SURFACE TOPOLOGY

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CS 468 – Lecture 2

1-21-4

OVERVIEW

- Last lecture:
  - Manifolds: locally Euclidean (locally flat)
  - Homeomorphisms: bijective bi-continuous maps
  - Topology studies invariant properties
  - Classification?
- This lecture:
  - Fun with homeomorphisms
  - Classifying 2-Manifolds
  - Sphere Eversions

PARTITIONS

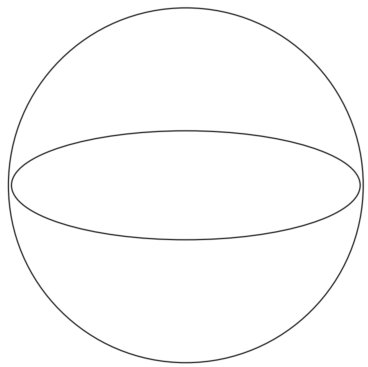
- A partition of a set is a decomposition of the set into subsets (cells) such that every element of the set is in one and only one of the subsets.
- Let $\sim$ be a relation on a nonempty set S so that for all $a, b, c \in S$:
  1. (Reflexive) $a \sim a$.
  2. (Symmetric) If $a \sim b$, then $b \sim a$.
  3. (Transitive) If $a \sim b$ and $b \sim c$, $a \sim c$.

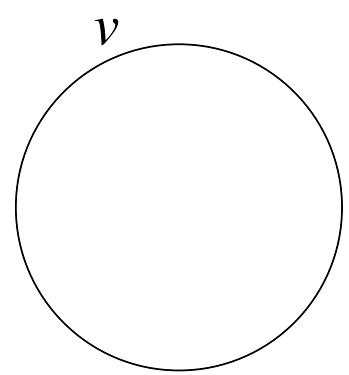
  Then, $\sim$ is an equivalence relation on S.
- Homeomorphism is an equivalence relation.

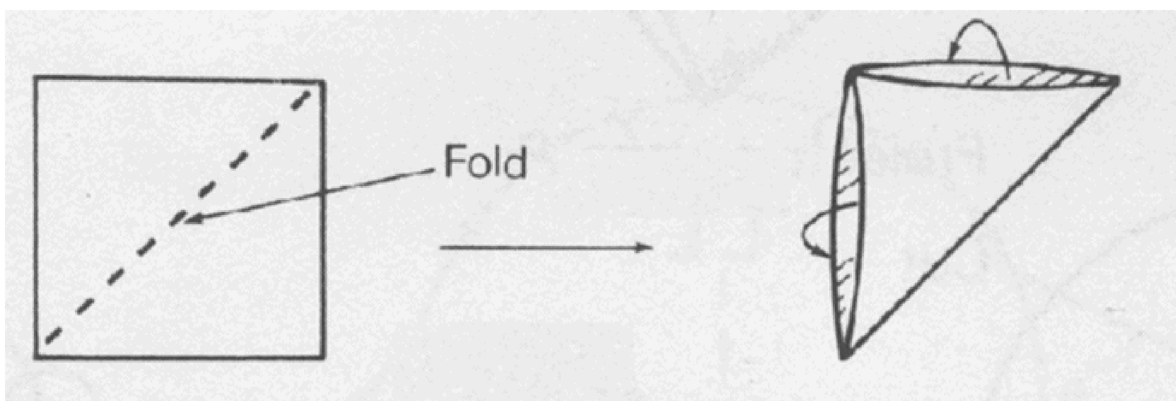
TOPOLOGICAL TYPE

- **(Theorem)** Let S be a nonempty set and let $\sim$ be an equivalence relation on S. Then, $\sim$ yields a natural partition of S, where $\bar{a} = \{x \in S \mid x \sim a\}$. $\bar{a}$ represents the subset to which a belongs to. Each cell $\bar{a}$ is an equivalence class.
- Homeomorphism partitions manifolds with the same topological type.
- Can we compute this?
  - $n = 1$: too easy
  - $n = 2$: yes (this lecture)
  - $n = 3$: ?
  - $n \geq 4$: undecidable! [Markov 1958]

