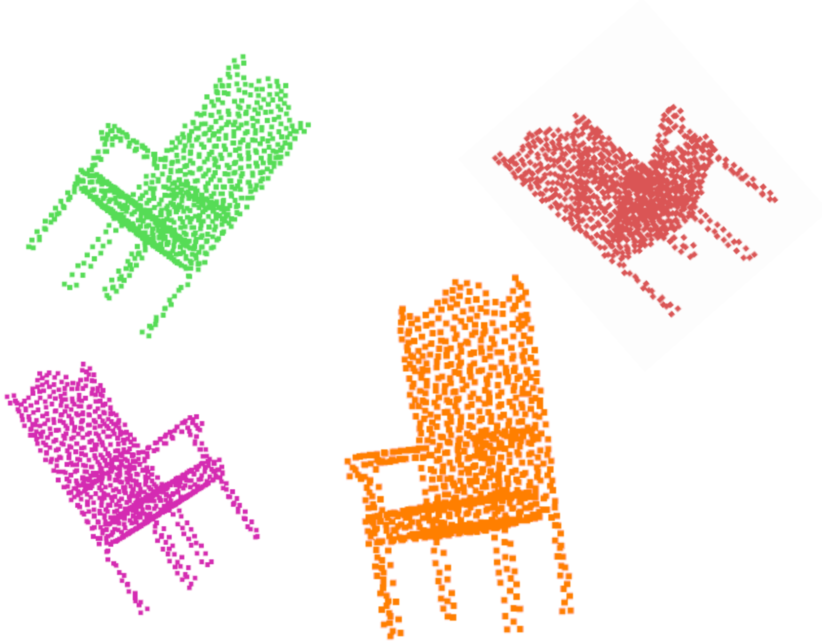
- Many datasets are **aligned** (shapes under canonical poses)
- Networks trained aligned data **cannot generalize** to arbitrary poses



But Neural Networks Struggle…

Identical objects coming in different poses, scales, ratios, colors…
– viewed as totally irrelevant entities by classical neural networks



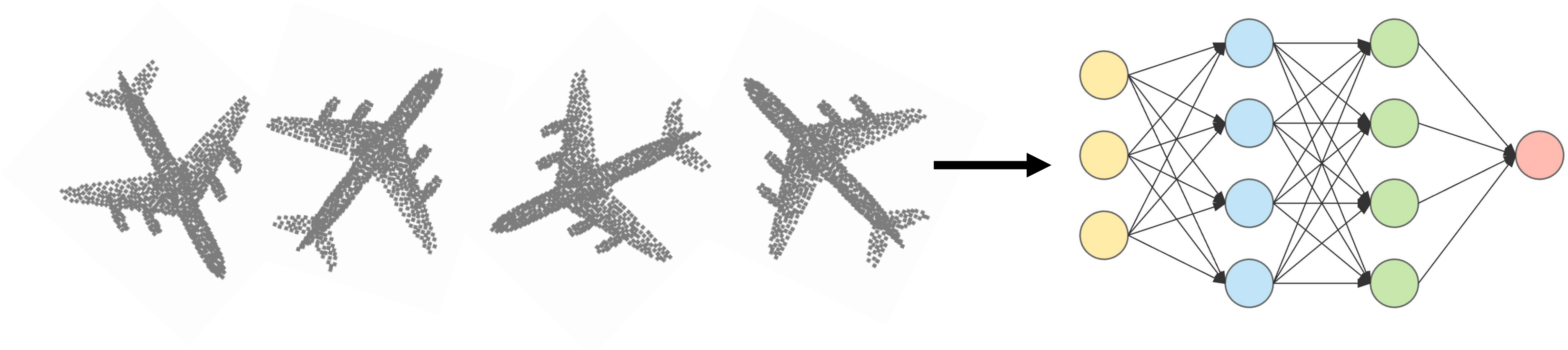


A Naïve Solution: Data Augmentation

Apply random rotation to the training data

So we let the network “see” and learn from all possible poses

- Reducing the generalization gap – but not eliminating it
- Sacrificing data-efficiency – longer training time
- Statistically equivariant/invariant – not guaranteed

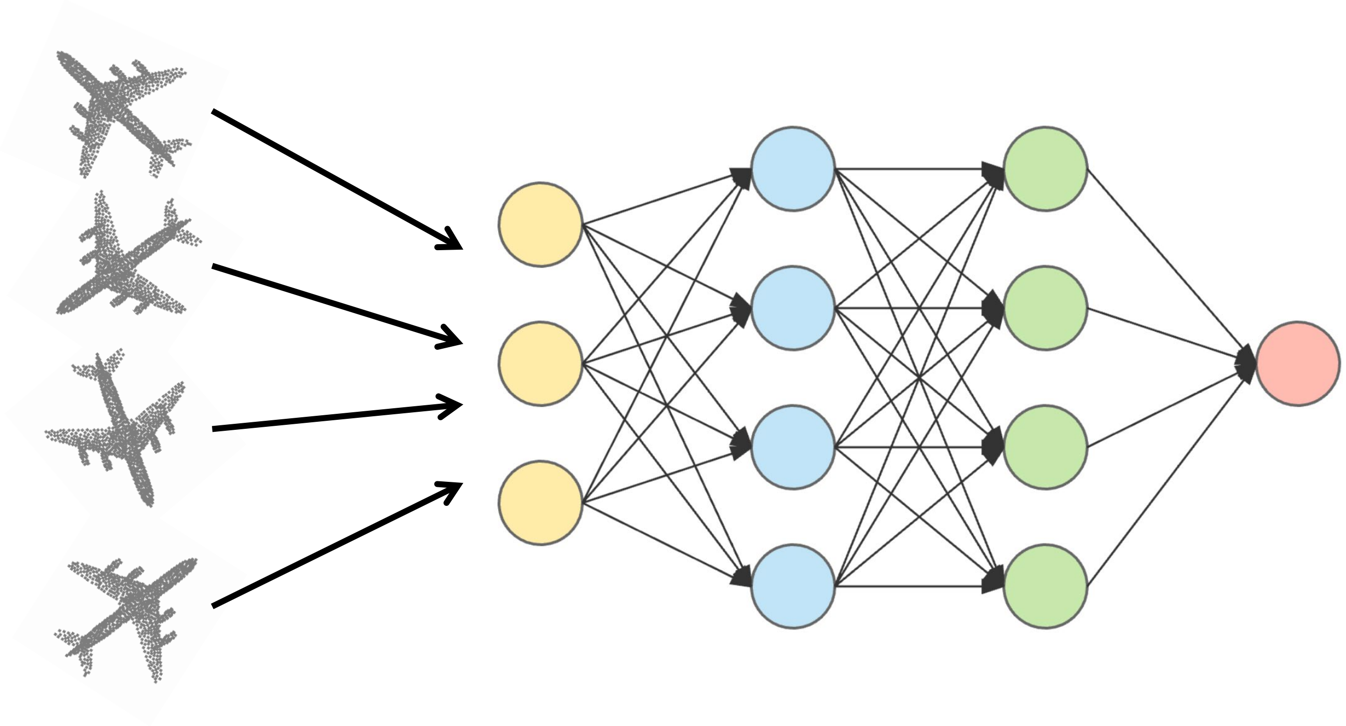


A Less Naïve Solution: Multi-view Approach

Feed multiple poses of the same object to the network **at once**

“If we don’t know what pose to look at, why not just look at **all** poses!”

- Can reproduce fairly good results
- Sacrifice data-efficiency – more memory consumption
- Theoretically equivariant – up to precision errors caused by discretization

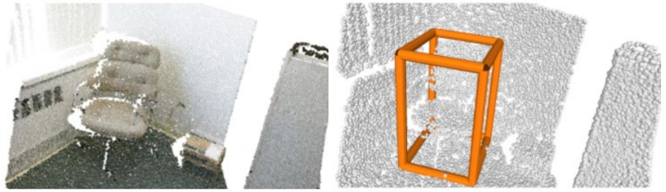


Summary: Why Equivariance / Invariance?

✓**"Intelligence":** More aligned with human perception

✓**Generalization:** Eliminate the prior that all shapes in a dataset (e.g., ShapeNet) are aligned

✓**Real-world applications:** Shapes (3D scans) in the wild may not have or come with canonical poses

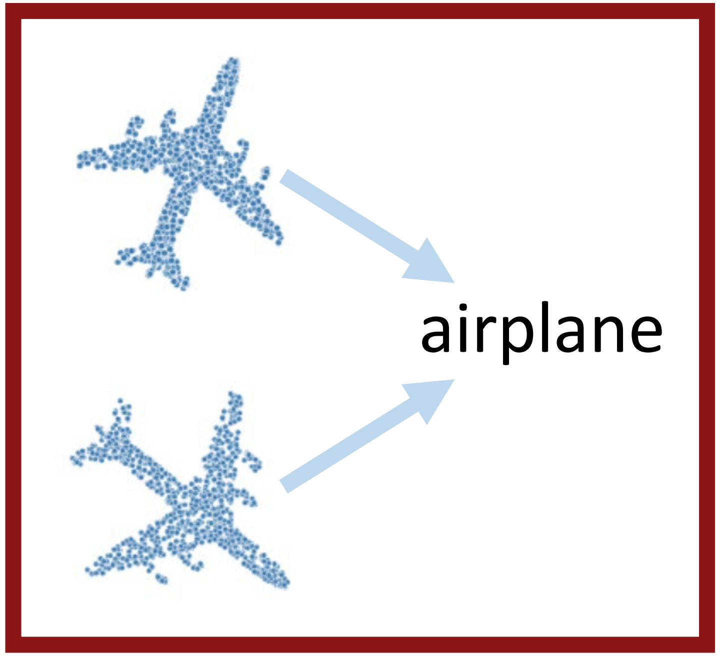


✓**Data-efficiency:** Avoid exhaustive data augmentation

[A. X. Chang, T. Funkhouser, L. Guibas, P. Hanrahan, Q. Huang, Z. Li, ... F. Yu, arXiv:1512.03012 (2015)]

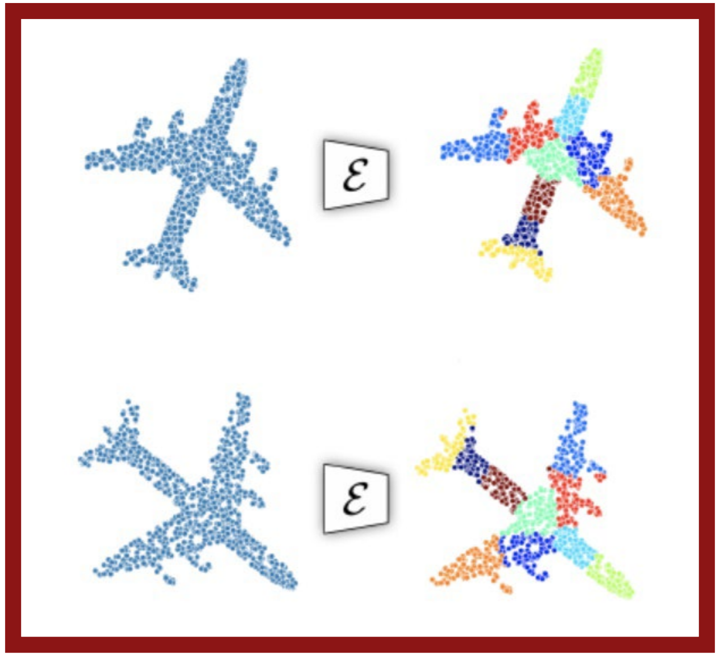
So We're Looking for…

classification



invariant

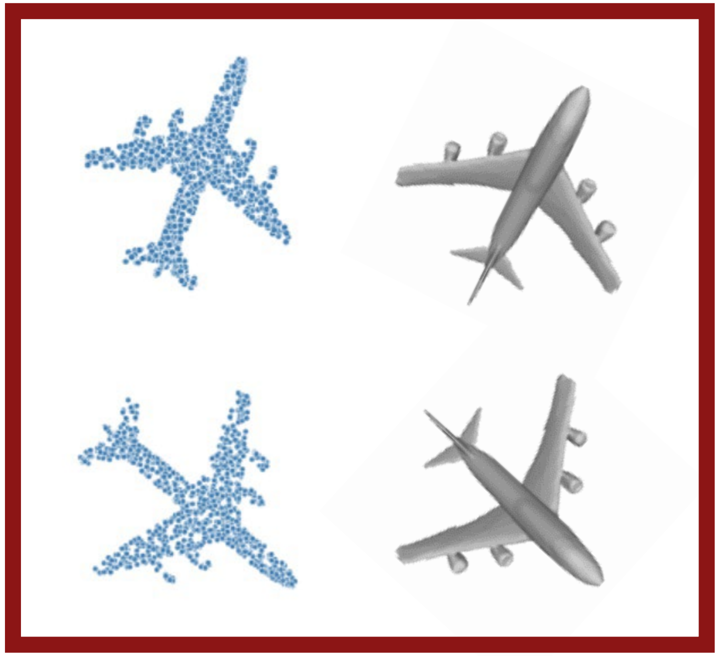
segmentation



equivariant

(or **invariant**, depending on data representation)

reconstruction



equivariant encoder

invariant decoder

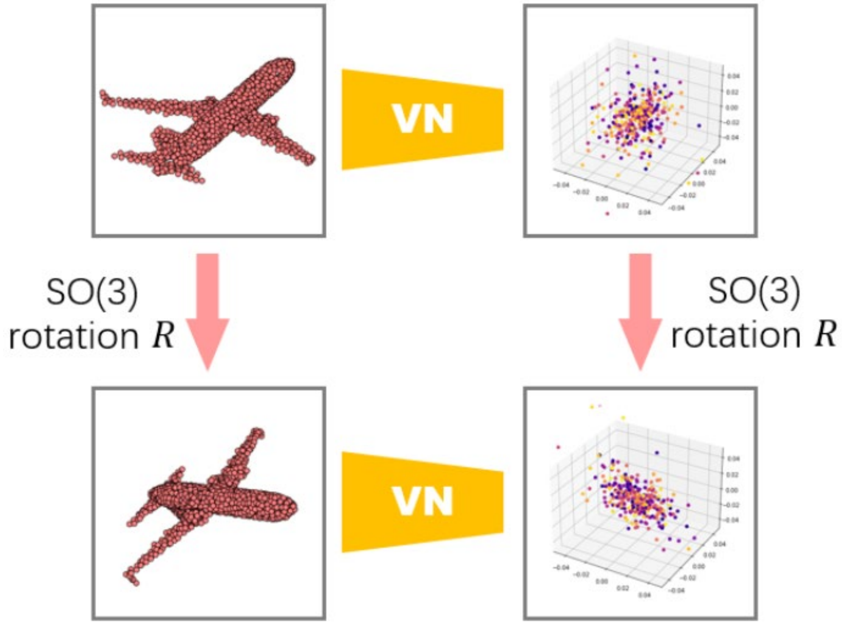
[W. Sun, A. Tagliasacchi, B. Deng, S. Sabour, S. Yazdani, G. Hinton, K. M. Yi, arXiv:2012.04718 (2020)]
[J. J. Park, P. Florence, J. Straub, R. Newcombe, S. Lovegrove, CVPR 2019]

Equivariance!

