Combinatorics is the slums of topology.
— J. H. C. Whitehead (attr.)

3 Simplicial Complexes

In the first lecture, we looked at concepts from *point set topology*, the branch of topology that studies continuity from an analytical point of view. This view does not have a computational nature: we cannot represent infinite point sets or their associated infinite open sets on a computer. Starting with this lecture, we will look at concepts from another major branch of topology: *combinatorial topology*. This branch also studies connectivity, but does so by examining constructing complicated objects out of simple blocks, and deducing the properties of the constructed objects from the blocks. While our view of the world–our *ontology*–will be mostly combinatorial in nature, we will see concepts from point set topology reemerging under disguise, and we will be careful to expose them!

In this lecture, we begin by learning about simple building blocks from which we may construct complicated spaces. Simplicial complexes are combinatorial objects that represent topological spaces. With simplicial complexes, we separate the topology of a space from its geometry, much like the separation of syntax and semantics in logic. Given the finite combinatorial description of a space, we are able to count, and the miracle of combinatorial topology is that counting alone enables us to make statements about the connectivity of a space. We shall experience a first instance of this marvelous theory in the *Euler characteristic*. This topological invariant gives a simple constructive procedure for classifying 2-manifolds, completing our treatment from the last lecture.

3.1 Geometric Definition

We begin with a definition of simplicial complexes that seems to mix geometry and topology. Combinations allow us to represent regions of space with very few points. In other words, allow us to describe simple cells which become our building blocks later.

Definition 3.1 (combinations) Let $S = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{R}^d$. A *linear combination* is $x = \sum_{i=0}^{k} \lambda_i p_i$, for some $\lambda_i \in \mathbb{R}$. An *affine combination* is a linear combination with $\sum_{i=0}^{k} \lambda_i = 1$. A *convex combination* is a an affine combination with $\lambda_i \geq 0$, for all i. The set of all convex combinations is the *convex hull*.

You may have seen the concept of *independence* in studying linear algebra.

Definition 3.2 (independence) A set S is *linearly (affinely) independent* if no point in S is a linear (affine) combination of the other points in S.

Figure 1 shows the linear, affine, and convex combinations of three affinely independent points in $\mathbb{R}^3$. We may now define our basic building block.

Definition 3.3 (k-simplex) A k*-simplex* is the convex hull of $k+1$ affinely independent points $S = \{v_0, v_1, \ldots, v_k\}$. The points in S are the *vertices* of the simplex.

A k-simplex is a k-dimensional subspace of $\mathbb{R}^d$, $\dim \sigma = k$. We show low-dimensional simplices with their names in Figure 2. Since all the points defining a simplex are affinely independent, so is any subset of them. This causes the simplex to have an interesting structure: it is composed of simplices of lower-dimension, or its *faces*.

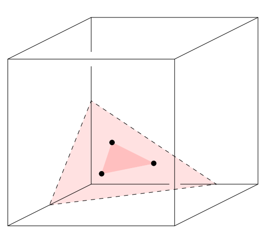


Figure 1. Combinations. The linear combinations of three affinely independent points in $\mathbb{R}^3$ covers the whole space. The affine combinations fill the plane defined by the three points. The convex hull is the triangle defined by the three points.

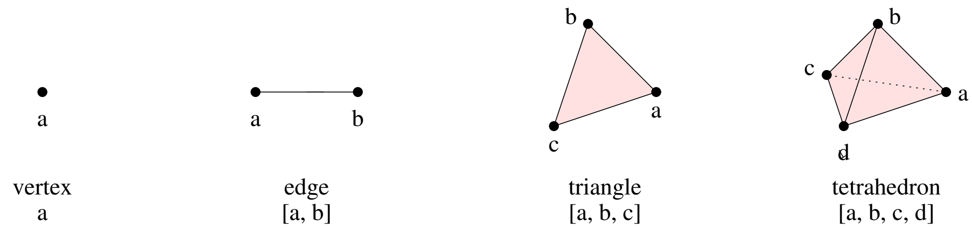


Figure 2. k-simplices, for each $0 \leq k \leq 3$.