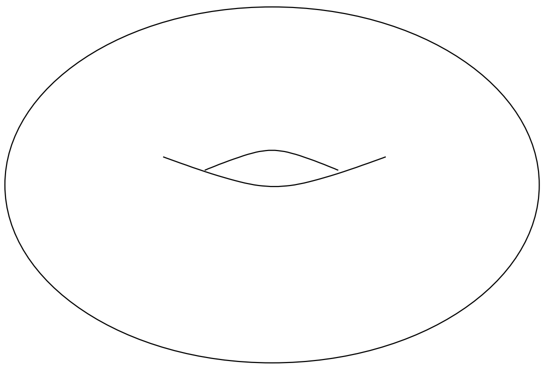
BASIC 2-MANIFOLDS:
SPHERE $\mathbb{S}^2$

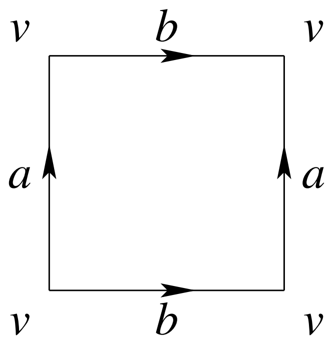


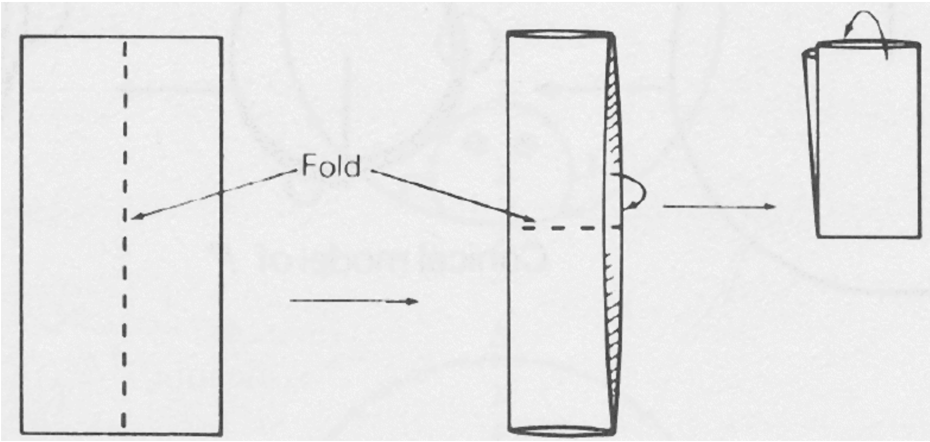




BASIC 2-MANIFOLDS:
TORUS $\mathbb{T}^2$

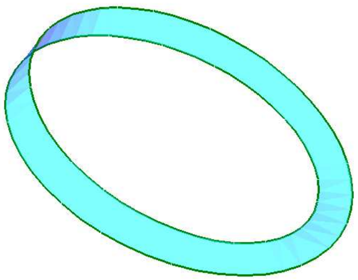


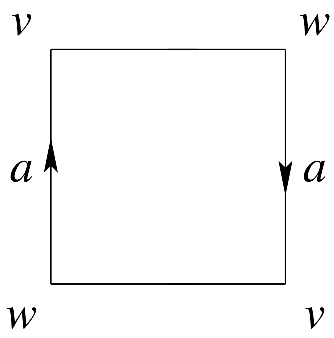


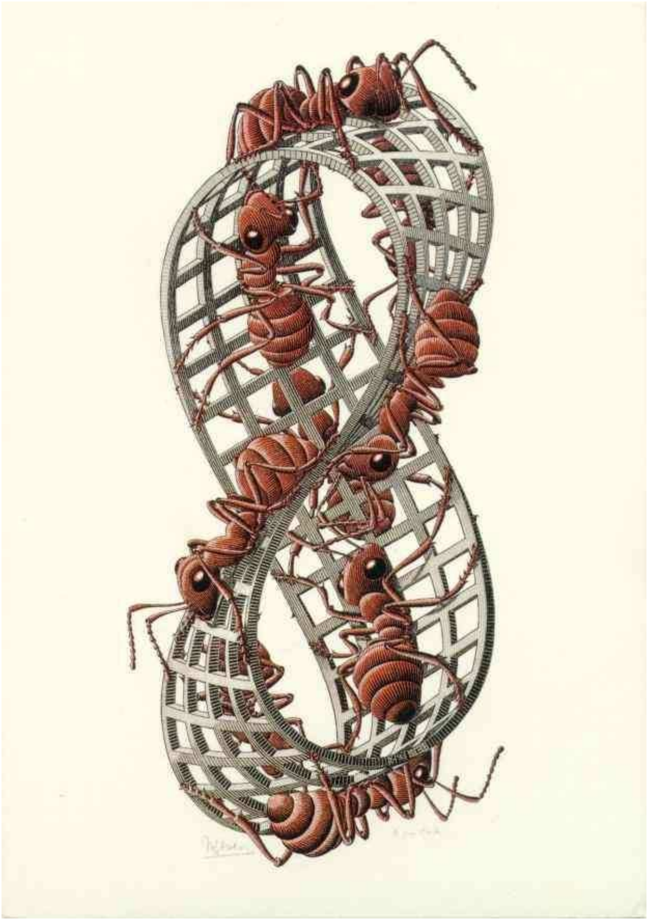


$$T(u,v) = ((1+\cos u)\cos v, (1+\cos u)\sin v, \sin(u))$$

