Overview of Appearance Models

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1 Review of Scattering

Last time, we looked at a few basic quantities and functions that characterize the scattering due to a participating medium. Here are a few of the most relevant:

Name	**Symbol**	**Units**	**Description**
Scattering Coeff.	σ_s	$(\text{length})^{-1}$	Probability of scattering per unit length
Absorption Coeff.	σ_a	$(\text{length})^{-1}$	Probability of absorbtion per unit length
Phase Function	$p(x, \vec{\omega}', \vec{\omega})$		Angular distribution of scattering
Extinction Coeff.	σ_t	$(\text{length})^{-1}$	$\sigma_a + \sigma_s$
(Scattering) Albedo	A		${}^{\sigma_s}/_{\sigma_t}$
Optical Depth	$\tau(0, d)$		$\int_0^d \sigma_t dx$
Transmittance	$t(0, d)$		$e^{-\tau(0,d)}$

Given these, we looked at the case of single scattering, and saw how the volume scattering equation may be solved both analytically and numerically. We also looked at a variety of phase functions that have been studied, including isotropic, Henyey-Greenstein, Schlick, Lambertian sphere, Rayleigh, and Mie. We will now move to the multiple scattering case, including some simplified analytic results (the "two-flux" theory) and algorithms for solving the general case.

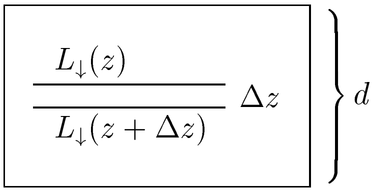
2 Two-flux Theory

We simplify the scattering problem to be essentially one-dimensional, and consider only the radiance transmitted and scattered in the z direction.

First, let us define some terms:

$L_\downarrow$	Radiance going down
$L_\uparrow$	Radiance going up
$p_{\downarrow\uparrow}$	Probability that a particle going down is scattered up
$p_{\uparrow\downarrow}$	Probability that a particle going up is scattered down
$p_{\downarrow\downarrow}$	Probability that a particle going down is scattered down
$p_{\uparrow\uparrow}$	Probability that a particle going up is scattered up

We now look at the change in $L_\downarrow$ as it travels through a small thickness Δz:



The following equation expresses the energy balance through the thickness Δz:

$$L_\downarrow(z) + \sigma_s \Delta z p_{\uparrow\downarrow} L_\uparrow(z+\Delta z) = \sigma_a \Delta z L_\downarrow(z) + \sigma_s \Delta z p_{\downarrow\uparrow} L_\downarrow(z) + L_\downarrow(z+\Delta z) \quad (1)$$

The two terms on the left side of this equation are the downward radiance at z and the part of the upward radiance at $z + \Delta z$ that is scattered downward. On the right, the three terms are the amount of light absorbed, the amount of light initially headed down but scattered up, and the downward radiance leaving the small region. Thus, the equation states that the downward-headed radiance, whether incident on the slab or scattered, is either absorbed, scattered upwards, or transmitted. An analogous equation could be written for the case of upward-heading radiance.

Rewriting this in differential form and rearranging terms, we obtain:

$$\frac{dL_\downarrow}{dz} = -\sigma_a L_\downarrow - \sigma_s p_{\downarrow\uparrow} L_\downarrow + \sigma_s p_{\uparrow\downarrow} L_\uparrow \quad (2)$$

$$\frac{dL_\uparrow}{dz} = \sigma_a L_\uparrow + \sigma_s p_{\uparrow\downarrow} L_\uparrow - \sigma_s p_{\downarrow\uparrow} L_\downarrow \quad (3)$$

We now make some additional assumptions, namely that the medium is isotropic and that only scattering that occurs is in the z direction. The first assumption implies that

$$p_{\downarrow\uparrow} = p_{\uparrow\downarrow} \quad (4)$$

$$p_{\uparrow\uparrow} = p_{\downarrow\downarrow} \quad (5)$$

while the second lets us write

$$p_{\uparrow\downarrow} + p_{\uparrow\uparrow} = p_{\downarrow\uparrow} + p_{\downarrow\downarrow} = 1 \quad (6)$$

Also, we define an “asymmetry parameter”:

$$g = p_{\downarrow\downarrow} - p_{\downarrow\uparrow} \quad (7)$$

Note that the case of perfect forward scattering corresponds to $g = 1$ and perfect backscattering corresponds to $g = -1$.