We say a neural network $f(\cdot\,;\theta)$ is rotation equivariant, if for any 3D rotation $R \in \mathrm{SO}(3)$ applied to its input $\mathbf{x}$, it is explicitly related to a transformation $D(R)$ on the network output satisfying

$$f(\mathbf{x}R;\theta) = f(\mathbf{x};\theta)D(R)$$

- $D(R)$ should be independent of $\mathbf{x}$
- **Special case:** when $D(R) = R$ is the identity mapping, it is the common-sense “equivariance”
- **Special case:** when $D(R) = \mathrm{I}$ is the constant mapping, it is invariance



Vector Neurons (VN)

[C. Deng, O. Litany, Y. Duan, A. Poulenard, A. Tagliasacchi, L. J. Guibas, ICCV 2021]

Vector Neurons

Classical (scalar) feature $\boldsymbol{z} = [z_1, z_2, \cdots, z_C]^\top \in \mathbb{R}^C$, **with** $z_i \in \mathbb{R}$

Vector-list feature $\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_C]^\top \in \mathbb{R}^{C\times 3}$, **with** $\boldsymbol{v}_i \in \mathbb{R}^3$

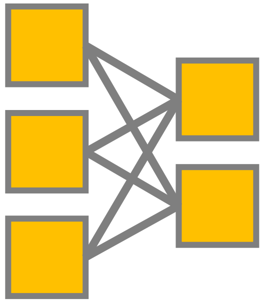
- For pointcloud with N points $\mathcal{V} = \{\boldsymbol{V}_1, \boldsymbol{V}_2, \cdots, \boldsymbol{V}_N\} \in \mathbb{R}^{N\times C\times 3}$

Mapping between network layers:

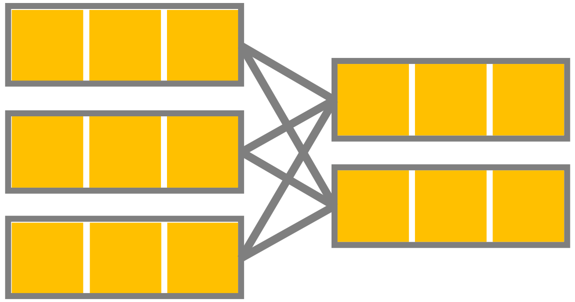
$$f(\cdot;\theta) : \mathbb{R}^{N\times C^{(d)}\times 3} \to \mathbb{R}^{N\times C^{(d+1)}\times 3}$$

? Equivariance to rotation $R \in \mathrm{SO}(3)$:

$$f(\mathcal{V}R;\theta) = f(\mathcal{V};\theta)R$$



(classical) scalar neurons

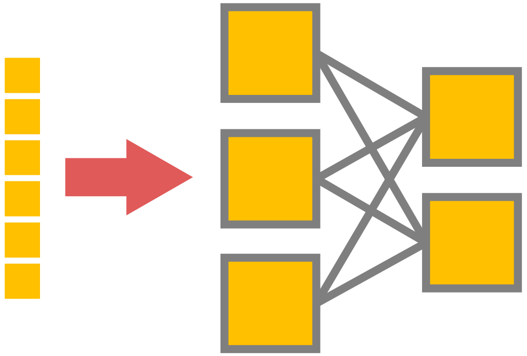


vector neurons

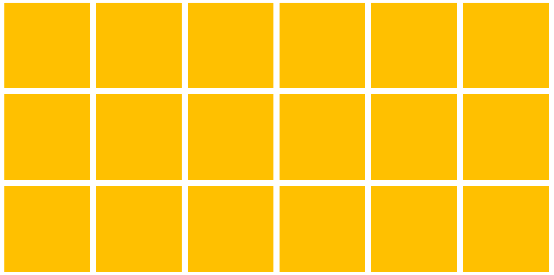
VN Features

Classical: scalar channels

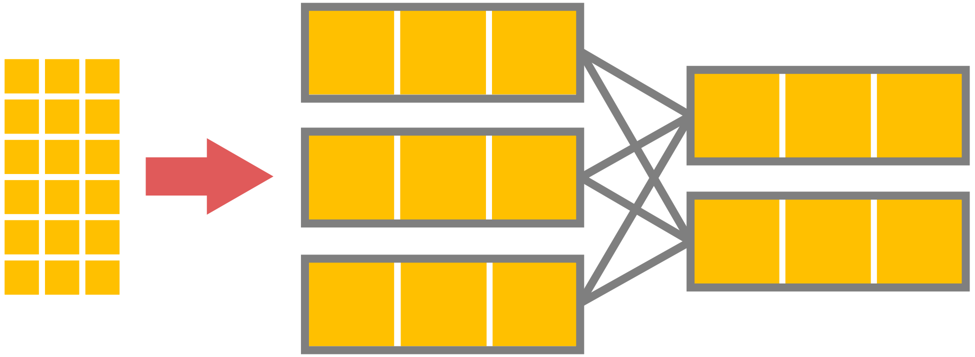
$C \times 1$ feature



VN: 3D vector channels

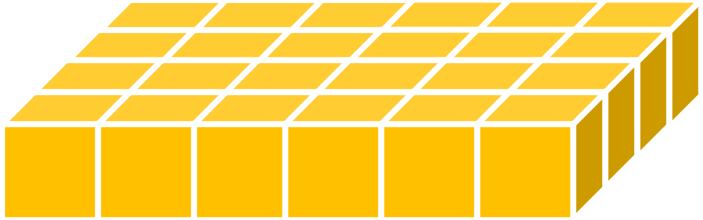


$C \times 3$ feature



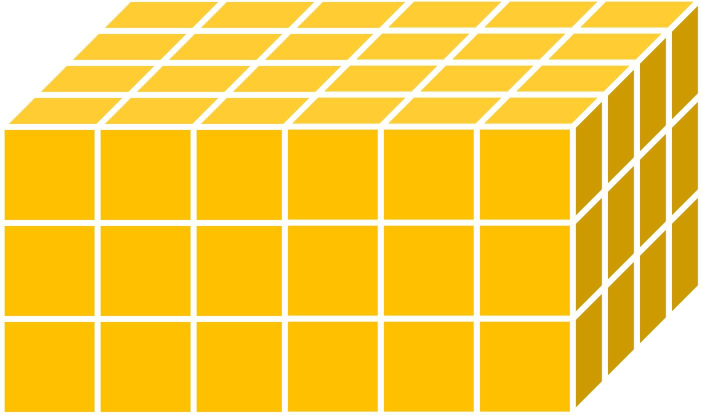
VN Features (for Point Cloud)

Classical:

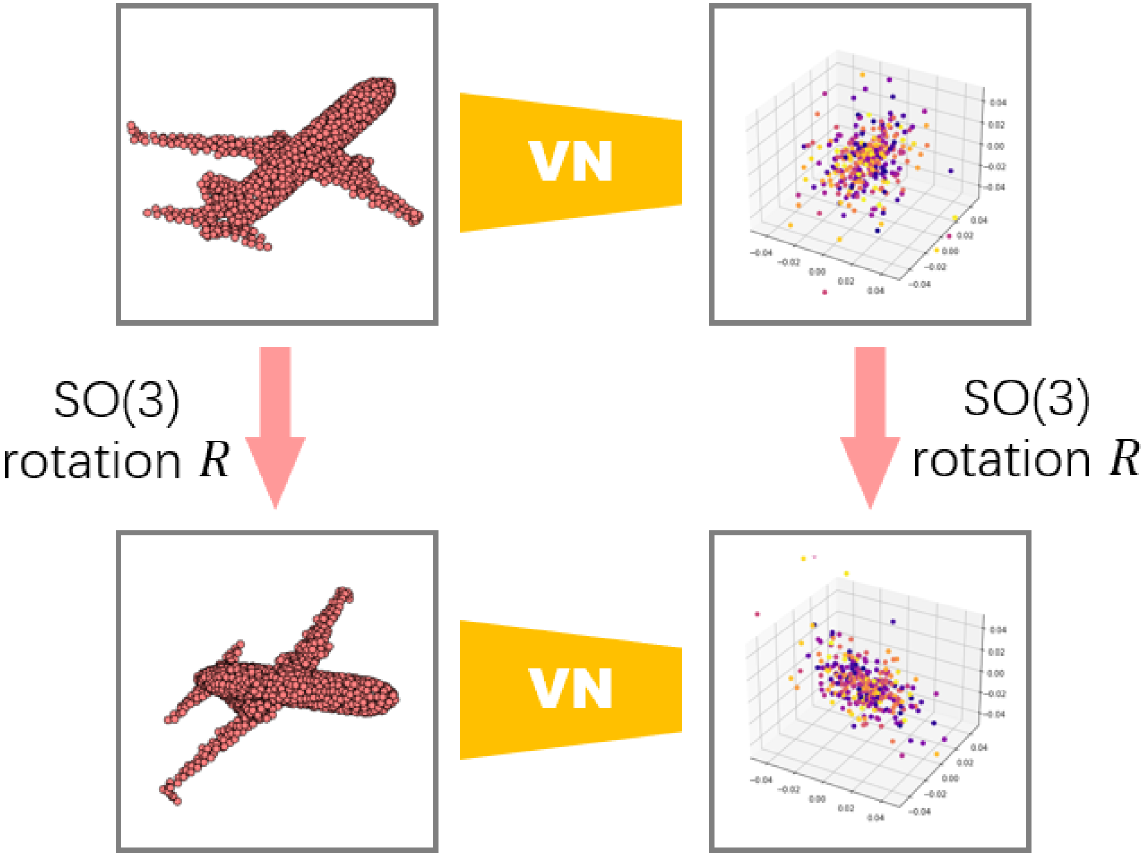


$N \times C \times 1$ feature

VN:

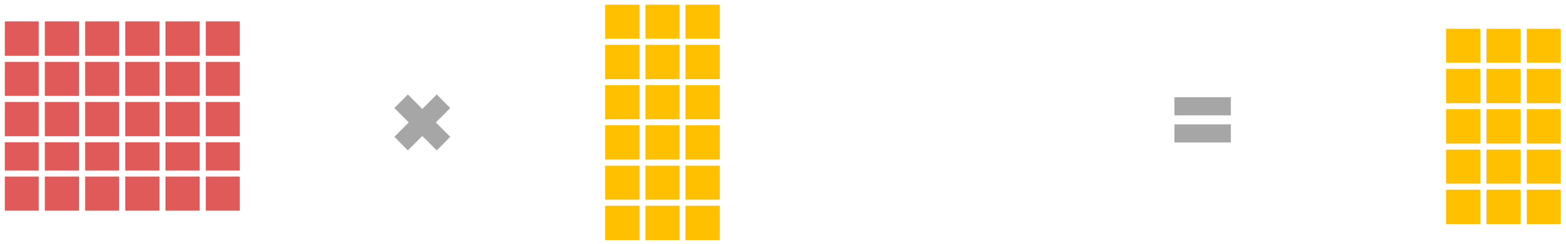


$N \times C \times 3$ feature



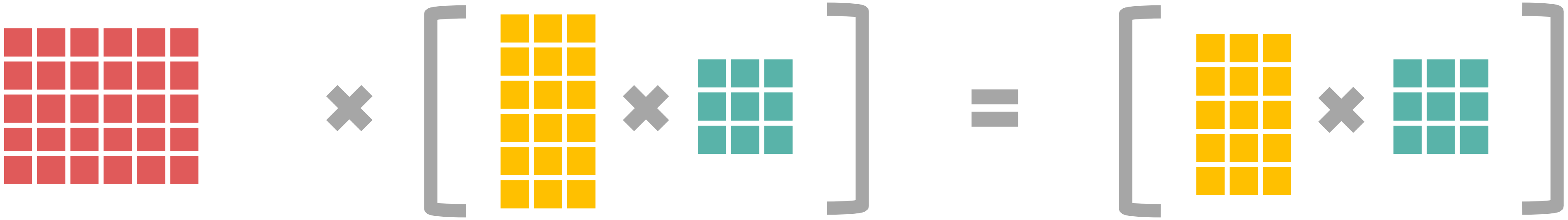
VN Linear Layer

Linear operator: left multiply by the learnable weight matrix



$C' \times C$ weight $C \times 3$ feature $C' \times 3$ feature

Equivariance: right multiply by the SO(3) rotation matrix



$C' \times C$ weight $C \times 3$ feature $C' \times 3$ feature

VN Linear Layer

Vector-list feature $\boldsymbol{V} \in \mathbb{R}^{C\times 3}$

Linear operator $f_{\mathrm{lin}}(\,\cdot\,;\mathbf{W})$ with learnable weights $\mathbf{W} \in \mathbb{R}^{C'\times C}$:

$$\boldsymbol{V}' = f_{\mathrm{lin}}(\boldsymbol{V};\mathbf{W}) = \mathbf{W}\boldsymbol{V} \in \mathbb{R}^{C'\times 3}$$

Equivariance to rotation $R \in \mathrm{SO}(3)$:

$$f_{\mathrm{lin}}(\boldsymbol{V}R;\mathbf{W}) = \mathbf{W}\boldsymbol{V}R = f_{\mathrm{lin}}(\boldsymbol{V};\mathbf{W})R = \boldsymbol{V}'R$$

- $\mathbf{W}$- left multiplication, R- right multiplication
- Note the absence of a bias term