Definition 3.4 (face, coface) Let σ be a k-simplex defined by $S = \{v_0, v_1, \ldots, v_k\}$. A simplex τ defined by $T \subseteq S$ is a *face* of σ and has σ as a *coface*. The relationship is denoted with $\sigma \geq \tau$ and $\tau \leq \sigma$. Note that $\sigma \leq \sigma$ and $\sigma \geq \sigma$.

Note that a simplex is always a face of itself by this definition.

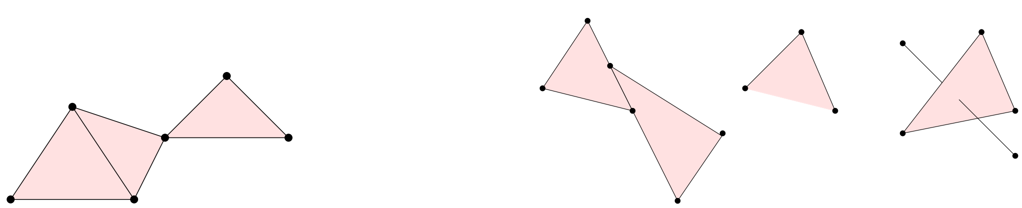
We attach simplices together to represent spaces. This attaching is very much like using lego blocks to build castles: we can only attach lego blocks on the special interfaces. Similarly, we may only attach simplices along their special interfaces: their faces. The following definition formally defines our structures, which we call *complexes*. All that the following cryptic definition states is that if a simplex is part of the complex, so are all its faces; and if two simplices intersect, the intersection is part of the complex. It is good to see this formal definition, however, as we will encounter similar ones in reading current research in computational topology, and we should lose our fear of them!

Definition 3.5 (simplicial complex) A *simplicial complex* K is a finite set of simplices such that

1. $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$,
2. $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma' \leq \sigma, \sigma'$ or $\sigma \cap \sigma' = \emptyset$.

The *dimension* of K is $\dim K = \max\{\dim \sigma \mid \sigma \in K\}$. The *vertices* of K are the zero-simplices in K. A simplex is *principal* if it has no proper coface in K.

Here, *proper* has the same definition as for sets. So, a simplicial complex is a collection of simplices that fit together nicely, as shown in Figure 3 (a), as opposed to simplices in (b).



(a) The middle triangle shares an edge with the triangle on the left, and a vertex with the triangle on the right.

(b) In the middle, the triangle is missing an edge. The simplices on the left and right intersect, but not along shared simplices.

Figure 3. A simplicial complex (a) and disallowed collections of simplices (b).

3.2 Size of a Simplex

As already mentioned, combinatorial topology derives its power from counting. Now that we have a finite description of a space, we can count easily. So, let's use Figure 2 to count the number of faces of a simplex. For example, an edge has two vertices and an edge as its faces (recall that a simplex is a face of itself.) A tetrahedron has four vertices, six edges, four triangles, and a tetrahedron as faces. These counts are summarized in Table 1. What should the numbers be for a 4-simplex? The numbers in the table may look really familiar to you. If we add a 1 to the left of each row, we get *Pascal's triangle*, as shown in Figure 4. Recall that Pascal's triangle encodes the binomial coefficients: the

k/l	0	1	2	3
0	1	0	0	0
1	2	1	0	0
2	3	3	1	0
3	4	6	4	1
4	?	?	?	?

Table 1. Number of l-simplices in each k-simplex.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Figure 4. If we add a 1 to the left side of each row in Table 1, we get Pascal's triangle.

number of different combinations of l objects out of k objects or $\binom{k}{l}$. Here, we have $k+1$ points representing an k-simplex, any $l+1$ of which defines a l-simplex. To make the relationship complete, we define the empty set $\emptyset$ as the (-1)-*simplex*. This simplex is part of every simplex and allows us to add a column of 1's to the left side of Table 1 to get Pascal's triangle. It also allows us to eliminate the underlined part of Definition 3.5, as the empty set of part of both simplices for non-intersecting simplices. To get the total size of a simplex, we sum each row of Pascal's triangle. A k-simplex has $\binom{k+1}{l+1}$ faces of dimension l and

$$\sum_{l=-1}^{k} \binom{k+1}{l+1} = 2^{k+1}$$

faces in total. A simplex, therefore, is a very large object. Mathematicians often do not find it appropriate for "computation", when computation is being done by hand. Simplices are very uniform and simple in structure, however, and therefore provide an ideal computational gadget for computers.