We now rewrite equations 2 and 3 in terms of the asymmetry g, the albedo A, and the optical depth τ:

$$\frac{dL_\downarrow}{d\tau} = -L_\downarrow + A\frac{1+g}{2}L_\downarrow + A\frac{1-g}{2}L_\uparrow \quad (8)$$

$$\frac{dL_\uparrow}{d\tau} = L_\uparrow - A\frac{1+g}{2}L_\uparrow - A\frac{1-g}{2}L_\downarrow \quad (9)$$

The above equations may also be written as a sum and difference:

$$\frac{d(L_\downarrow - L_\uparrow)}{d\tau} = -(1-A)(L_\downarrow + L_\uparrow) \tag{10}$$
$$\frac{d(L_\downarrow + L_\uparrow)}{d\tau} = -(1-gA)(L_\downarrow - L_\uparrow) \tag{11}$$

2.1 Albedo = 1

We now consider the case in which $A = 1$. That is, no absorption occurs, and light is only transmitted or scattered. Though simplified, this is still an interesting scenario, since it is reasonably approximated by media such as snow or clouds.

Substituting $A = 1$, equations 10 and 11 become

$$\frac{d(L_\downarrow - L_\uparrow)}{d\tau} = 0 \tag{12}$$
$$\frac{d(L_\downarrow + L_\uparrow)}{d\tau} = -(1-g)(L_\downarrow - L_\uparrow) \tag{13}$$

which have the solutions

$$L_\downarrow = D + C(1 - \tau(1-g)) \tag{14}$$
$$L_\uparrow = D - C(1 + \tau(1-g)) \tag{15}$$

for some constants D and C.

Perfect Mirror:

Let us consider what happens when we place a perfect mirror at the bottom of our volume (that is, at $z = d$). This implies that

$$L_\downarrow(d) = L_\uparrow(d) \tag{16}$$

Solving equations 14 and 15, we obtain

$$C = 0 \tag{17}$$
$$D = I_0 \tag{18}$$
$$L_\downarrow(z) = I_0 \tag{19}$$
$$L_\uparrow(z) = I_0 \tag{20}$$

where I_0 is $L_\downarrow(0)$, i.e. the light incident on the top of the surface.

Thus, we see that all light incident on the surface is eventually scattered out. Moreover, we see that illumination, both upwards and downwards, is the same everywhere throughout the medium.

Nothing Below:

Let us consider the opposite case, namely that there is nothing below our medium. In this case, we have

$$L_{\uparrow}(d) \;=\; 0 \tag{21}$$

Solving for the amount of light reflected and transmitted, we obtain

$$R = \frac{L_{\uparrow}(0)}{I_0} \;=\; \frac{(1-g)\tau(d)}{2+(1-g)\tau(d)} \tag{22}$$

$$T = \frac{L_{\downarrow}(d)}{I_0} \;=\; \frac{2}{2+(1-g)\tau(d)} \tag{23}$$

Some simple conclusions:

- If $g = 1$ (perfect forward scattering), $R = 0$ and $T = 1$. That is, all the light is transmitted.
- If $g = -1$ (perfect backscattering), there is still some transmission, though it approaches 0 as $\tau \to \infty$.
- If $g \neq 1$, $R \to 1$ and $T \to 0$ as $\tau \to \infty$.

Snow-Covered Ground:

We can use the preceding equations to understand some of the behavior of snow-covered ground. We start with the assumption that snow is a perfect scatterer and the empirical observation that

$$(1-g)\tau(d) \;\approx\; 200d \tag{24}$$

where d is in meters. We can use this to determine the contrast between the light reflected from a white surface covered by snow (given by Equation 20) and a black surface covered by the same thickness of snow (Equation 22):

$$R_w - R_b \;=\; 1 - \frac{(1-g)\tau(d)}{2+(1-g)\tau(d)} \tag{25}$$

$$\approx\; \frac{2}{2+200d} \tag{26}$$

We see that this quantity approaches 1 very rapidly as d increases, meaning that even a relatively thin layer of snow is enough to obscure any contrasting colors on the ground below.

Color of Multiply-Scattered Light:

We can also look at the effect of multiple scattering on the color of scattered light. Starting with Equation 22 and differentiating with respect to wavelength, we obtain

$$\frac{dR}{d\lambda} = \frac{2}{(2+(1-g)\tau(d))^2}\frac{d\tau}{d\lambda} \tag{27}$$

This approaches 0 quickly as $\tau \to \infty$, meaning that, deep within a scattering medium, the color of scattered light tends to white.