VN Non-Linearity

ReLU Non-Linearity

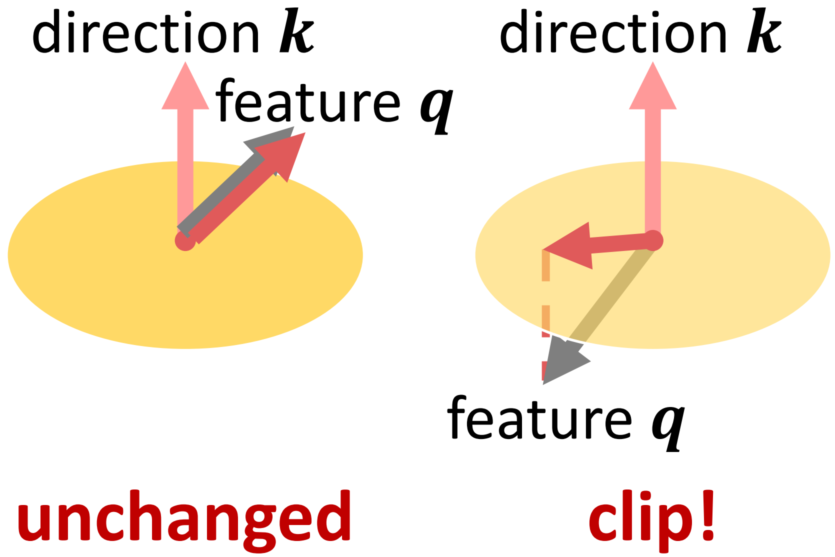
Weights $\mathbf{W} \in \mathbb{R}^{1\times C}$ and $\mathbf{U} \in \mathbb{R}^{1\times C}$

Learn a feature $\boldsymbol{q} = \mathbf{W}\boldsymbol{V} \in \mathbb{R}^{1\times 3}$

Learn a direction $\boldsymbol{k} = \mathbf{U}\boldsymbol{V} \in \mathbb{R}^{1\times 3}$

For each output vector neuron $\boldsymbol{v}' \in \boldsymbol{V}'$

$$\boldsymbol{v}' = \begin{cases} \boldsymbol{q} & \text{if } \langle \boldsymbol{q}, \boldsymbol{k} \rangle \geqslant 0 \\ \boldsymbol{q} - \langle \boldsymbol{q}, \frac{\boldsymbol{k}}{\|\boldsymbol{k}\|} \rangle \frac{\boldsymbol{k}}{\|\boldsymbol{k}\|} & \text{otherwise} \end{cases}$$



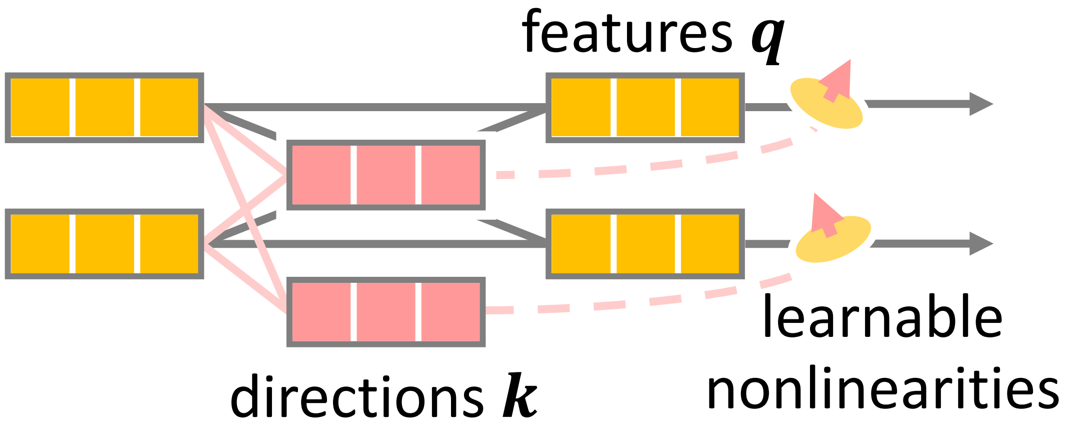
VN Non-Linearity

ReLU Non-Linearity

- **Non-linear layer** (with built-in linear layer)
  = **input linear transformation q + non-linearity k**

- A single non-linear layer detached from linear layer: $\boldsymbol{q} = \boldsymbol{v}$
  - introduce additional network depth!

- Other non-linearities



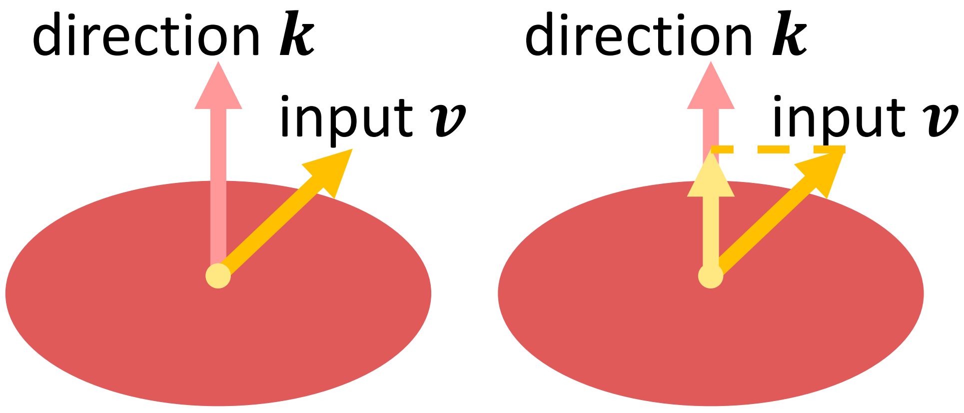
overall structure

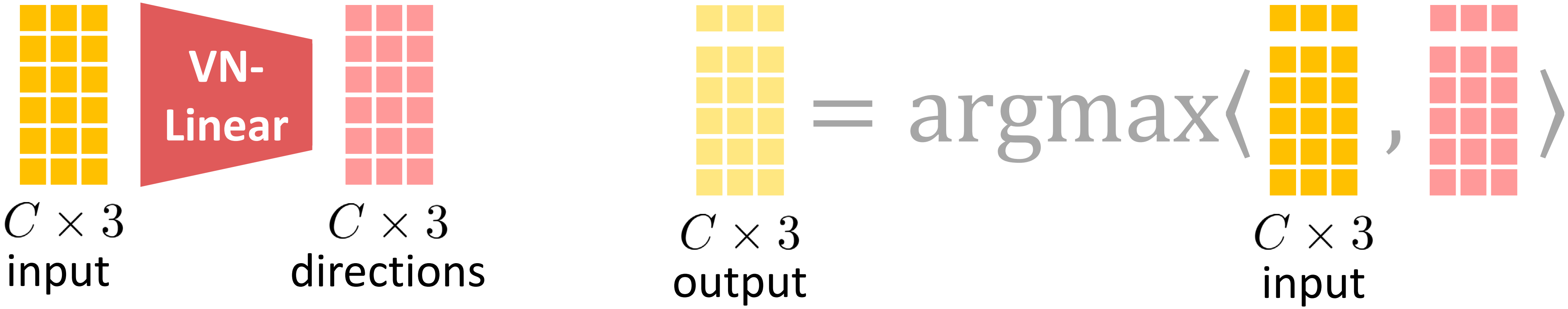
VN Pooling

✓ **Mean pooling**

? Max pooling

- (Similar to non-linearity)
- argmax alone learned directions





VN Pooling

✓**Mean pooling**

? Max pooling

- similar to non-linearities → use learnable directions

Given a set of vector-lists $\mathcal{V} = \{\boldsymbol{V}_1, \boldsymbol{V}_2, \cdots, \boldsymbol{V}_N\} \in \mathbb{R}^{N \times C \times 3}$

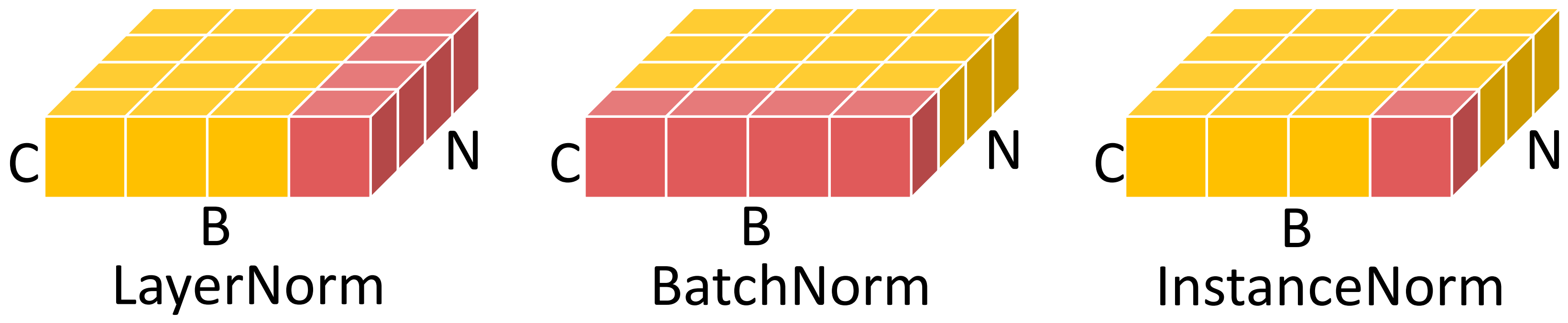
Learn data-dependent directions $\mathcal{K} = \{\mathbf{W}\boldsymbol{V}_n\}_{n=1}^{N} \in \mathbb{R}^{N \times C \times 3}$ with learnable weights $\mathbf{W} \in \mathbb{R}^{C \times C}$

Max pool along $\mathcal{K}$ directions: for each channel $c \in [C]$

$$f_{\mathrm{MAX}}(\mathcal{V})[c] = \boldsymbol{V}_{n^*}[c] \quad \text{where } n^* = \operatorname{argmax}\langle \mathbf{W}\boldsymbol{V}_n[c], \boldsymbol{V}_n[c] \rangle$$

VN Normalizations

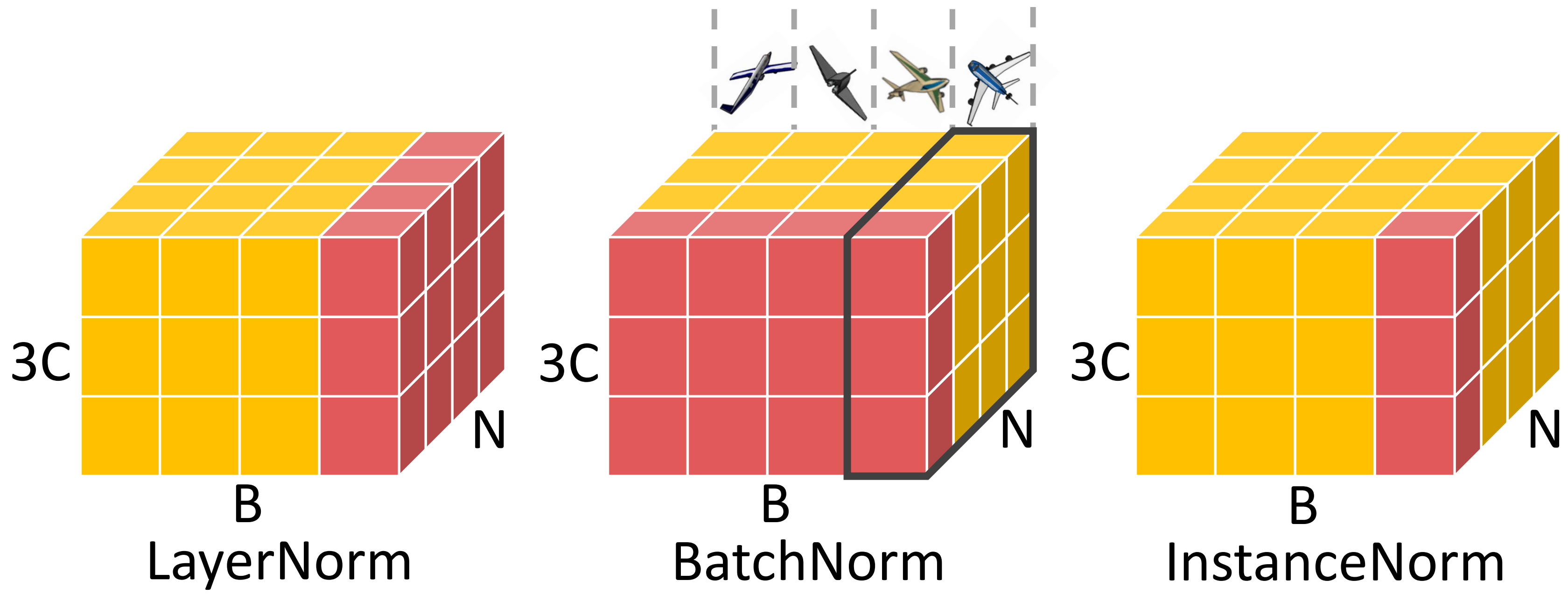
- ✓ **LayerNorm**
- ✓ **InstanceNorm**
- ✓ **Dropout**



(classical) scalar neurons

? BatchNorm

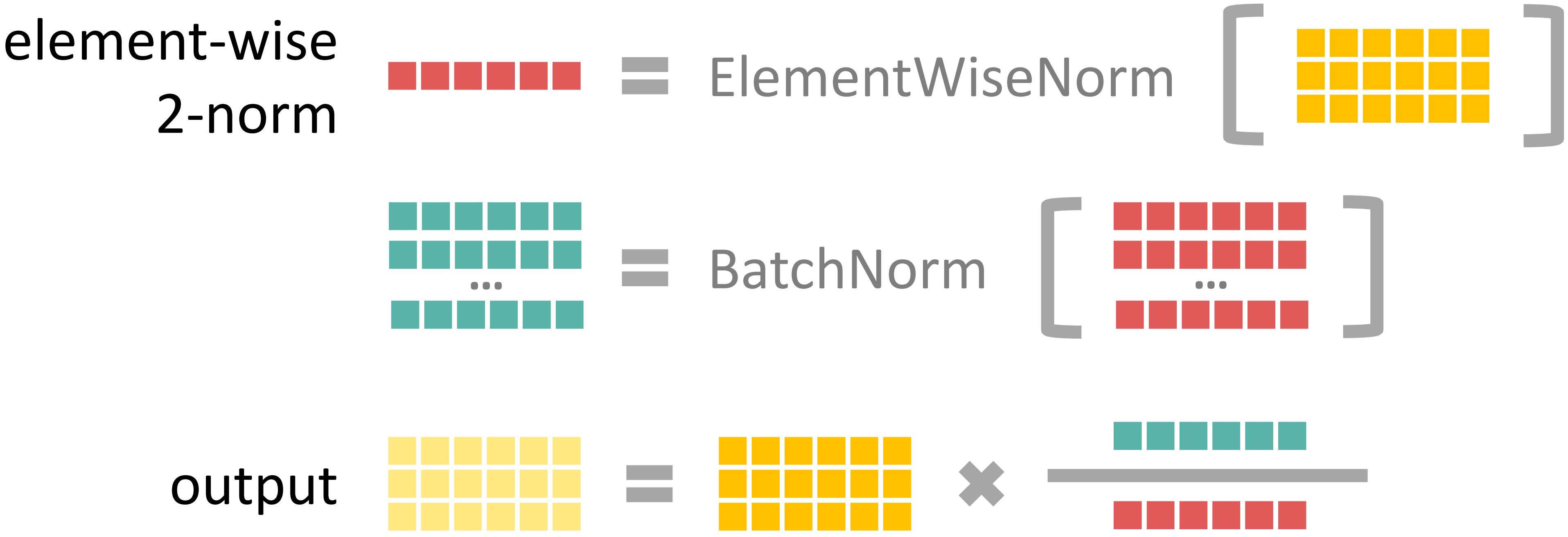
averaging across arbitrarily rotated inputs would not necessarily be meaningful



