Optimally Cutting a Surface into a Disk

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Problem

Given:

- Polyhedral representation of a 2-manifold (with boundary)

Task:

- Cut the surface along the edges into a topological disk, minimize total length of cut edges

Applications:

- Surface parameterization
- Texture mapping
- Geometric algorithms

NP hard problem

Problem

To transform a surface into a disk, we need to

- Cut along the non-bounding cycles
- Cut to connect the boundaries

Idea

- Decompose into topologically simpler, but not trivial components, making small number of cuts
- Handle the pieces separately

Cut graph

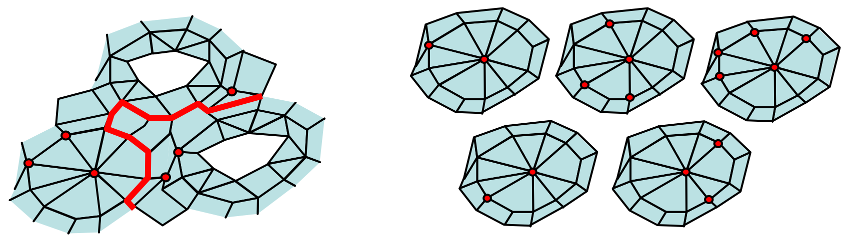
- A collection of edges G in M such that $M \setminus G$ is a topological disk

Method Overview

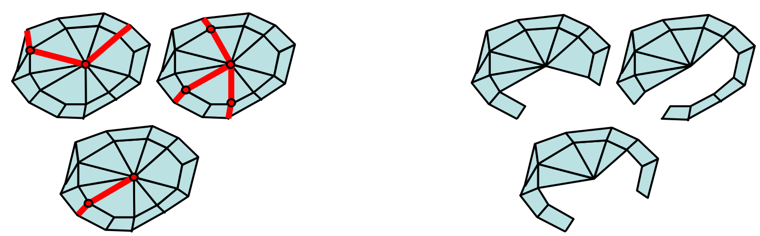
Approximate minimal cut graph, approximation ratio $O(\log^2 g)$

- Convert boundaries into *punctures*

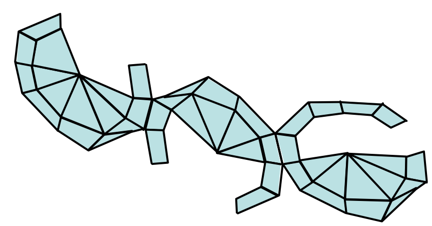
- Cut along *essential cycles,* to get a collection of *punctured spheres*