Inside a Cloud:

Let us now consider the effect of multiple scattering within a cloud. Let D be the multiply-scattered light. It can be shown that

$$\frac{D_\uparrow}{I_0} = \frac{\frac{1}{2}(1-g)(\tau(d)-\tau)}{1+\frac{1}{2}(1-g)\tau(d)} \tag{28}$$

$$\frac{D_\downarrow}{I_0} = \frac{1+\frac{1}{2}(1-g)(\tau(d)-\tau)}{1+\frac{1}{2}(1-g)\tau(d)} - e^{-\tau} \tag{29}$$

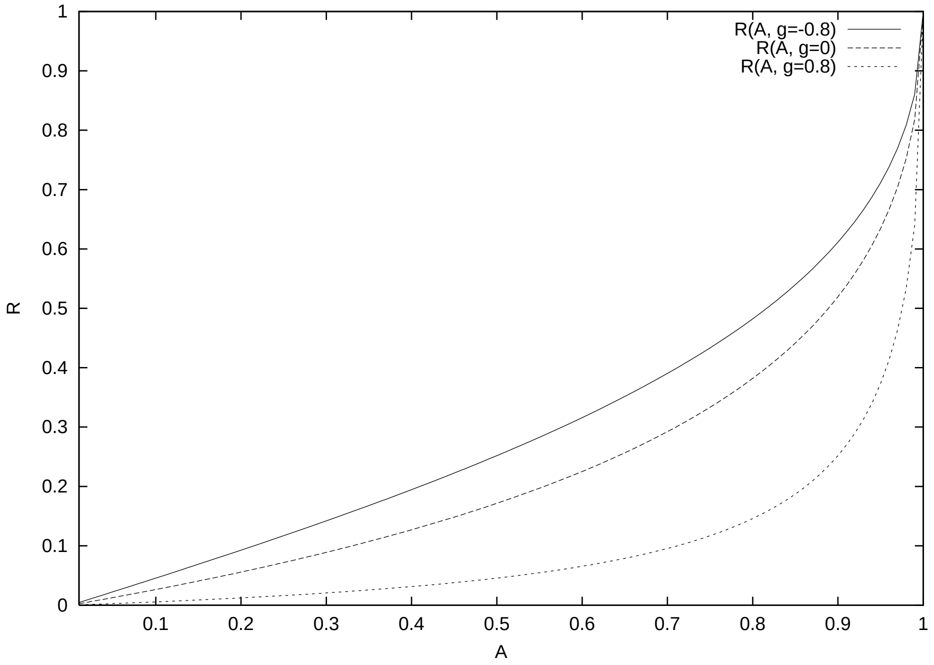
Therefore, $D_\uparrow \approx D_\downarrow$ in the range $\frac{2}{1-g} < \tau < \tau(d) - \frac{2}{1-g}$. This means that, except for near the boundaries of the cloud, multiply-scattered light contributes equally in all directions. In other words, inside a cloud you can't see which way is up.

2.2 Albedo $\neq$ 1

If the participating medium is not a perfect scatterer (i.e., albedo is not 1), then the expression for reflection becomes

$$R = \frac{\sqrt{1-Ag}-\sqrt{1-A}}{\sqrt{1-Ag}+\sqrt{1-A}} \tag{30}$$

Let us look at how this function looks for several values of g:



We see that the reflectance decreases dramatically when albedo is only slightly below 1, especially for the case of forward scattering. This predicts the darker appearance of wet sand (the presence of water reduces the difference in indices of refraction at the interface, making the scattering greater in the forward direction) and the occasional blue appearance of snow (even if there is a tiny amount of absorption, the overall reflectance can be significantly decreased).

3 Rendering Methods for Multiple Scattering

There are three commonly-used methods for simulating multiple scattering in physically-correct rendering systems:

- Recursive ray marching
- Random walk simulation
- Photon mapping simulation

3.1 Recursive Ray Marching

This is a fairly straightforward extension of the ray marching technique we saw in the single-scattering case. As before, we step along a ray through the medium and at each step we attenuate the light from the previous step and collect inscattered light. The difference is that in order to evaluate the amount of inscattering we can't just shoot a ray at the light source and attenuate it based on the thickness of the medium through which it passes. Instead, we simulate multiple scattering events by recursively invoking the ray marching algorithm for each inscattered ray:

$$
\begin{aligned}
L_{n+1}(x,\vec{\omega}) \;=\;& L_d(x,\vec{\omega}'))p(x,\vec{\omega}',\vec{\omega})\sigma_s\Delta x + \\
& \left\{\frac{1}{N}\sum_{s=1}^{N} L_s(x,\vec{\omega}_s')p(x,\vec{\omega}_s',\vec{\omega})\right\}\sigma_s\Delta x + \\
& e^{-\sigma_t\Delta x}L_n(x+\Delta x,\vec{\omega}) \qquad (31)
\end{aligned}
$$

The recursive call appears as the second term in this equation. Note that we can play all the usual Monte Carlo tricks (importance sampling, Russian roulette, stratified sampling, etc.) in order to reduce the variance of the estimated scattering.