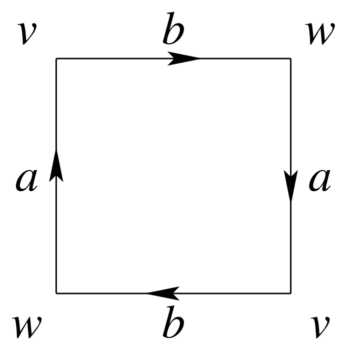
BASIC 2-MANIFOLDS:
MÖBIUS STRIP

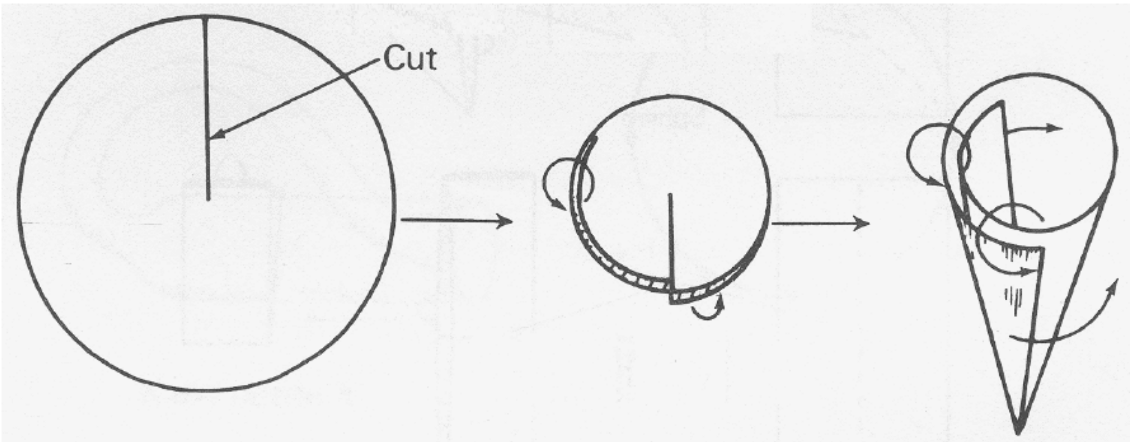




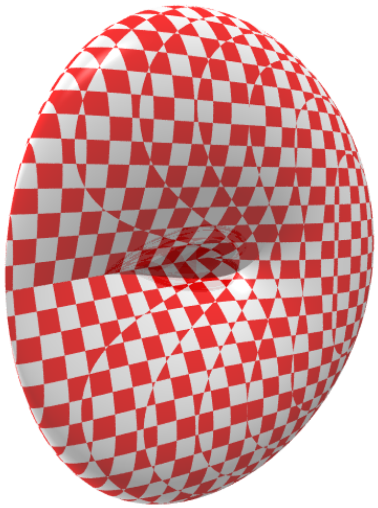


BASIC 2-MANIFOLDS:
PROJECTIVE PLANE $\mathbb{RP}^2$

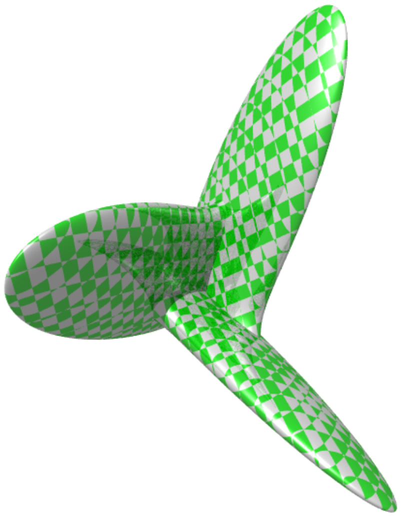




Basic 2-Manifolds:
Models of $\mathbb{R}P^2$



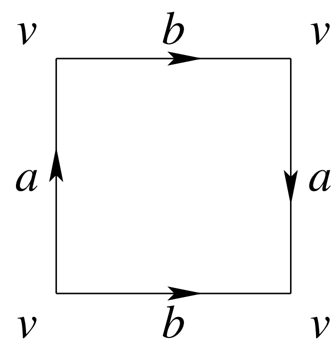
(a) Cross cap