- Connect the punctures on each sphere by cutting along a MST



- Re-glue components (disks) if necessary, to get a single disk

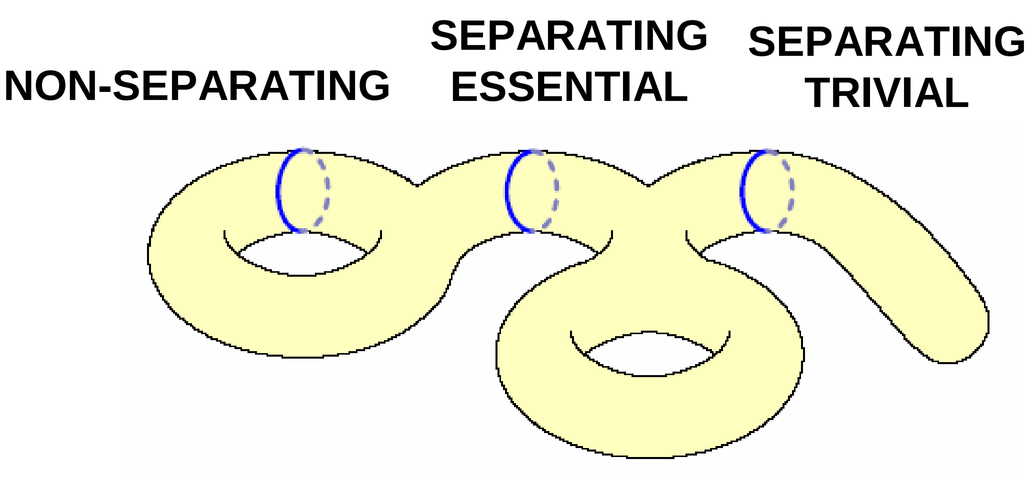


Essential Cycle

Essential cycle

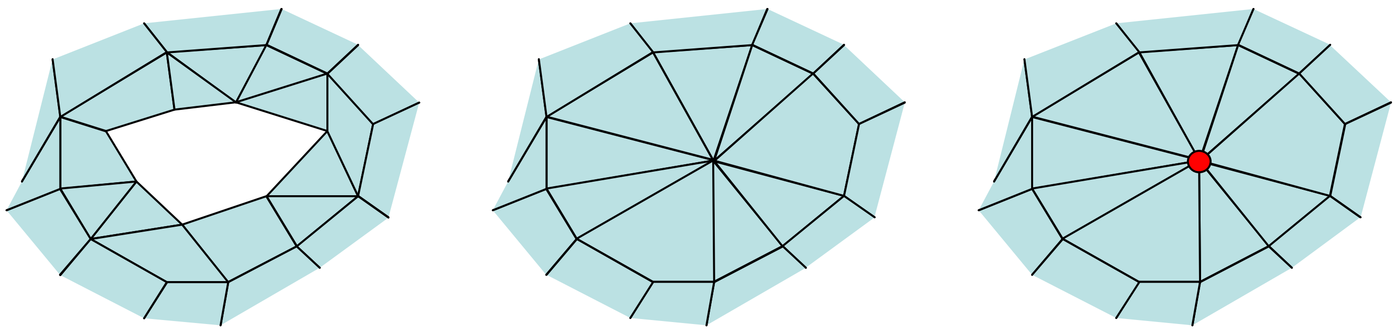
- Simple cycle that does not bound a (punctured) sphere

Essential cycles are either non-bounding, or they bound a topologically non-trivial sub-manifold



Converting boundaries to punctures

Contract into a point, close the boundary, mark the point as a puncture

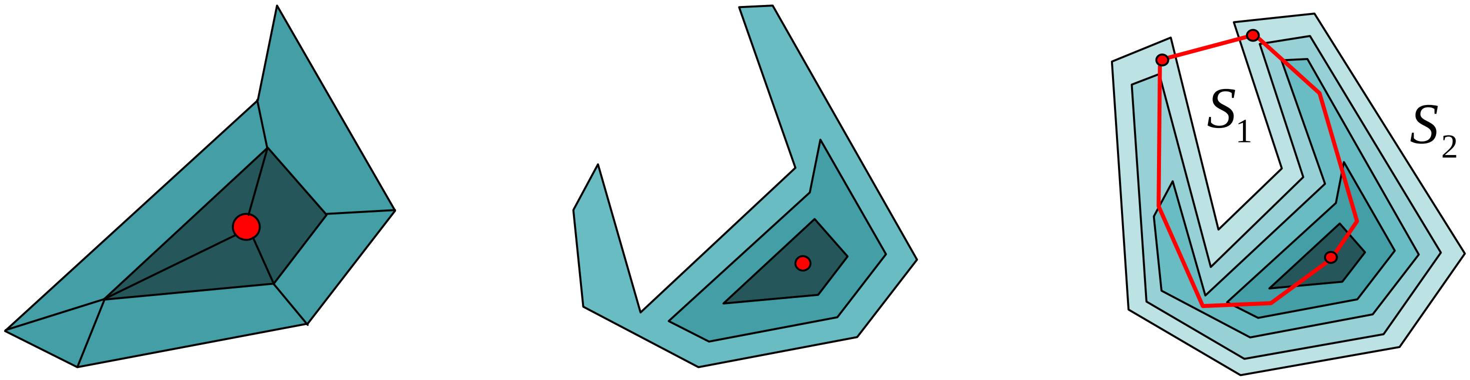


OLD MESH = NEW MESH + SET OF PUNCTURES

Lemma This does not change the length of the shortest cut graph, if as long as the cut graph in the modified manifold includes all the punctures

Shortest essential cycle containing a given vertex

Simulating a circular wave expanding from the source vertex
Add edges one by one, by Dijkstra-like greedy rule