Vector neurons

VN Normalizations

BatchNorm



VN Normalizations

BatchNorm

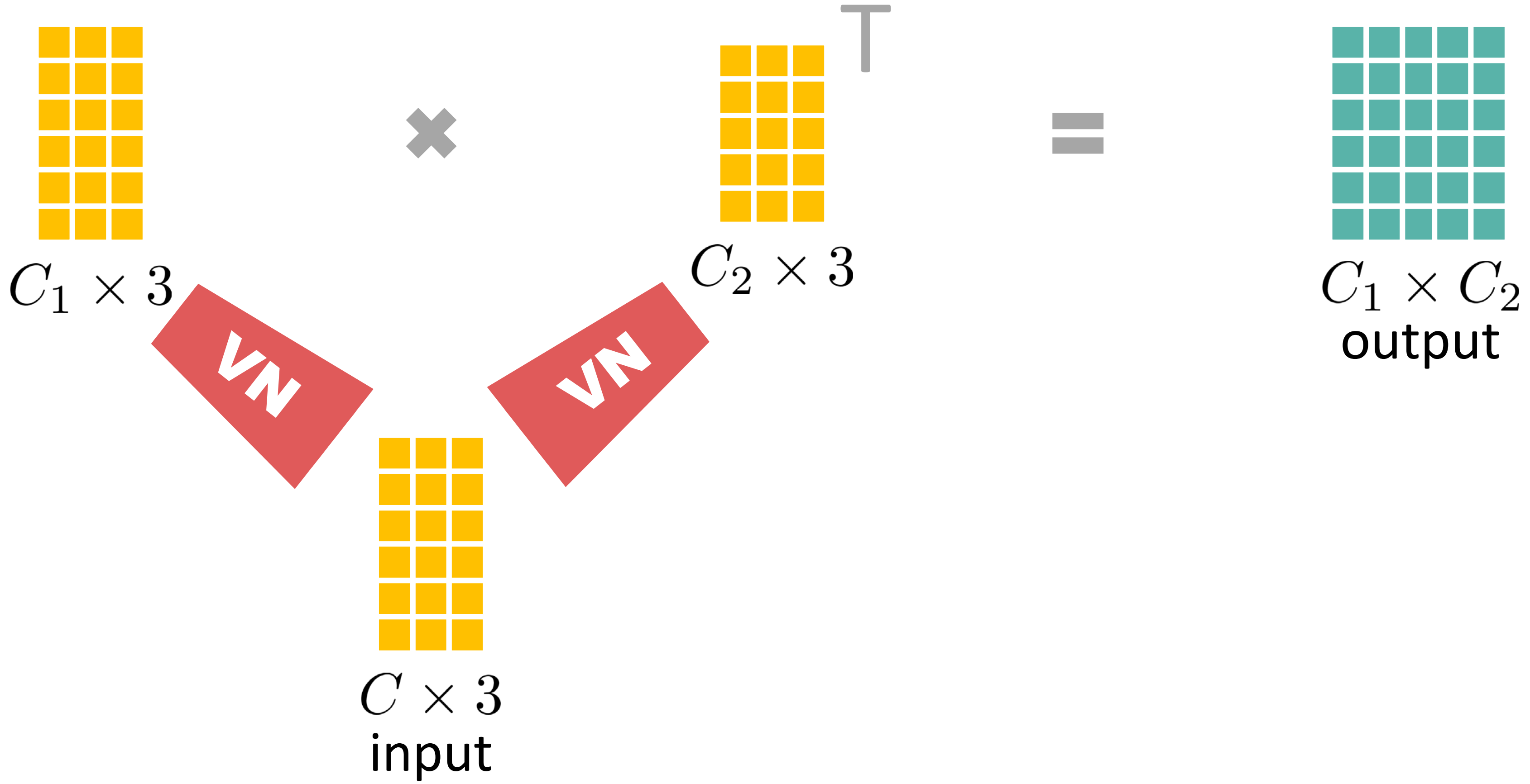
- Normalize the **2-norm (invariant component)** of the vector-list feature

$$\boldsymbol{N}_b = \mathrm{ElementWiseNorm}(\boldsymbol{V}_b) \in \mathbb{R}^{N\times 1}$$
$$\{\boldsymbol{N}'_b\}_{b=1}^{B} = \mathrm{BatchNorm}\left(\{\boldsymbol{N}_b\}_{b=1}^{B}\right)$$
$$\boldsymbol{V}'_b = \boldsymbol{V}_b[c]\frac{\boldsymbol{N}'_b[c]}{\boldsymbol{N}_b[c]} \quad \forall c \in [C]$$

- Element-wise norm: 2-norm for each vector $\boldsymbol{v}_c \in \boldsymbol{V}_b$

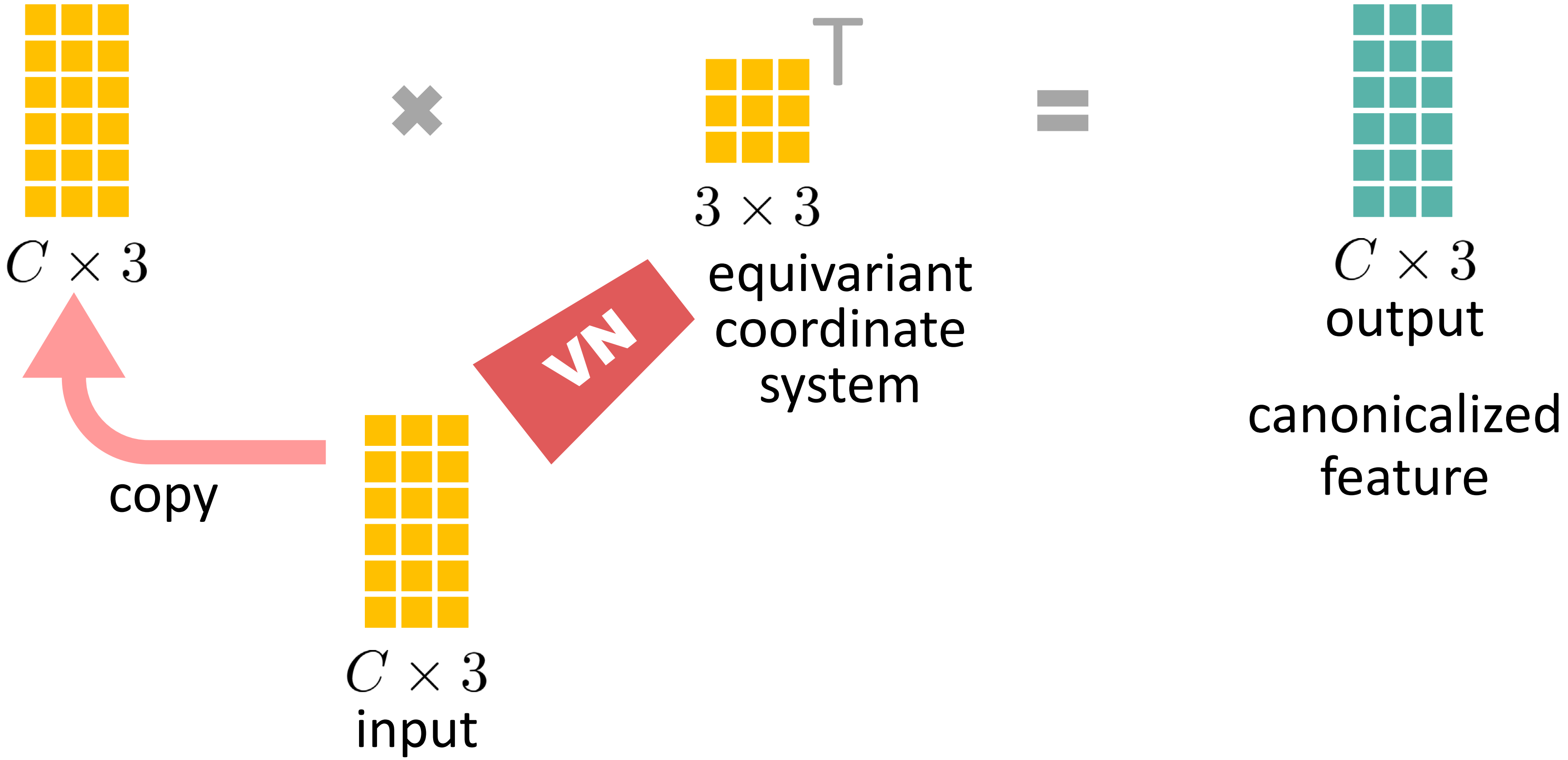
VN Invariant Layer

(equivariant feature) × (equivariant feature)$^{\top}$ = (invariant feature)



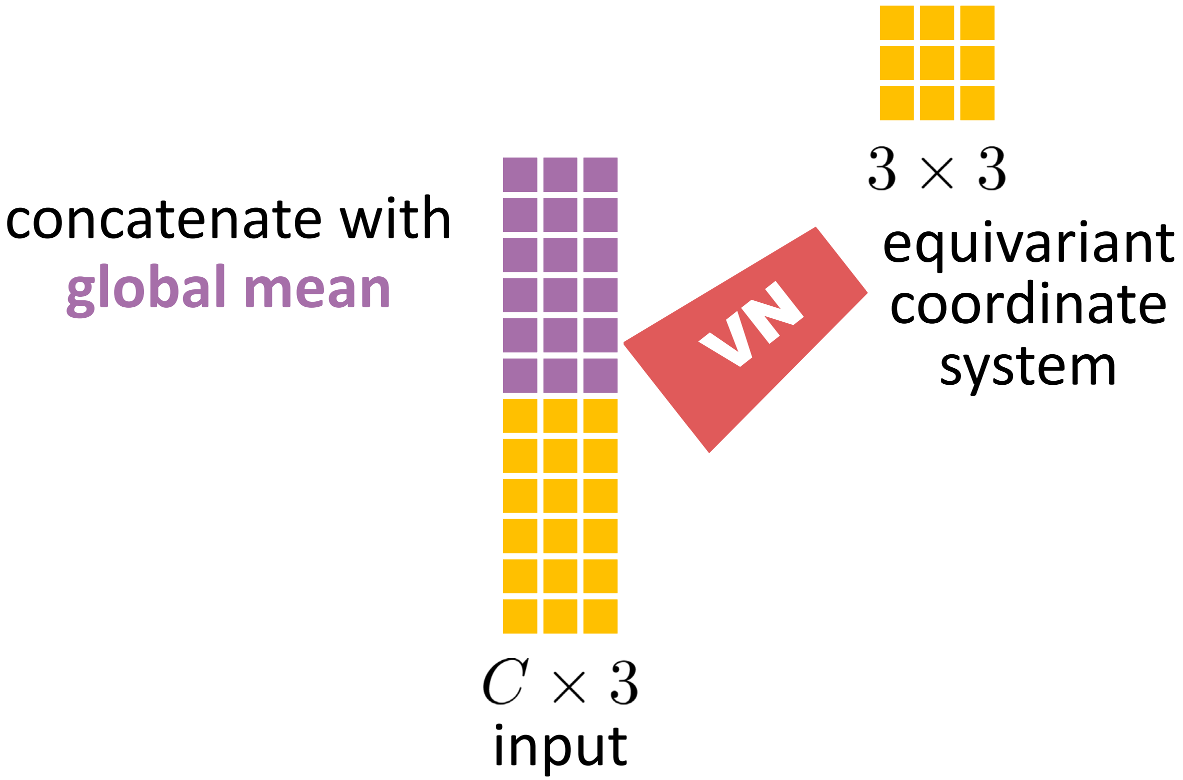
VN Invariant Layer

Specifically…



VN Invariant Layer

For pointcloud: combine **global information** with local features



VN Invariant Layer

- Product of an equivariant signal $\boldsymbol{V} \in \mathbb{R}^{C\times 3}$ by the transpose of another equivariant signal $\boldsymbol{T} \in \mathbb{R}^{C'\times 3}$ → invariant signal

- **Special case:** $\boldsymbol{T} \in \mathbb{R}^{3\times 3}$ - an equivariant coordinate system
- For pointcloud, concatenate local feature $\boldsymbol{V} \in \mathbb{R}^{C\times 3}$ with global mean $\overline{\boldsymbol{V}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{V}_n \in \mathbb{R}^{C\times 3}$

Invariant layer:

$$\boldsymbol{T}_n = \text{VN-MLP}([\boldsymbol{V}_n, \overline{\boldsymbol{V}}])$$

$$\text{VN-In}(\boldsymbol{V}_n) = \boldsymbol{V}_n \boldsymbol{T}_n^\top$$

Build VN Networks: VN-DGCNN

DGCNN

Edge feature: $${}'_{nm} = \mathrm{ReLU}(\Theta({}_m - {}_n) + \Phi\,{}_n)$$

Aggregation: $${}'_{n} = \mathrm{Pool}_{m:(n,m)\in\mathcal{E}}({}'_{nm})$$

VN-DGCNN

Edge feature: $${}'_{nm} = \text{VN-ReLU}(\Theta({}_m - {}_n) + \Phi\,{}_n)$$

Aggregation $${}'_{n} = \text{VN-Pool}_{m:(n,m)\in\mathcal{E}}({}'_{nm})$$

[Y. Wang, Y. Sun, Z. Liu, S. E. Sarma, M. M. Bronstein, J. M. Solomon, TOG 2019]

Build VN Networks: VN-DGCNN

Convolution in latent spaces, graph edges not embedded in $\mathbb{R}^3$

DGCNN: $\boldsymbol{x} \in \mathbb{R}^{C}, \boldsymbol{e}'_n, \boldsymbol{x}'_n \in \mathbb{R}^{C'}$ scalar features

Edge conv $\boldsymbol{e}'_{nm} = \mathrm{ReLU}(\Theta(\boldsymbol{x}_m - \boldsymbol{x}_n) + \Phi \boldsymbol{x}_n)$