3.3 Abstract Definition

Our discussion on the size of a simplex shows that we can view a simplex as a set along and its power set (the collection of all its subsets. This view of a simplex is attractive because it avoids references to geometry in defining a simplicial complex. It also should give you eerie feelings of déjà vu, as it matches the definition of a topology

Definition 3.6 (abstract simplicial complex) An *abstract simplicial complex* is a set K, together with a collection $\mathcal{S}$ of subsets of K called *(abstract) simplices* such that:

1. For all $v \in K, \{v\} \in \mathcal{S}$. We call the sets $\{v\}$ the *vertices* of K.
2. If $\tau \subseteq \sigma \in \mathcal{S}$, then $\tau \in \mathcal{S}$.

When it is clear from context what $\mathcal{S}$ is, we refer to K as a complex. We say σ is a k-*simplex* of *dimension* k if $|\sigma| = k+1$. If $\tau \subseteq \sigma$, τ is a *face* of σ, and σ is a *coface* of τ.

Note that the definition automatically allows for $\emptyset$ as a (-1)-simplex. We will often abuse notation and refer to $\mathcal{S}$ as the complex. The abstract definition affirms the notion that topology only cares about how the simplices are connected, and not how they are placed within a space. We now relate this abstract set-theoretic definition to the geometric one by extracting the combinatorial structure of a (geometric) simplicial complex.

Definition 3.7 (vertex scheme) Let K be a simplicial complex with vertices V and let $\mathcal{S}$ be the collection of all subsets $\{v_0, v_1, \ldots, v_k\}$ of V such that the vertices $v_0, v_1, \ldots, v_k$ span a simplex of K. The collection $\mathcal{S}$ is called the *vertex scheme* of K.

Clearly, the set K and the the collection $\mathcal{S}$ together form an abstract simplicial complex. It allows us to compare simplicial complexes easily, using isomorphisms between sets.

Definition 3.8 (isomorphism) Let K_1, K_2 be abstract simplicial complexes with vertices V_1, V_2 and subset collections $\mathcal{S}_1, \mathcal{S}_2$, respectively. An *isomorphism* between K_1, K_2 is a bijection $\varphi : V_1 \to V_2$, such that the sets in S_1 and S_2 are the same under the renaming of the vertices by φ and its inverse.

Theorem 3.1 *Every abstract complex $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K. Two simplicial complexes are isomorphic iff their vertex schemes are isomorphic as abstract simplicial complexes.*

Definition 3.9 (Geometric Realization) If the simplices $\mathcal{S}$ of an abstract simplicial complex K_1 isomorphic with the vertex scheme $\mathcal{S}$ of the simplicial complex K_2, we call K_2 a *geometric realization* of K_1. It is uniquely determined up to an isomorphism, linear on the simplices.

A geometric realization of an abstract simplicial complex is the analog of an immersion of a manifold, as the simplices may intersect once we place the complex inside a space. Simplices are convex hulls which are compact, so we do not have to worry about other "nasty" immersions.

Having constructed a finite simplicial complex, we will divide it into topological and geometric components. The former will be a abstract simplicial complex, a purely combinatorial object, easily stored and manipulated in a computer system. The latter is a map of the vertices of the complex into the space in which the complex is realized. Again, this map is finite, and can be approximately represented in a computer using a floating point representation.