Do a topology check on S_1 and S_2 when a loop forms

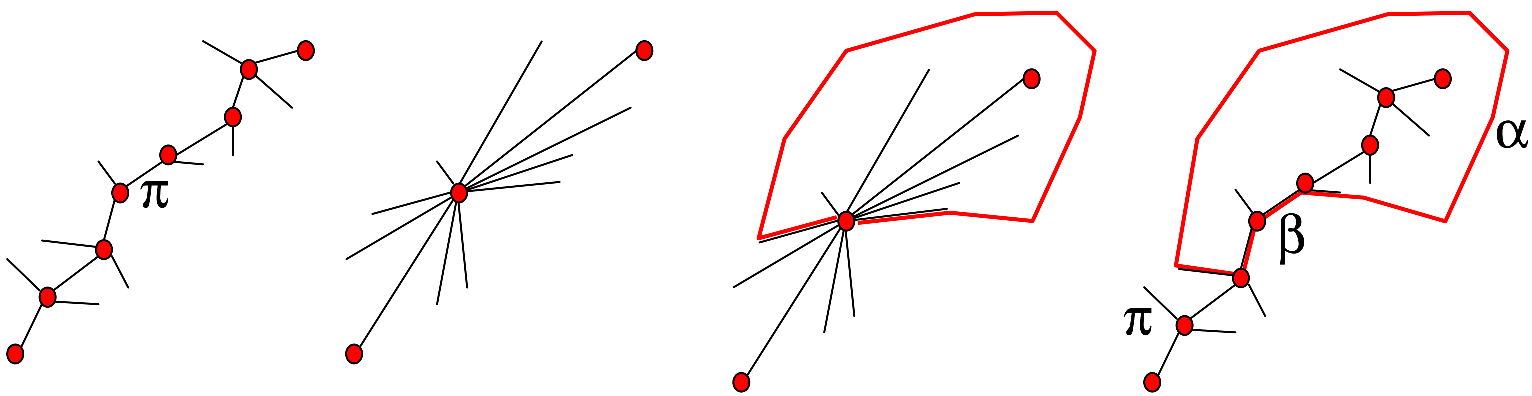
- S_1 and S_2 are connected – essential
- S_1 and S_2 are non-trivial – essential
- Either S_1 and S_2 is trivial – non-essential

Running time $O(n \log n)$

Nearly Shortest Essential Cycles

Lemma Shortest essential cycle that intersects a *shortest path* between any two vertices, can be 2-approximated in time $O(n \log n)$

Proof (in pictures)



$$|\gamma| = |\alpha| + |\beta| \leq 2|\alpha| \leq 2|\gamma^*|$$

Lemma A set of $O(g)$ shortest paths that intersects all essential cycles can be computed in $O(gn \log n)$

Corollary A 2-approximation to a shortest essential cycle can be computed in $O(gn \log n)$

Puncture Spanning Trees

Now we have a collection of punctured spheres
Make minimal set of cuts to connect the punctures

Steiner tree problem - punctures are the “special” nodes

2-approximation algorithm

- Compute shortest paths between all pairs of “special” nodes
- Find a MST in the complete graph of distances
- Project MST back to the original graph

Running time $O(n \log n)$, with careful implementation

Analysis

Running time

- At most g non-separating cuts and $g-1$ separating cuts

$$(2g-1)O(gn\log n) = O(g^2 n\log n)$$

Approximation ratio

Lemma The shortest essential cycle in a manifold without boundary contains $O(\log g / g)$ fraction of the minimum cut graph

- Carries over to the case of manifold with multiple components of the same genus
- g phases, delimited by consecutive genus-decreasing cuts

$$\sum_{i=1}^{g} O(\log i / i)\,|G_i^*| \le \sum_{i=1}^{g} O(\log i / i)\,|G^*| \le O(\log^2 g)\,|G^*|$$

Conclusion

Method

- Divide into simpler, topologically non-trivial components, tractable by the MST approach
- Process the components separately, and re-glue at the end

No distinction between bounding and non-bounding cycles, orientable and non-orientable manifolds

Open problems

- Complexity for constant parameters
- Better approximation ratio