Aggregation $\boldsymbol{x}'_n = \mathrm{Pool}_{m:(n,m)\in\mathcal{E}}(\boldsymbol{e}'_{nm})$

VN-DGCNN: $\boldsymbol{V} \in \mathbb{R}^{C\times 3}, \boldsymbol{E}'_n, \boldsymbol{V}'_n \in \mathbb{R}^{C'\times 3}$ vector-list features

Edge conv $\boldsymbol{E}'_{nm} = \text{VN-ReLU}(\Theta(\boldsymbol{V}_m - \boldsymbol{V}_n) + \Phi \boldsymbol{V}_n)$

Aggregation $\boldsymbol{V}'_n = \text{VN-Pool}_{m:(n,m)\in\mathcal{E}}(\boldsymbol{E}'_{nm})$

[Y. Wang, Y. Sun, Z. Liu, S. E. Sarma, M. M. Bronstein, J. M. Solomon, TOG 2019]

Build VN Networks: VN-PointNet

PointNet

$$' = \mathrm{Pool}_{\boldsymbol{x}_n \in \mathcal{X}}\left(h(\ _1), h(\ _2) \cdots, h(\ _N)\right)$$

VN-PointNet

$$' = \text{VN-Pool}_{\boldsymbol{V}_n \in \mathcal{V}}\left(f(\ _1), f(\ _2), \cdots, f(\ _N)\right)$$

[C. R. Qi, H. Su, K. Mo, L. J. Guibas, CVPR 2017]

Build VN Networks: VN-PointNet

No convolutions, only point-wise feature transformations

PointNet: $$\boldsymbol{x}' = \mathrm{Pool}_{\boldsymbol{x}_n \in \mathcal{X}}(h(\boldsymbol{x}_1), \cdots, h(\boldsymbol{x}_N))$$

VN-PointNet: $$\boldsymbol{V}' = \text{VN-Pool}_{\boldsymbol{V}_n \in \mathcal{V}}(f(\boldsymbol{V}_1), \cdots, f(\boldsymbol{V}_N))$$

- h MLP, f VN-MLP
- First layer single channel → edge conv to lift dimensionality
  - “Cannot apply per-channel transformation to gray-scale images”

[C. R. Qi, H. Su, K. Mo, L. J. Guibas, CVPR 2017]

Classification

Classification results on ModelNet40

VN Networks

Rotation sensitive methods

Rotation robust methods

Methods	z/z	$z/\mathrm{SO}(3)$	$\mathrm{SO}(3)/\mathrm{SO}(3)$
Point / mesh inputs			
PointNet [25]	85.9	19.6	74.7
DGCNN [35]	90.3	33.8	88.6
VN-PointNet	77.5	77.5	77.2
VN-DGCNN	89.5	**89.5**	**90.2**
PCNN [2]	92.3	11.9	85.1
ShellNet [40]	**93.1**	19.9	87.8
PointNet++ [26]	91.8	28.4	85.0
PointCNN [20]	92.5	41.2	84.5
Spherical-CNN [11]	88.9	76.7	86.9
a^3S-CNN [21]	89.6	87.9	88.7
SFCNN [27]	91.4	84.8	90.1
TFN [32]	88.5	85.3	87.6
RI-Conv [39]	86.5	86.4	86.4
SPHNet [24]	87.7	86.6	87.6
ClusterNet [6]	87.1	87.1	87.1
GC-Conv [41]	89.0	89.1	89.2
RI-Framework [18]	89.4	89.4	89.3

Classification

Classification results on ModelNet40

- VN networks are **robust to (seen & unseen) rotations**

- Excellent performance compared with other methods

- **SO(3)/SO(3):** equivariance by construction is better than rotation augmentation

Methods	z/z	$z/\mathrm{SO}(3)$	$\mathrm{SO}(3)/\mathrm{SO}(3)$
Point / mesh inputs			
PointNet [25]	85.9	19.6	74.7
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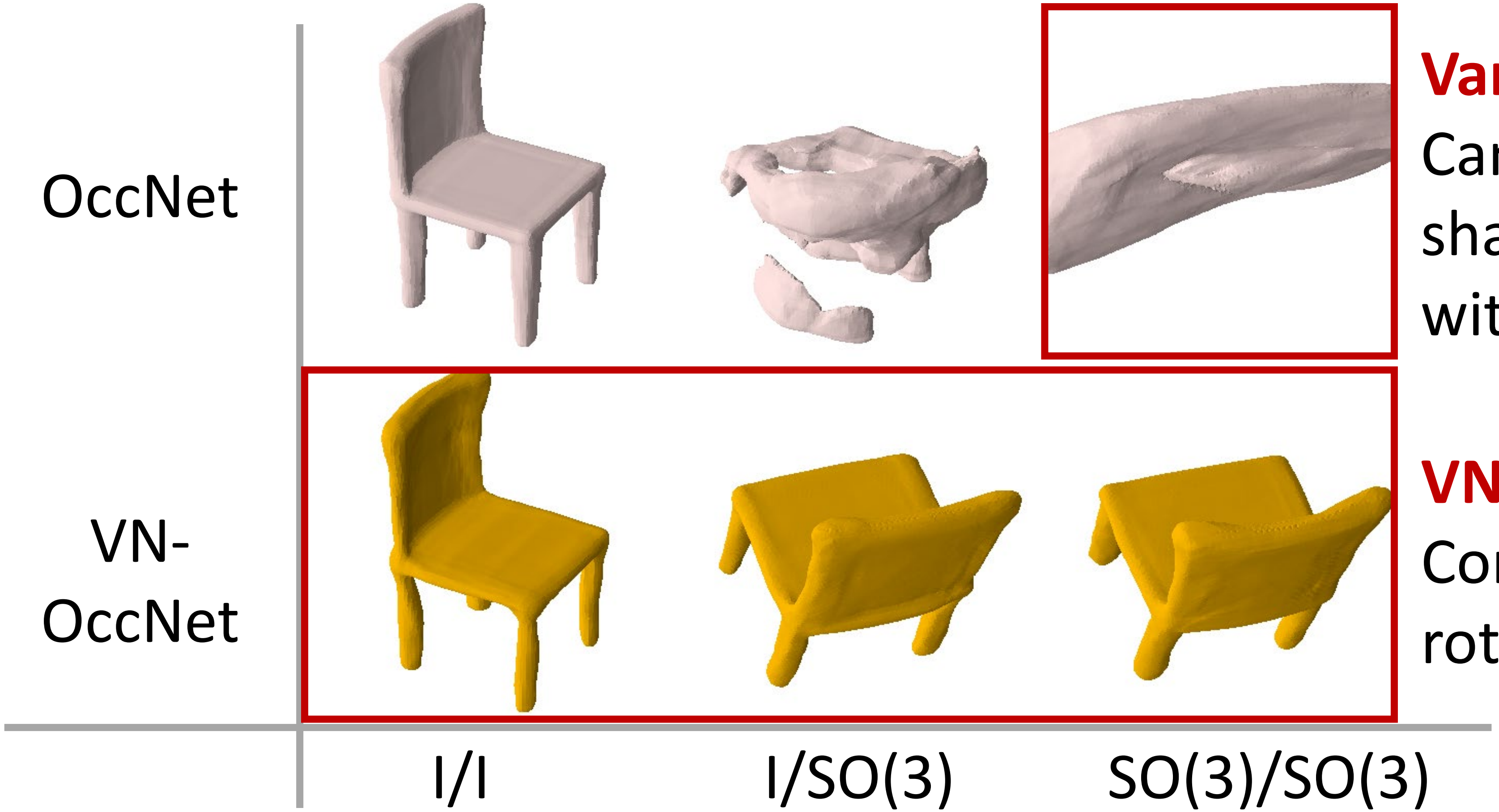
Neural Implicit Reconstruction

Results on ShapeNet (Examples)

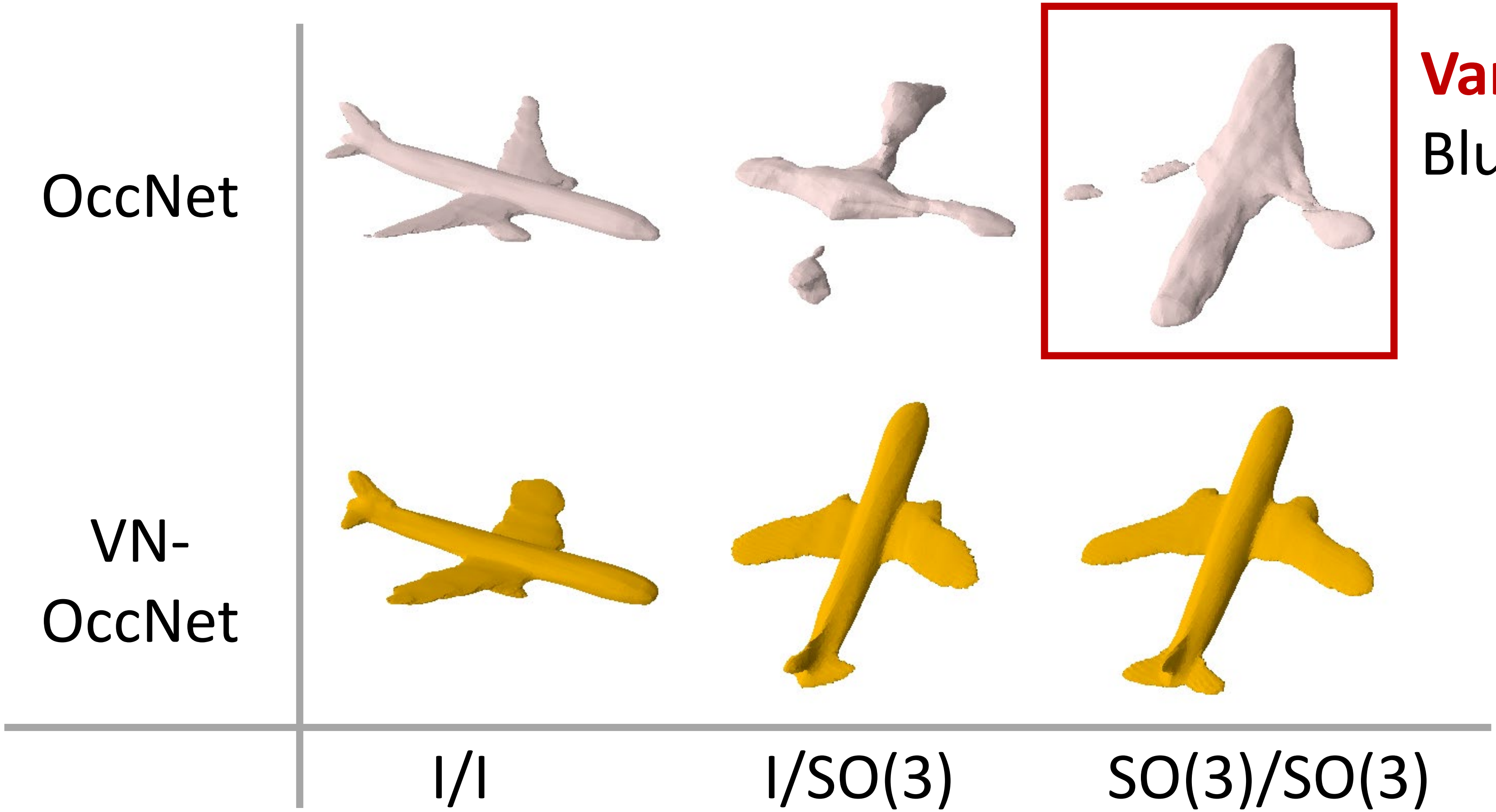


Vanilla OccNet:
Can’t learn rotated shapes even trained with augmentation

VN-OccNet:
Consistent across rotations

Neural Implicit Reconstruction

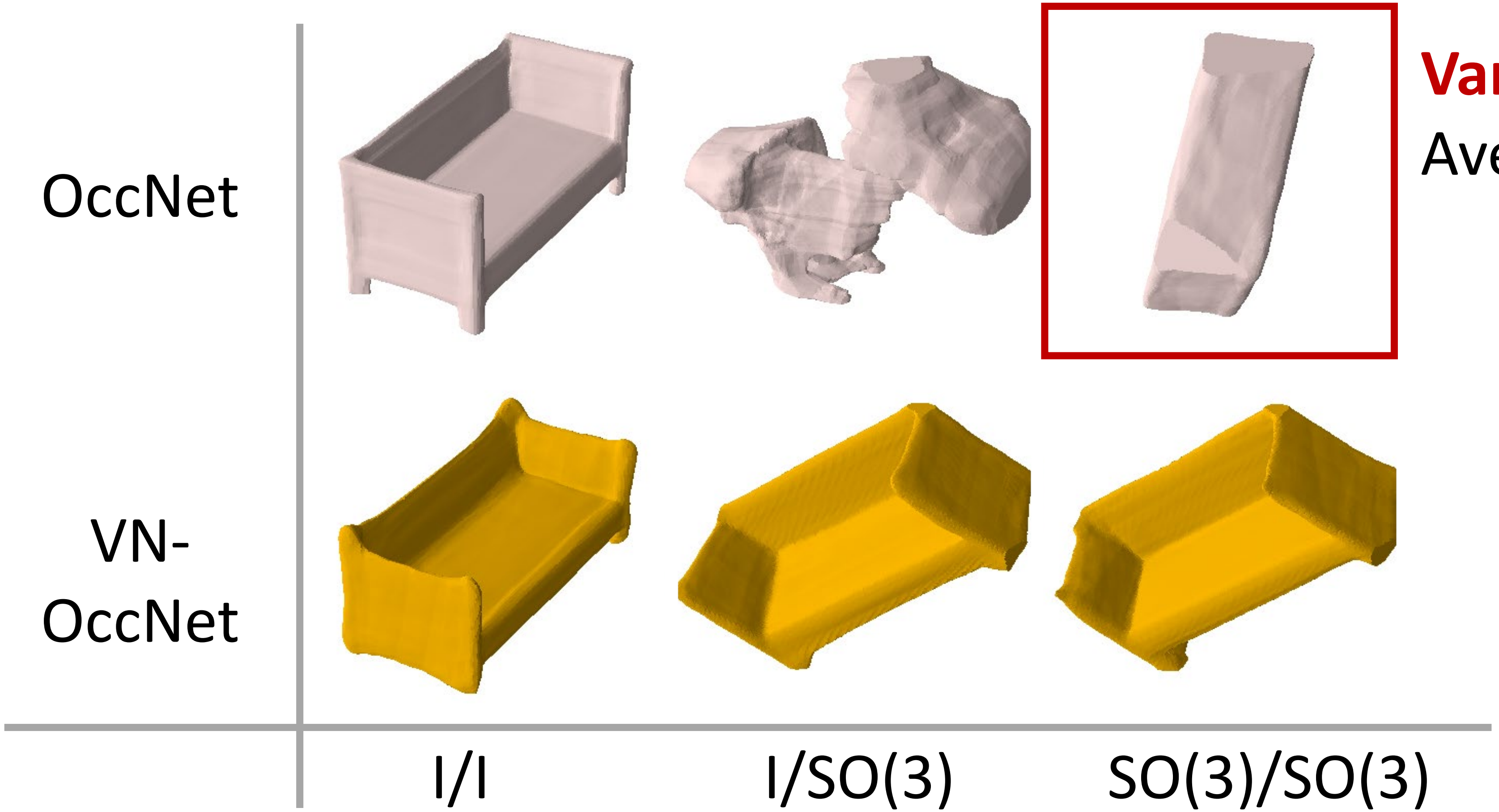
Results on ShapeNet (Examples)