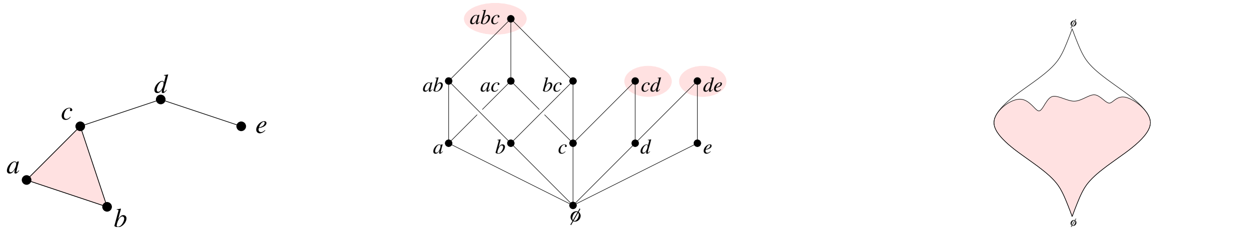
Example 3.1 (Wavefront OBJ format) This representation of a simplicial complex translates word for word into most common file formats for storing surfaces. One standard format is the OBJ format from *Wavefront*. The format first describes the map which places the vertices in $\mathbb{R}^3$. A vertex with location $(x, y, z) \in \mathbb{R}^3$ gets the line "v x y z" in the file. After specifying the map, the format describes an simplicial complex by only listing its triangles, which are the principal simplices (see Definition 3.5.) The vertices are numbered according to their order in the file and numbered from 1. A triangle with vertices v_1, v_2, v_3 is specified with line "f v_1 v_2 v_3". The description in an OBJ file is often called a "triangle soup", as the topology is specified implicitly and must be extracted.

```
v -0.269616 0.228466 0.077226
v -0.358878 0.240631 0.044214
v -0.657287 0.527813 0.497524
v 0.186944 0.256855 0.318011
v -0.074047 0.212217 0.111664
...
f 19670 20463 20464
f 8936 8846 14300
f 4985 12950 15447
f 4985 15447 15448
...
```

Figure 5. Portions of an OBJ file specifying the surface of the Stanford Bunny.

3.4 Subcomplexes

Recall that a simplex is the power set of its simplices. Similarly, a natural view of a simplicial complex is that it is special subset of the power set of all its vertices. The subset is *special* because of the requirements in Definition 3.6. Consider the small complex in Figure 6 (a). The diagram (b) shows how the simplices connect within the complex: it has a node for each simplex, and an edge indicating a face-coface relationship. The marked principal simplices are the "peaks" of the diagram. This diagram is, in fact, a *poset*.



(a) A small complex

(b) Poset of the small complex, with principal simplices marked.

(c) An abstract poset: the "water level" of the poset is defined by principal simplices.

Figure 6. Poset view of a simplicial complex

Definition 3.10 (poset) Let S be a finite set. A *partial order* is a binary relation $\leq$ on S that is reflexive, antisymmetric, and transitive. That is for all $x, y, z \in S$,

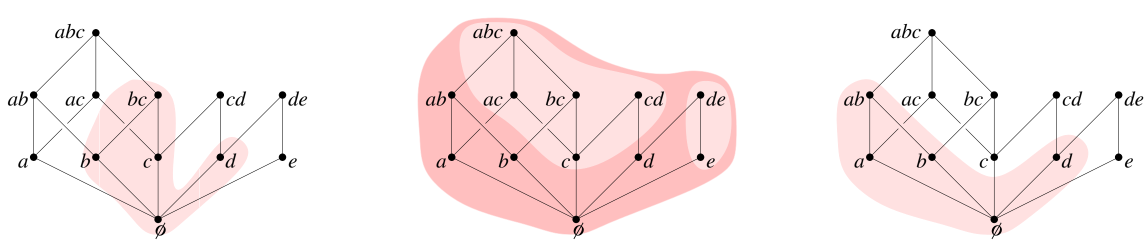
1. $x \leq x$,
2. $x \leq y$ and $y \leq x$ implies $x = y$, and
3. $x \leq y$ and $y \leq z$ implies $x \leq z$.

A set with a partial order is a *partially ordered set* or *poset* for short.

It is clear from the definition that the face relation on simplices is a partial order. Therefore, the set of simplices with the face relation forms a poset. We often abstractly imagine a poset as in Figure 6 (c). The set is fat around its waist because the number of possible simplices $\binom{n}{k}$ is maximized for $k \approx n/2$. The principal simplices form a level beneath which all simplices must be included. Therefore, we may recover a simplicial complex by simply storing its principal simplices, as in the case with triangulations in Example 3.1. This view also gives us intuition for extensions of concepts in point set theory to simplicial complexes. A simplicial complex may be viewed as a closed set (it *is* a closed point set, if it is geometrically realized.)