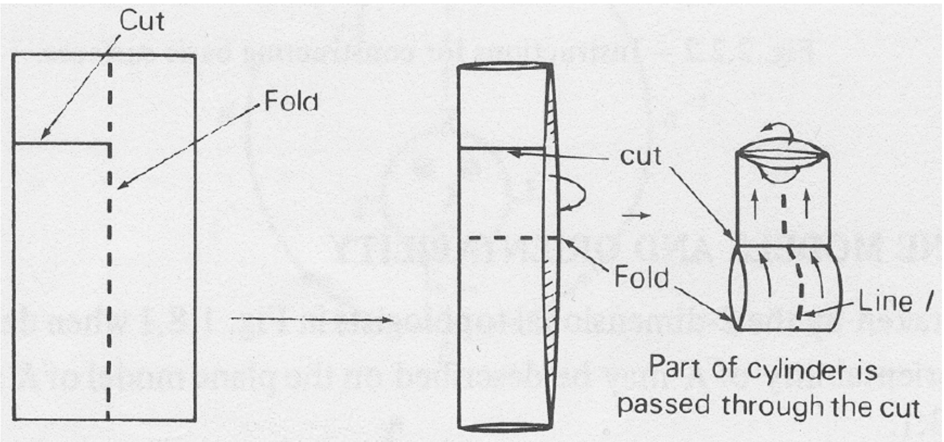


(b) Boy's Surface

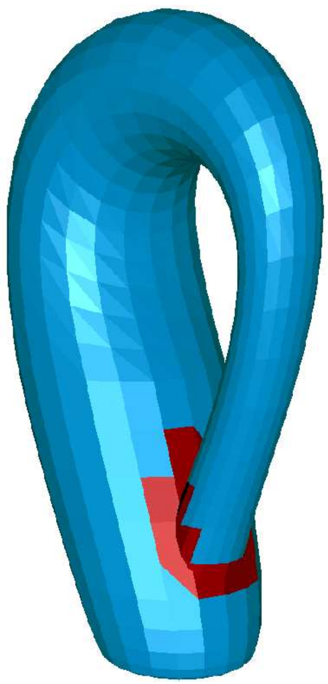
(c) Steiner's Roman Surface

Basic 2-Manifolds:
Klein Bottle $\mathbb{K}^2$

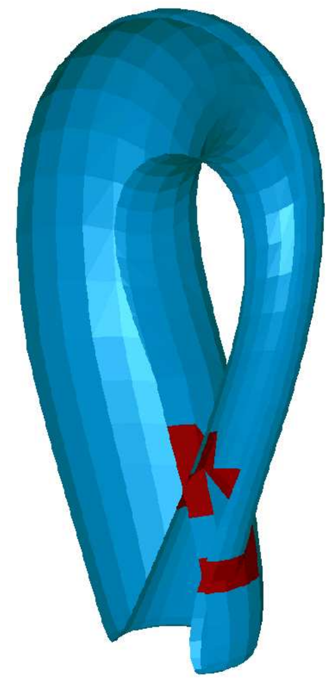




BASIC 2-MANIFOLDS:
IMMERSION OF $\mathbb{K}^2$



(a) Klein Bottle



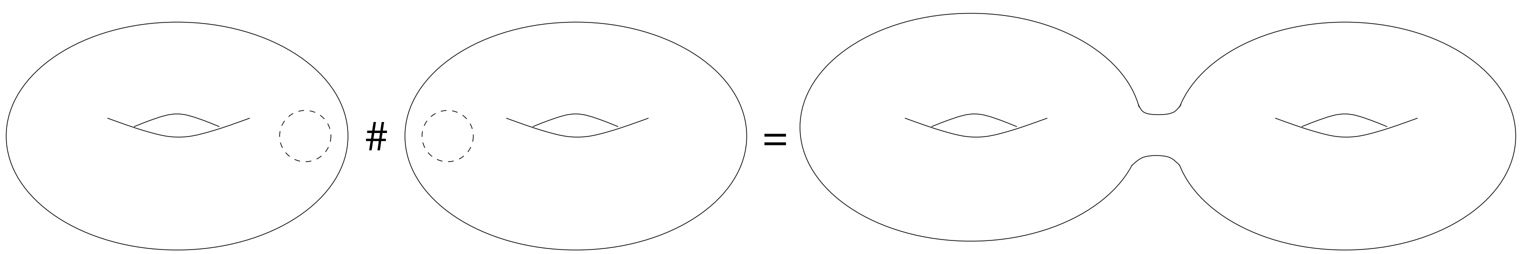
(b) Möbius Strip

CONNECTED SUM