Vanilla OccNet:
Blurry shapes

Neural Implicit Reconstruction

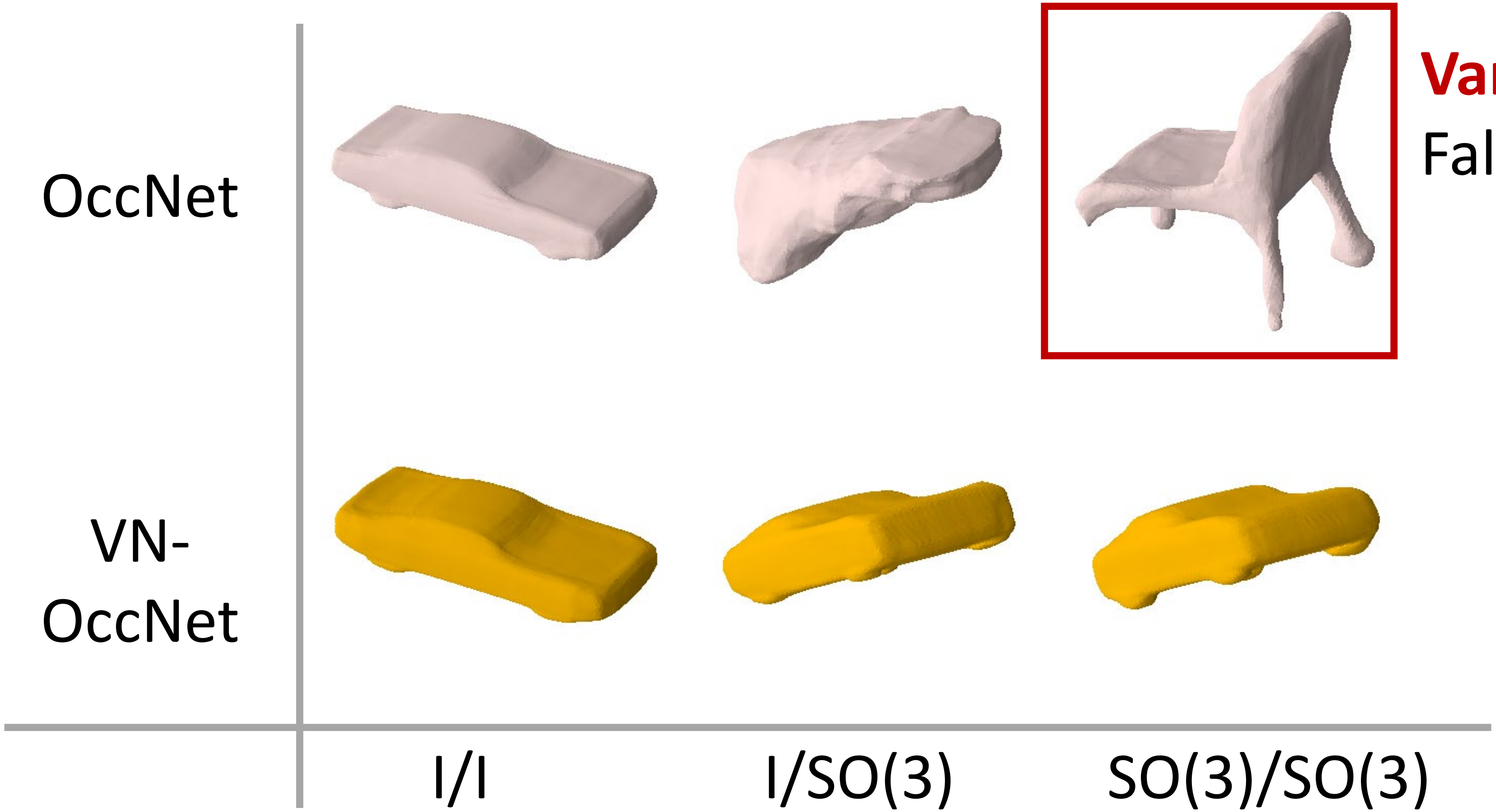
Results on ShapeNet (Examples)



Vanilla OccNet:
Averaged shapes

Neural Implicit Reconstruction

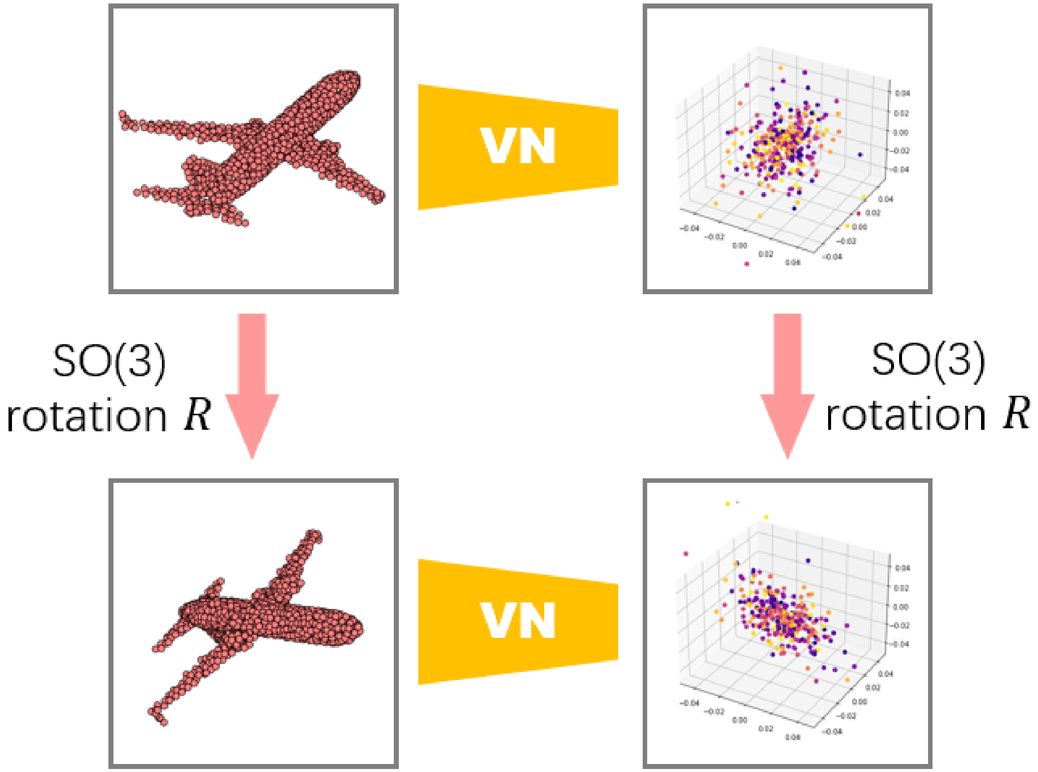
Results on ShapeNet (Examples)



Vanilla OccNet:
False shape priors

Summary: Vector Neurons

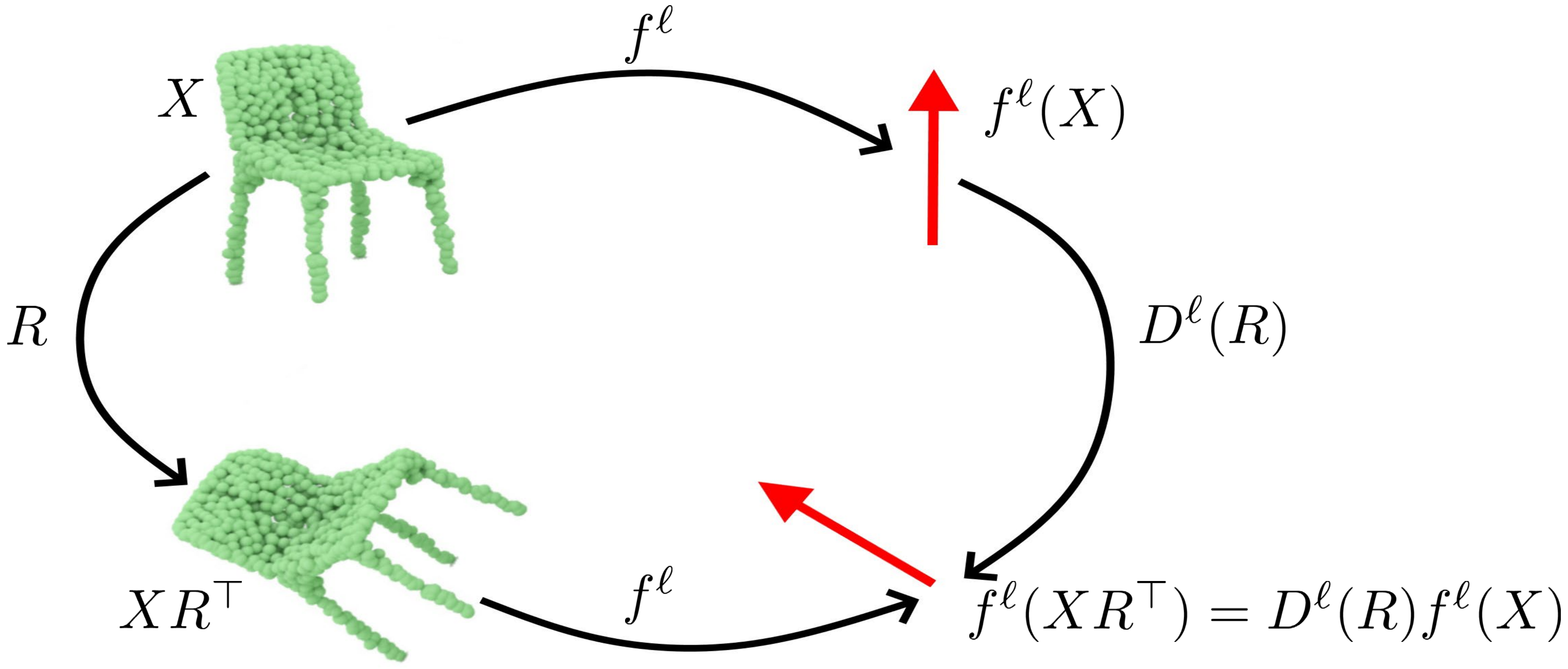
- **Vector Neurons:**
  Lift latent features to 3D vector lists
- **Building blocks:**
  - Linear layer
  - Non-linearity (ReLU)
  - Pooling (MaxPool)
  - Normalizations (BatchNorm)
  - Invariance
- **Network examples:**
  - VN-DGCNN
  - VN-PointNet



Tensor Field Networks and Equivariant CNNs

SO(3) Equivariant Features

- A type $\ell \in \mathbb{N}$ equivariant feature map $f^\ell : \mathbb{R}^{n\times 3} \to \mathbb{R}^{2\ell+1}$ satsfying $f^\ell(XR^\top) = D^\ell(R) f^\ell(X)$ for $R \in \mathrm{SO}(3)$, where $D^\ell(R) \in \mathrm{SO}(2\ell+1)$ is the type ℓ wigner matrix.



Tensor Field Networks (TFN)

- Given a pointcloud X a TFN can produce pointwise or global equivariant features of different types ($\ell \in \mathbb{N}$):

Pointwise features

$$\underbrace{f^{\ell}(X)_{i,c,:}}$$

$$f^{\ell}(X)_{\underset{\text{point}}{i},\underset{\text{channel}}{c},:}$$

global features

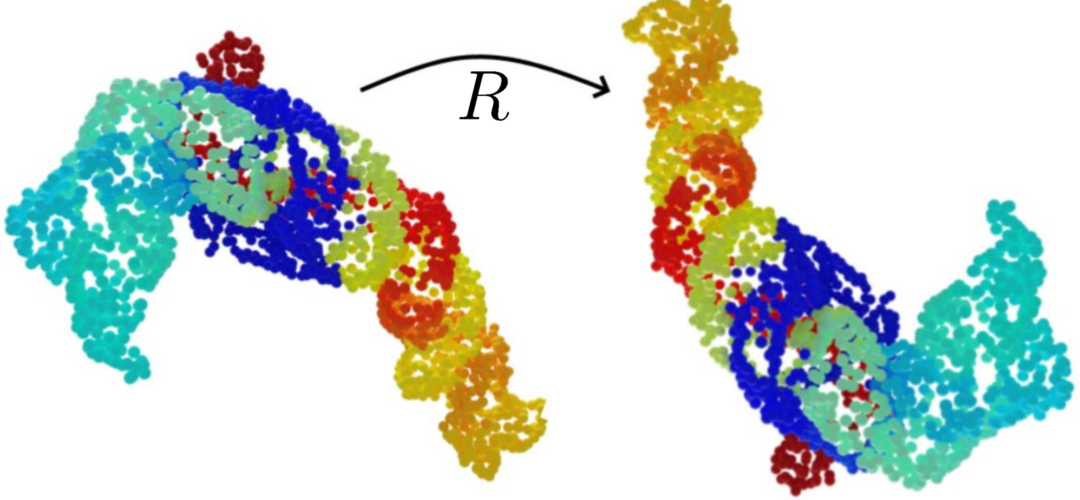
$$g^{\ell}(X)_{\underset{\text{channel}}{c},:}$$

Examples of Type 0 Features

- Type 0 features are rotation invariant as $D^0(R) = 1$:

Segmentation

$$f^0(X)_{i,c,:} = f^0(XR^\top)$$



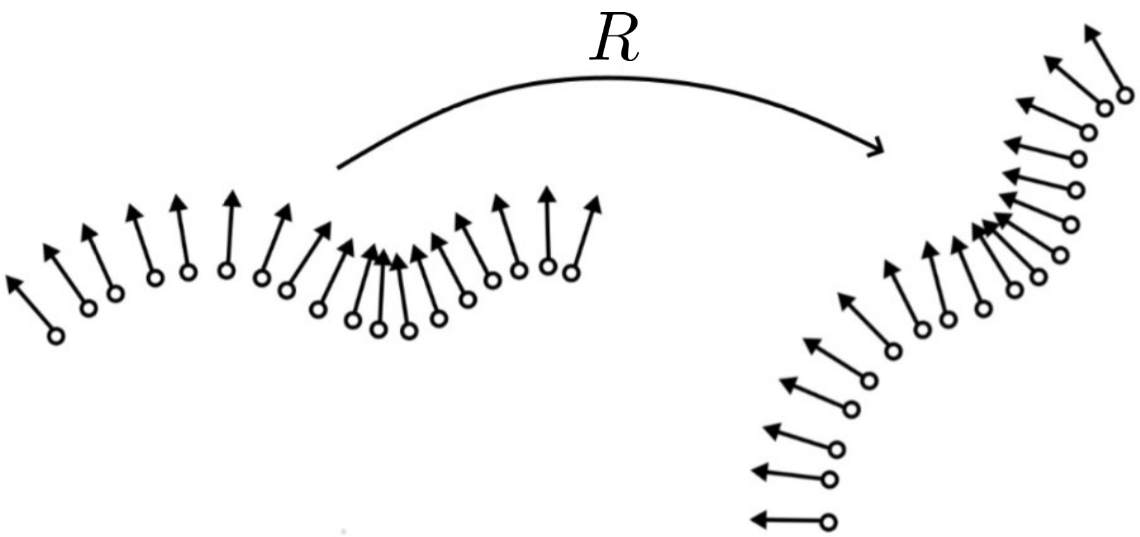
Classification

$g^0($  $) = \text{Chair}$

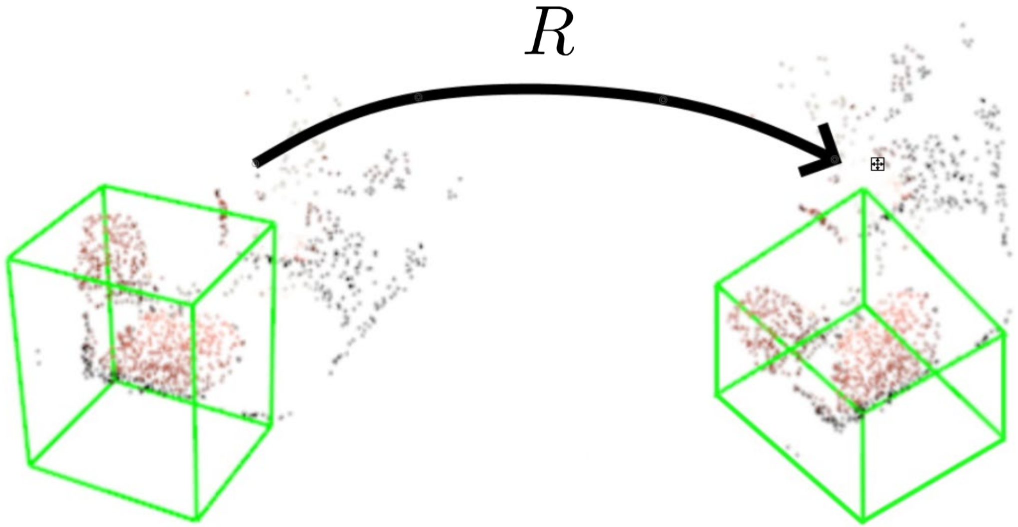
$g^0($  $) = \text{Chair}$

Examples of type 1 Features

- We have $D^1(R) = R$, thefore type 1 features are 3D vectors rotating with the pointcloud X.



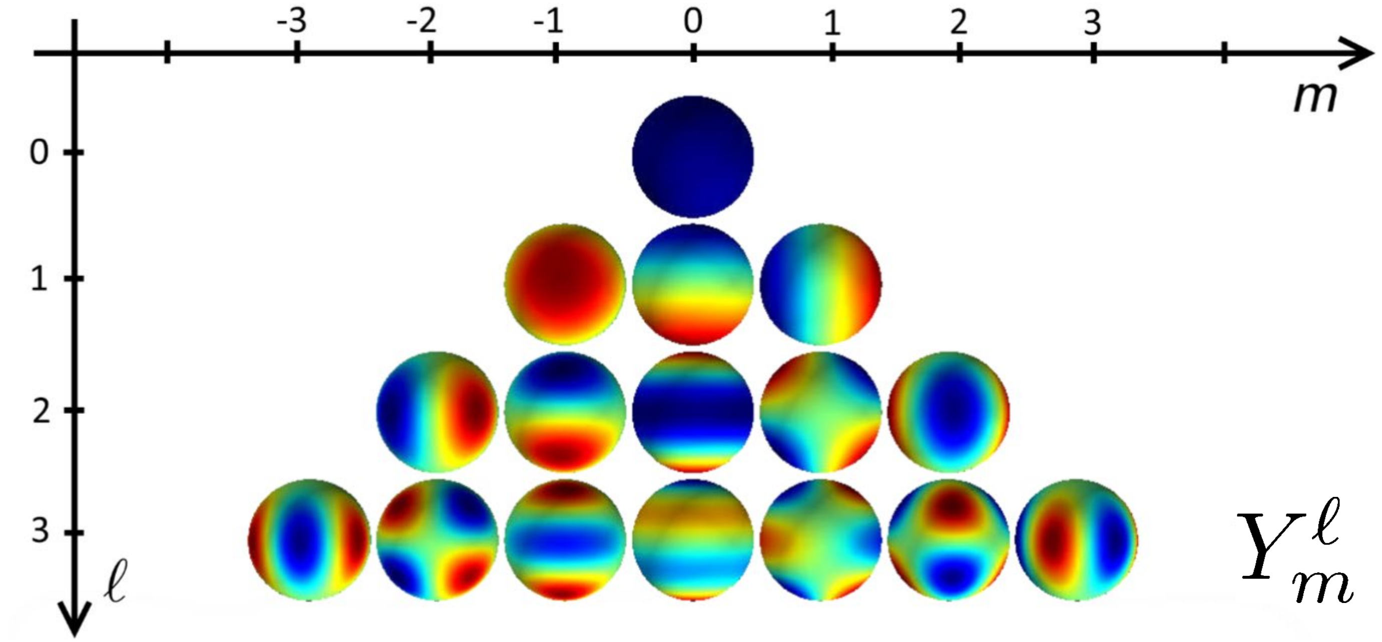
Pointcloud normals are type 1 features.



Bounding box center and principal directions are type 1 features, lengths are type 0 features

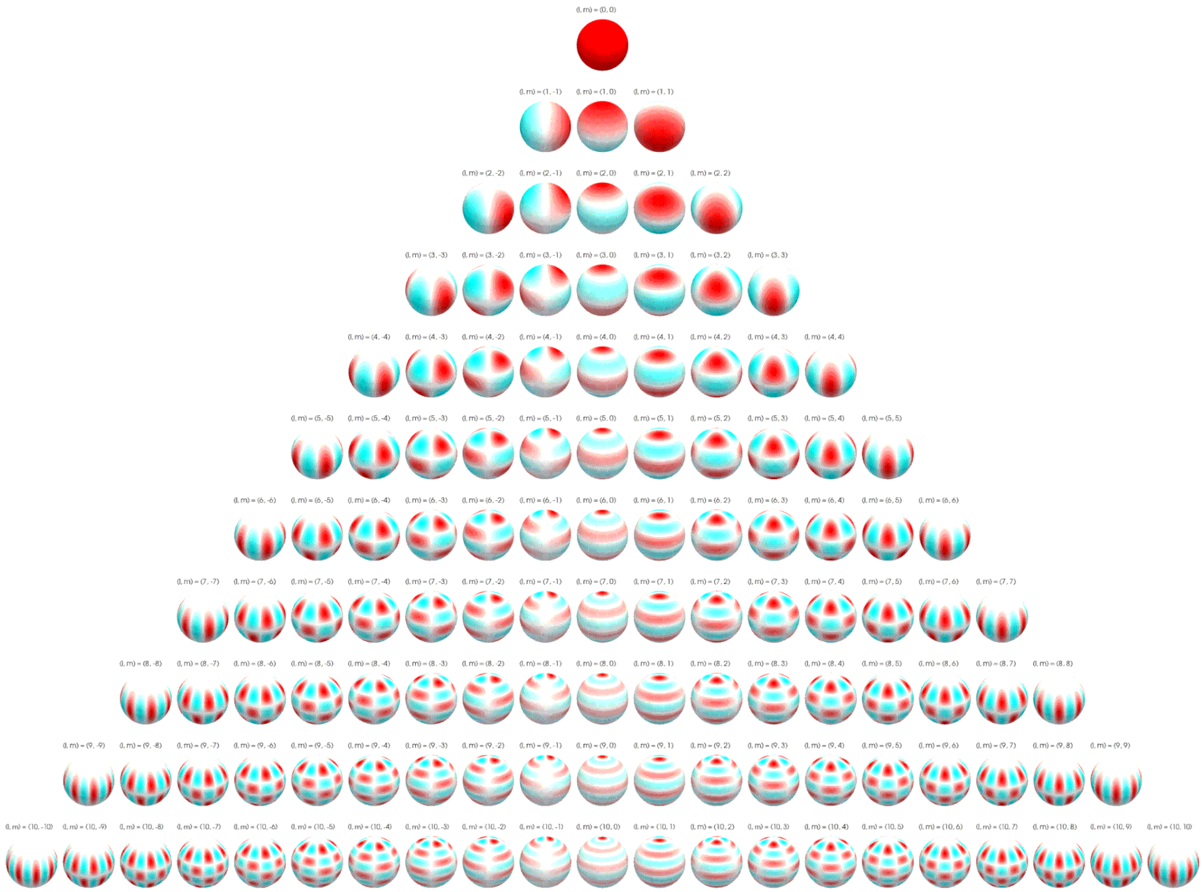
Spherical Harmonics & Higher Degree Features

- Spherical harmonics are homogeneous polynomials on $\mathbb{R}^3$, their restriction to $\mathcal{S}_2$ form an orthonormal basis of $L^2(\mathcal{S}_2)$.
- Just like type ℓ equivariant features the vector of degree ℓ spherical harmonics $Y^\ell(x) \in \mathbb{R}^{2\ell+1}$, satisfies $Y^\ell(Rx) = D^\ell(R)Y^\ell(x)$.



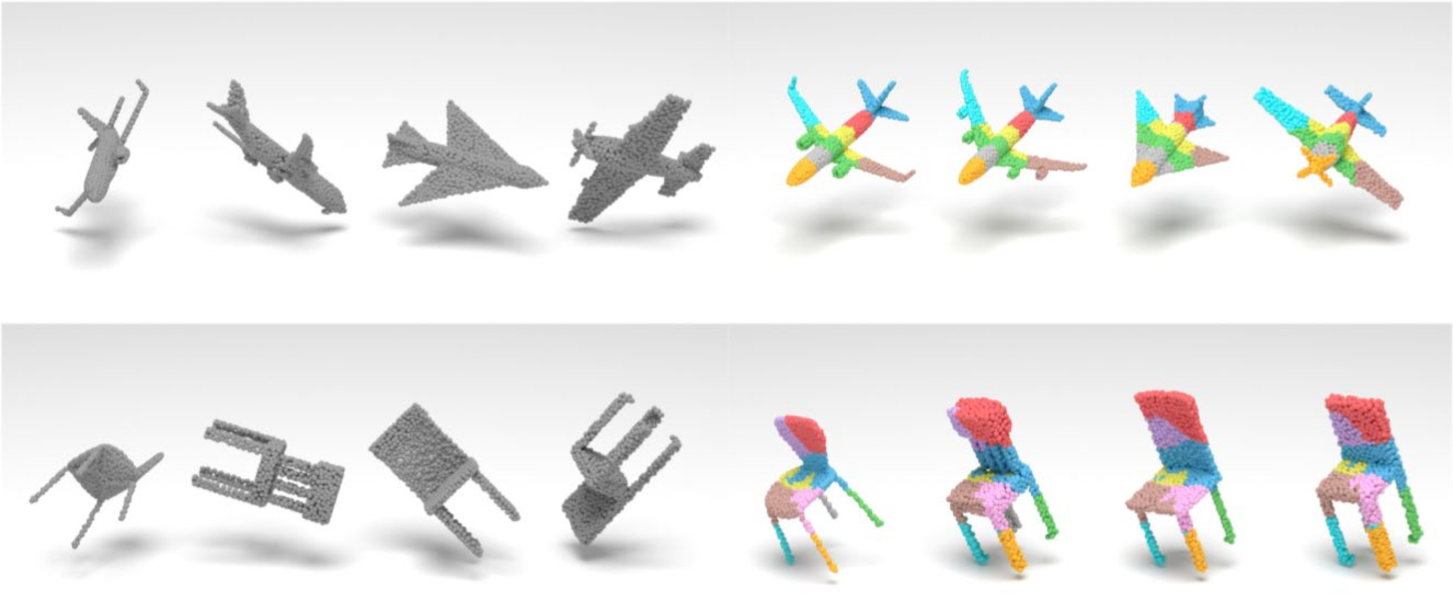
Spherical Harmonics & Higher Degree Features

- Spherical harmonics rotating arround the z axis:



SH & TFN Feature Applications (ConDor)

- Dot products of type ℓ features are invariant.
- We can view TFN features as coefficients of functions in the SH basis.
- The dot prods $\langle f^{\ell}(X)_{ic}, Y^{\ell}(X_i) \rangle$ give an invariant embedding of X.