Definition 3.11 (subcomplex, link, star) A *subcomplex* is a simplicial complex $L \subseteq K$. The smallest subcomplex containing a subset $L \subseteq K$ is its closure, $\text{Cl}\, L = \{\tau \in K \mid \tau \leq \sigma \in L\}$. The *star of* L contains all of the cofaces of L, $\text{St}\, L = \{\sigma \in K \mid \sigma \geq \tau \in L\}$. The *link of* L is the boundary of its star, $\text{Lk}\, L = \text{Cl}\,\text{St}\, L - \text{St}\,(\text{Cl}\, L - \{\emptyset\})$.

Figure 7 demonstrates these concepts within the poset for our complex in Figure 6. A subcomplex is the analog of a subset for a simplicial complex. Given a set of simplices, we take all the simplices "below" the set within the poset to get its closure (a), and all the simplices "above" the set to get its star (b). The face relation is the partial order



(a) $\text{Cl}\,\{bc, d\}$

(b) $\text{St}\,\{c, e\}$ (light) and its closure $\text{Cl}\,\text{St}\,\{c, e\}$ (dark)

(c) $\text{Lk}\,\{c, e\}$

Figure 7. Closure, star, and link of simplices

that defines "above" and "below". Most of the time, the star of a set is an open set (viewed as a point set) and not a simplicial complex. The star corresponds to the notion of a neighborhood for a simplex, and like a neighborhood, it is open. The closure operation completes the boundary of a set as before, making the star a simplicial complex (b). The link operation gives us the boundary. In our example, $\text{Cl}\,\{c, e\} - \emptyset = \{c, e\}$, so we remove the simplices from the light regions from those in the dark region in (b) to get the link (c). Therefore, the link of c and e is the edge ab and the vertex d. Check on Figure 6 (a) to see if this matches your intuition of what a boundary should be.

3.5 Triangulations

The primary reason we study simplicial complexes is to represent manifolds.

Definition 3.12 (underlying space) The *underlying space* $|K|$ of a simplicial complex K is $|K| = \cup_{\sigma \in K} \sigma$.

Note that $|K|$ is a topological space, as defined in the last lecture.

Definition 3.13 (triangulation) A *triangulation* of a topological space $\mathbb{X}$ is a simplicial complex K such that $|K| \approx \mathbb{X}$.

For example, the boundary of a 3-simplex (tetrahedron) is homeomorphic to a sphere and is a triangulation of the sphere, as shown in Figure 8.

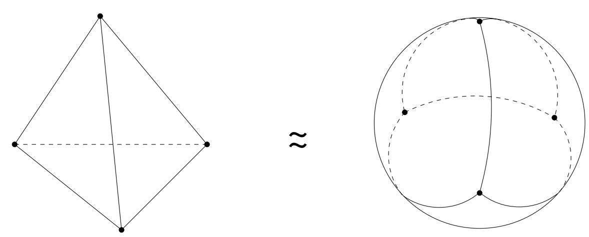


Figure 8. The boundary of a tetrahedron is a triangulation of a sphere, as its underlying space is homeomorphic to the sphere.

The term "triangulation" is used by different fields with different meanings. For example, in computer graphics, the term most often refers to "triangle soup" descriptions of surfaces. The finite element community often refers to triangle soups as a *mesh*, and may allow other elements, such as quadrangles, as basic building blocks. In areas, three-dimensional meshes composed of tetrahedra are called *tetrahedralizations*. Within topology, a triangulation refers to complexes of *any* dimension, however.

3.6 Orientability

We had a definition of orientability in the notes for the first lecture that depended on differentiability. We now extend this definition to simplicial complexes, which are not smooth. This extension further affirms that orientability is a topological property not dependent on smoothness.

Definition 3.14 (orientation) Let K be a simplicial complex. An *orientation* of a k-simplex $\sigma \in K$, $\sigma = \{v_0, v_1, \ldots, v_k\}, v_i \in K$ is an equivalence class of orderings of the vertices of σ, where

$$(v_0, v_1, \ldots, v_k) \sim (v_{\tau(0)}, v_{\tau(1)}, \ldots, v_{\tau(k)}) \tag{1}$$

are equivalent orderings if the parity of the permutation τ is even. We denote an *oriented simplex*, a simplex with an equivalence class of orderings, by $[\sigma]$.