3.2 Random walk

One of the problems with the recursive ray marching algorithm is that it is rather inefficient for the case of a dense medium illuminated and viewed from the same direction. If there is a large amount of scattering, we will waste a lot of effort stepping through the lower parts of the medium, when most of the scattering events actually take place in a small region near the top. Although one could add some special-case code to take case of this case, this is inelegant and not robust.

An alternative solution method uses a random walk. We first rewrite the volume scattering equation and pull the transmission term inside the integral:

$$
\begin{aligned}
L(x,\vec{\omega}) \;=\;& \int_0^d \int_{\Omega_{4\pi}} \sigma(x')p(x',\vec{\omega}',\vec{\omega})L(x',\vec{\omega}')t(x,x')\,d\omega'\,dx' + \\
& t(x,x+d\vec{\omega})L(x+d\vec{\omega},\vec{\omega}) \qquad (32) \\
\rightarrow\;& \int_0^\infty \Big[\, x'<d\ ? \\
& \frac{\sigma_s}{\sigma_t}\int_{\Omega_{4\pi}} p(x',\vec{\omega}',\vec{\omega})L(x',\vec{\omega}')\,d\omega' \ : \\
& \frac{1}{\sigma_t}L(x+d\vec{\omega},\vec{\omega})\,\Big]\,\sigma_t e^{-\sigma_t x'}\,dx' \qquad (33)
\end{aligned}
$$

We then perform a random walk through the medium, at each step evaluating how far to move and whether this movement stays within the medium or escapes. To figure out how far to move, we start with the probability of traveling a distance x:

$$p(x) = \sigma_t e^{-\sigma_t x} \qquad (34)$$

We integrate this to find the PDF

$$P(x) = 1 - e^{-\sigma_t x} \qquad (35)$$

then invert to find the length of a scattering event, given a uniform random variable $\xi \in [0..1]$:

$$x' = \frac{-\log(\xi)}{\sigma_t} \qquad (36)$$

3.3 Photon mapping

Photon mapping is another rendering method that can be used for scenes containing multiple scattering. It is a two-pass method. In the first phase, photons are sent out from all the light sources and are allowed to bounce around the scene. The path of each photon is recorded in a data structure called the *photon map.* The intensity of light at any point in the scene is proportional to the density of photons that passed through a neighborhood of that point. One optimization possible at this point is to only store *indirect* illumination, since this reduces the number of photons required and it is often easier to compute the direct illumination directly during rendering.

In the second phase, the scene is ray-traced as usual, direct lighting is computed, and the indirect illumination is estimated from the photon map.

In order to use a photon map to speed up ray marching, we start with the expression for the radiance stored in a photon map based on the density of photons:

$$L(x,\vec{\omega}) = \frac{d^2\Phi(x,\vec{\omega})}{\sigma_s(x)d\omega dV} \tag{37}$$

We can then compute the indirect illumination:

$$\begin{aligned} L_i(x,\vec{\omega}) &= \int_{\Omega_{4\pi}} p(x,\vec{\omega},\vec{\omega}')L(x,\vec{\omega}')d\omega' \\ &= \int_{\Omega_{4\pi}} p(x,\vec{\omega},\vec{\omega}')\frac{d^2\Phi(x,\vec{\omega}')}{\sigma_s(x)d\omega' dV}d\omega' \\ &= \frac{1}{\sigma_s(x)}\sum_{p=1}^{n} p(x,\vec{\omega},\vec{\omega}')\frac{\Delta\Phi_p(x,\vec{\omega}')}{\frac{4}{3}\pi r^3} \end{aligned} \tag{38}$$

Finally, we use the indirect illumination estimate during rendering:

$$\begin{aligned} L_{n+1}(x,\vec{\omega}) = \; & L_d(x,\vec{\omega}'))p(x,\vec{\omega}',\vec{\omega})\sigma_s\Delta x + \\ & L_i(x,\vec{\omega}'_s)p(x,\vec{\omega}'_s,\vec{\omega})\sigma_s\Delta x + \\ & e^{-\sigma_t\Delta x}L_n(x+\Delta x,\vec{\omega}) \end{aligned} \tag{39}$$