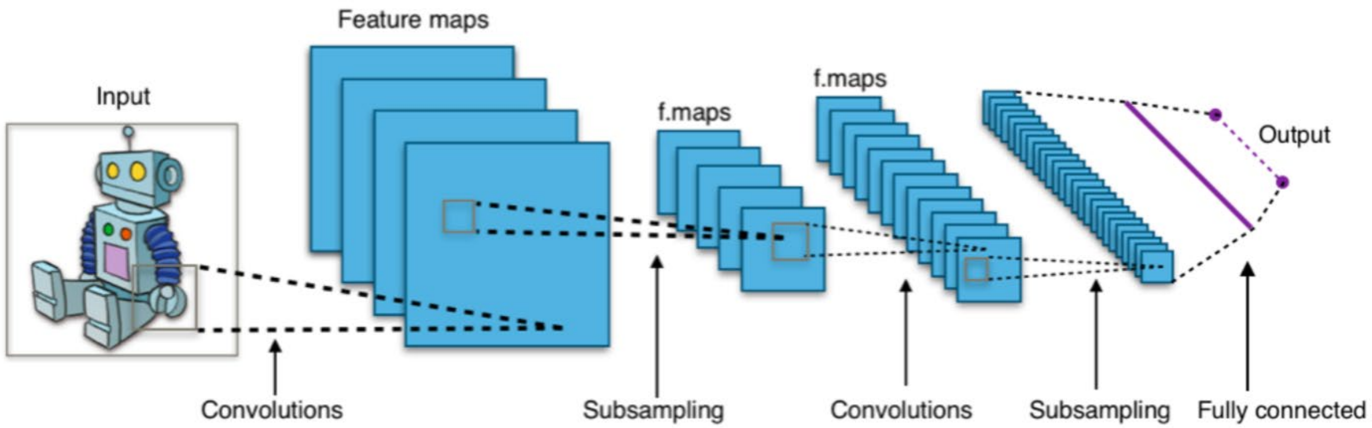
- Invariant embeddings can be used to:
  1. canonicalize shape pose.
  2. segment the shape

(see ConDor).

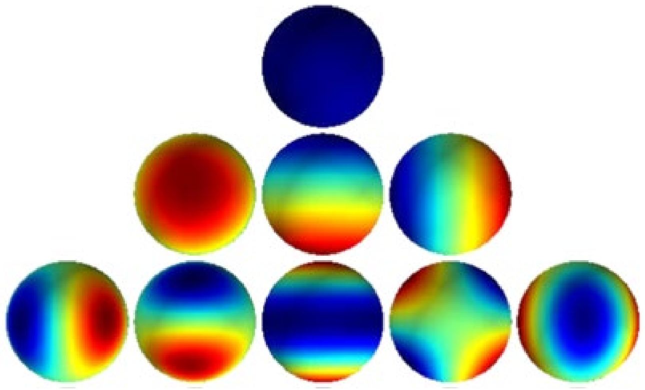
How does TFN work ?

- TFN is a convolutional architecture.
- It inherits its equivariance properties from SH kernels.

$$\mathrm{TFN} =$$



$$+$$



Convolutions on Different Domains

- The convolution between a function $f : \mathbb{R}^d \to \mathbb{R}$ and a kernel $\kappa : \mathbb{R}^d \to \mathbb{R}$ is defined by:

$$f * \kappa(x) := \int_{\mathbb{R}^d} f(t)\kappa(t - x)dt.$$

- Similarly the convolution a function f over a pointcloud X and a kernel $\kappa : \mathbb{R}^d \to \mathbb{R}$ is defined by:

$$f *_X \kappa(X_i) := \sum_j f_j \kappa(X_j - X_i).$$

- We can view images as functions over the pointcloud $\mathbb{Z}^d$ $(d = 2, 3)$.

Convolutional Layers in Neural Nets

- Typically, we are given a collection of functions $(f_i)_i$ on a pointcloud X and a collection of kernel filters $(\kappa_j)_j$ and a weight tensor W_{kij}.
- A convolutional layer outputs a collection of functions by taking linear combinations of convolutions:

$$\mathrm{Conv}_X(f, \kappa, W)_k := \sum_{ij} W_{kij} f_i *_X \kappa_j.$$

Convolutions and Rotations

- For any rotation R we have:

$$f *_{XR^\top} \kappa_j(X_i) = \sum_k f_k \kappa_j(R(X_k - X_i)).$$

- Problem what is the relation between $(f *_X \kappa_j)_j$ and $(f *_{XR^\top} \kappa_j)_j$?

Steerable Kernel Basis

- A steerable kernel basis, is a kernel basis $(\kappa_k)_k$ such that, the rotation of any kernel κ_j linearly decomposes onto the basis by a rotation matrix $D(R)$:

$$\kappa_j(Rx) = \sum_k D(R)_{jk}\kappa_k(x).$$

- Using a steerable basis we can relate convolutions on X and on $XR^\top$ by a simple linear relation:

$$f *_{XR^\top} \kappa_j(x) = \sum_j D(R)_{jk} f *_X \kappa_k(x)$$

3D Steerable Basis

- For each ℓ we have a sterrable basis of the form:

$$\kappa^{\ell}_{rm}(x) := \varphi_r(\|x\|_2) Y^{\ell}_m\left(\frac{x}{\|x\|_2}\right)$$

- Where φ_r is any radial function e.g. a gaussian shell:

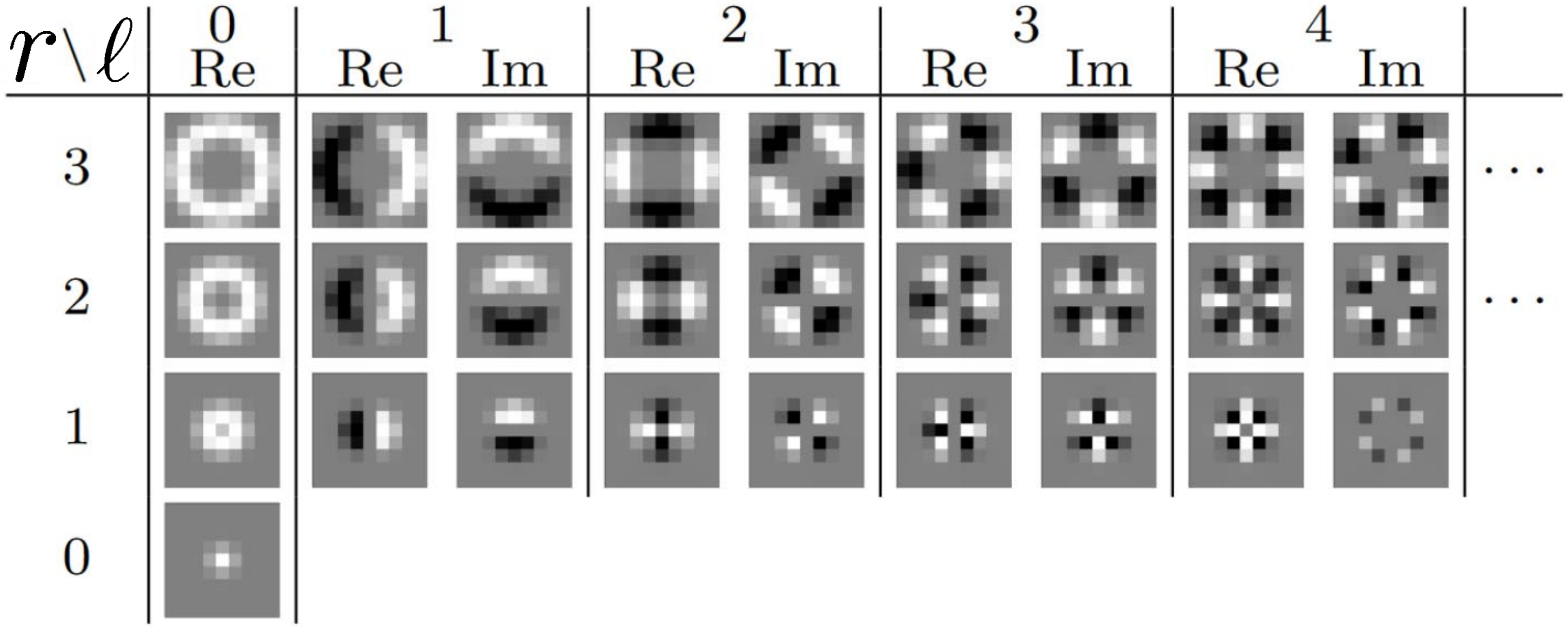
$$\varphi_r(y) := \exp\left(\frac{-(y-\rho_r)^2}{2\sigma^2}\right)$$

Example of Steerable Basis (2D)

- For each ℓ in $\mathbb{Z}$ we can build a steerable kernel basis on $\mathbb{R}^2$ using polar coordinates:

$$\kappa_r^\ell(\theta, \rho) := \varphi_r(\rho) e^{i\ell\theta}$$

  here we have $D^\ell(t) = e^{i\ell t}$.



Example of steerable kernels, (source Learning Steerable Filters with Rotation Equivariant CNNs)

Steerable Convolution

- We consider a collection of steerable kernel bases $(\kappa_r^\ell)_{\ell r}$ over $\mathbb{R}^d$ indexed by an equivariance type ℓ such that for any rotation $R \in \mathrm{SO}(d)$ we have:

$$\kappa^\ell(Rx) = D^\ell(R)\kappa^\ell(x)$$

- For each ℓ we define the associated steerable convolution operator:

$$\mathrm{SConv}_X^\ell(f, \kappa_r^\ell, W)_{km} := \sum_{ir} W_{kir} f_i *_X \kappa_{rm}^\ell.$$

Comparing Regular and Steerable Convolution

- Regular convolution takes linear combination over all indices of the kernel basis:

$$\mathrm{Conv}_X(f, \kappa, W)_k := \sum_{ij} W_{kij} f_i *_X \kappa_j.$$

- Steerable convolution does not take linear combination over the "steer-able" index m in order to preserve equivariance:

$$\mathrm{SConv}^{\ell}_X(f, \kappa^{\ell}_r, W)_{km} := \sum_{ir} W_{kir} f_i *_X \kappa^{\ell}_{rm}.$$

Steerable Convolution is Equivariant

- A rotation R of the pointcloud X induces a linear transform $D^\ell(R)$ which is independent of X:

$$\begin{aligned}
\mathrm{SConv}^\ell_{XR^\top}(f, \kappa^\ell_r, W)_k &= \sum_{ir} W_{kir} f_i *_{XR^\top} \kappa^\ell_r \\
&= \sum_{ir} W_{kir} f_i *_X D^\ell(R)\kappa^\ell_r \\
&= D^\ell(R)\mathrm{SConv}^\ell_X(f, \kappa^\ell_r, W)_k
\end{aligned}$$

Why not Mix Different Types?

- In 2D a type ℓ features makes ℓ turns when the input makes one turn:

$$v(e^{i\theta}X) = e^{i\ell\theta}v(X).$$

- There is no single rotation that rotates different types, or a linear combination of different types.

Stacking Steerable Convolutions

- We have described steerable convolution of rotation invariant pointwise features with a steerable basis.
- TFN extends steerable convolution to equivariant inputs, by linearly decomposing the resulting convolutional features into equivariant features of different types.

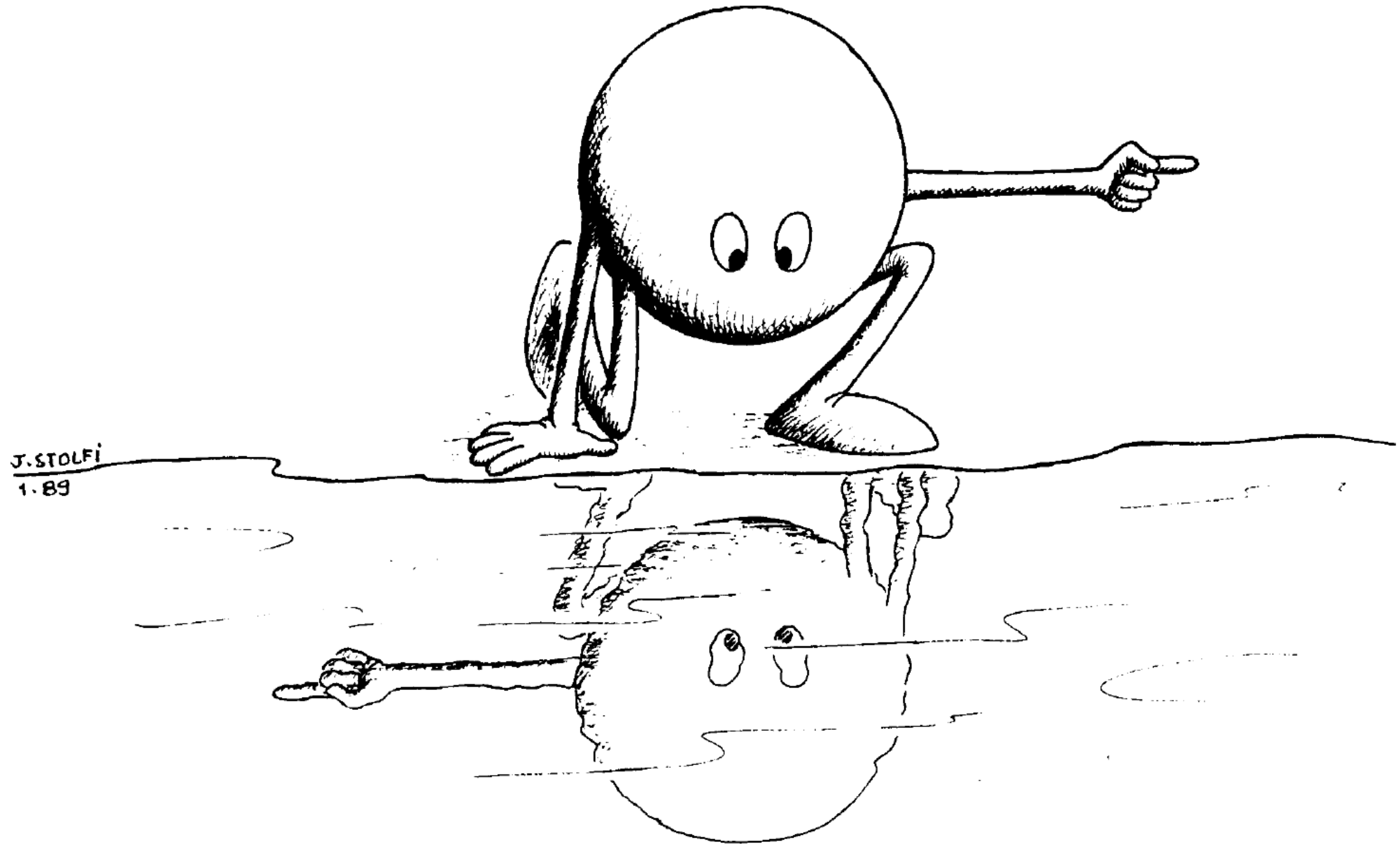
Equivariant non-Linearities

- Dot products and norms of type ℓ features are rotation invariant:

$$\langle f(R.X), g(R.X)\rangle = \langle D^\ell(R) f(X), D^\ell(R) g(X)\rangle = \langle f(X), g(X)\rangle.$$

- Non linearities can by applied to the norms of features as they are invariant:

$$\xi(f^\ell(X), b) = \xi(\|f^\ell(X)\|_2 + b) f^\ell(X).$$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