Note that the concept of orientation derives from that fact that permutations may be partitioned into two equivalence classes (if you have forgotten these concepts, you may review *permutations* and *partitions* in the notes from lecture 1 and 2, respectively.) Orientations may be shown graphically using arrows, as shown in Figure 9. We may use oriented simplices to define the concept of orientability to triangulated d-manifolds.

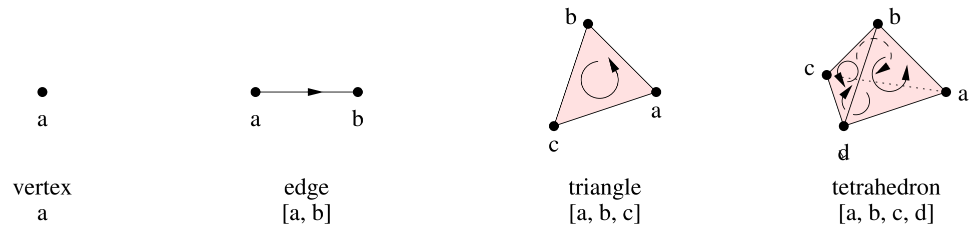


Figure 9. k-simplices, $0 \leq k \leq 3$. The orientation on the tetrahedron is shown on its faces.

Definition 3.15 (orientability) Two k-simplices sharing a $(k-1)$-face σ are *consistently oriented* if they induce different orientations on σ. A triangulable d-manifold is *orientable* if all d-simplices can be oriented consistently. Otherwise, the d-manifold is *non-orientable*

Last lecture, we saw two basic non-orientable 2-manifolds: the Klein bottle and the projective plane. Our exposition shows that non-orientable manifolds can exist in any dimensions, however.

Example 3.2 (Rendering) The surface of a three-dimensional object is a 2-manifold and may be modeled with a triangulation in a computer. In computer graphics, these triangulations are rendered using light models that assign color to each triangle according to how it is situation with respect to the lights in the scene, and the viewer. To do this, the model needs the normal for each triangle. But each triangle has two normals pointing in opposite directions. To get a correct rendering, we need the normals to be consistently oriented.

3.7 Euler Characteristic

Having seen orientability for simplicial surfaces, we finish this lecture by looking at our first topological invariant.

Definition 3.16 (invariant) A *(topological) invariant* is a map that assigns the same object to spaces of the same topological type.

Note that an invariant may assign the same object to spaces of different topological type. In other words, an invariant need not be *complete*. All that is required by the definition is that if the spaces have the same type, they are mapped to the same object. Generally, this characteristic of invariants implies their utility in contrapositives: if two spaces are assigned different objects, they have different topological types. On the other hand, if two spaces are assigned the same object, we usually cannot say anything about them. Let us formally state these statements for an invariant f:

$$\begin{aligned} \mathbb{X} \approx \mathbb{Y} &\implies f(\mathbb{X}) = f(\mathbb{Y}) \\ f(\mathbb{X}) \neq f(\mathbb{Y}) &\implies \mathbb{X} \not\approx \mathbb{Y} \quad \text{(contrapositive)} \\ f(\mathbb{X}) = f(\mathbb{Y}) &\implies \text{nothing} \end{aligned}$$

A good invariant, however, will have enough differentiating power to be useful through contrapositives. Here, we a famous invariant the Euler characteristic.

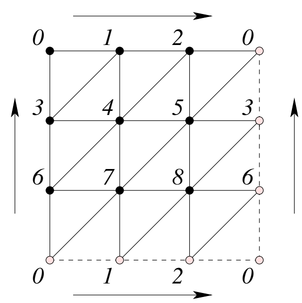
Definition 3.17 (Euler characteristic) Let K be a simplicial complex and $s_i = |\{\sigma \in K \mid \dim \sigma = i\}|$. The *Euler characteristic* $\chi(K)$ is

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i s_i = \sum_{\sigma \in K - \{\emptyset\}} (-1)^{\dim \sigma}. \tag{2}$$

While it is defined for a simplicial complex, the Euler characteristic is an integer invariant for $|K|$, the underlying space of K. Given *any* triangulation of a space $\mathbb{M}$, we always will get the same integer, which we will call the Euler characteristic of that space $\chi(\mathbb{M})$.

3.8 2-Manifolds

Armed with triangulations, orientability, and the Euler characteristic, we return to 2-manifolds to convert our "existential" proof from last lecture to a computational one. We begin with calculating the Euler characteristic for the basic surfaces from the last lecture. We have a triangulation of a sphere $\mathbb{S}^2$ in Figure 8, so $\chi(\mathbb{S}^2) = 4 - 6 + 4 = 2$. To compute the Euler characteristic of the other manifolds, we must build triangulations for them. This is simple, however, by triangulating the diagrams for constructing flat 2-manifolds from the last lecture, as in Figure 10. This triangulation gives us $\chi(\mathbb{T}^2) = 9 - 18 + 27 = 0$. We may complete the table in Figure 10 (b) in a similar fashion. As $\chi(\mathbb{T}^2) = \chi(\mathbb{K}^2) = 0$, the Euler characteristic by itself is not powerful enough to differentiate between surfaces.



(a) A triangulation for the diagram of the torus $\mathbb{T}^2$

2-Manifold	χ
Sphere $\mathbb{S}^2$	2
Torus $\mathbb{T}^2$	0
Klein bottle $\mathbb{K}^2$	0
Projective plane $\mathbb{R}\mathrm{P}^2$	1

(b) The Euler characteristics of our basic 2-manifolds

Figure 10. A triangulation of the diagram of the torus $\mathbb{T}^2$

Last lecture, we also discussed constructing more complicated surfaces using the connected sum. Suppose we form the connected sum of two surfaces $\mathbb{M}_1, \mathbb{M}_2$ by removing a single triangle from each, and identifying the two boundaries. Clearly, the Euler characteristic should be the sum of the Euler characteristics of the two surfaces, minus 2 for the two missing triangles. In fact, this is true for arbitrary shaped disks.

Theorem 3.2 *For compact surfaces* $\mathbb{M}_1, \mathbb{M}_2$, $\chi(\mathbb{M}_1 \# \mathbb{M}_2) = \chi(\mathbb{M}_1) + \chi(\mathbb{M}_2) - 2$.

For a compact surface $\mathbb{M}$, let $g\mathbb{M}$ be the connected sum of g copies of $\mathbb{M}$. Combining this theorem with the table in Figure 10 (b), we get the following.

Corollary 3.1 $\chi(g\mathbb{T}^2) = 2 - 2g$ *and* $\chi(g\mathbb{R}P^2) = 2 - g$.

These surfaces, along with the sphere, form the equivalence classes of 2-manifolds discussed in the last lecture.

Definition 3.18 (genus) The connected sum of g tori is called a surface with *genus* g.

The genus refers to how many "holes" the multi-donut surface has. We are now ready to give a complete answer to the homeomorphism problem for closed compact 2-manifolds.

Theorem 3.3 (Homeomorphism problem of 2-manifolds) *Closed compact surfaces* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *are homeomorphic,* $\mathbb{M}_1 \approx \mathbb{M}_2$ *iff*

1. $\chi(\mathbb{M}_1) = \chi(\mathbb{M}_2)$ *and*
2. *either both surfaces are orientable or both are non-orientable.*

Observe that the theorem is "if and only if". We can easily compute the Euler characteristic of any 2-manifold. Computing orientability is also easy by orienting one triangle and "spreading" the orientation throughout the manifold if it is orientable. Therefore, we have a full computational method for capturing topology of 2-manifolds. As we shall see in the future lectures, the problem is much harder in higher dimensions, forcing us to resort to more elaborate machinery.

Acknowledgments

The material for this lecture is mostly from Munkres [4] and Firby and Gardiner [2], with inspirations from Henle [3] and personal notes. The attributed quote is from Cameron [1]. Thanks to Daniel Russel and Niloy Mitra for proof-reading.

References

[1] CAMERON, P. J. *Combinatorics: topics, techniques, algorithms*. Cambridge University Press, New York, NY, 1994.

[2] FIRBY, P. A., AND GARDINER, C. F. *Surface Topology*. Ellis Horwood Limited, Chichester, England, 1982.

[3] HENLE, M. *A Combinatorial Introduction to Topology*. Dover Publications, Inc., New York, 1997.

[4] MUNKRES, J. R. *Elements of Algebraic Topology*. Addison-Wesley, Reading, MA, 1984.