- The connected sum of two n-manifolds $\mathbb{M}_1, \mathbb{M}_2$ is

$$\mathbb{M}_1 \# \mathbb{M}_2 \quad = \quad \mathbb{M}_1 - \mathring{D}_1^n \bigcup_{\partial \mathring{D}_1^n = \partial \mathring{D}_2^n} \mathbb{M}_2 - \mathring{D}_2^n,$$

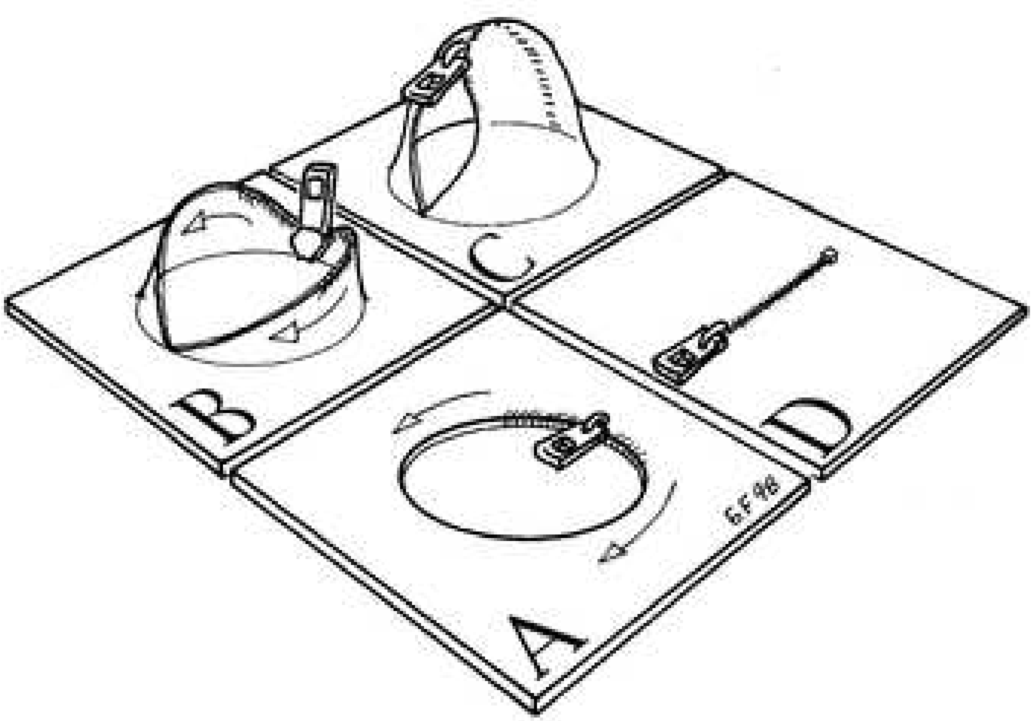
where D_1^n, D_2^n are n-dimensional closed disks in $\mathbb{M}_1, \mathbb{M}_2$, respectively.



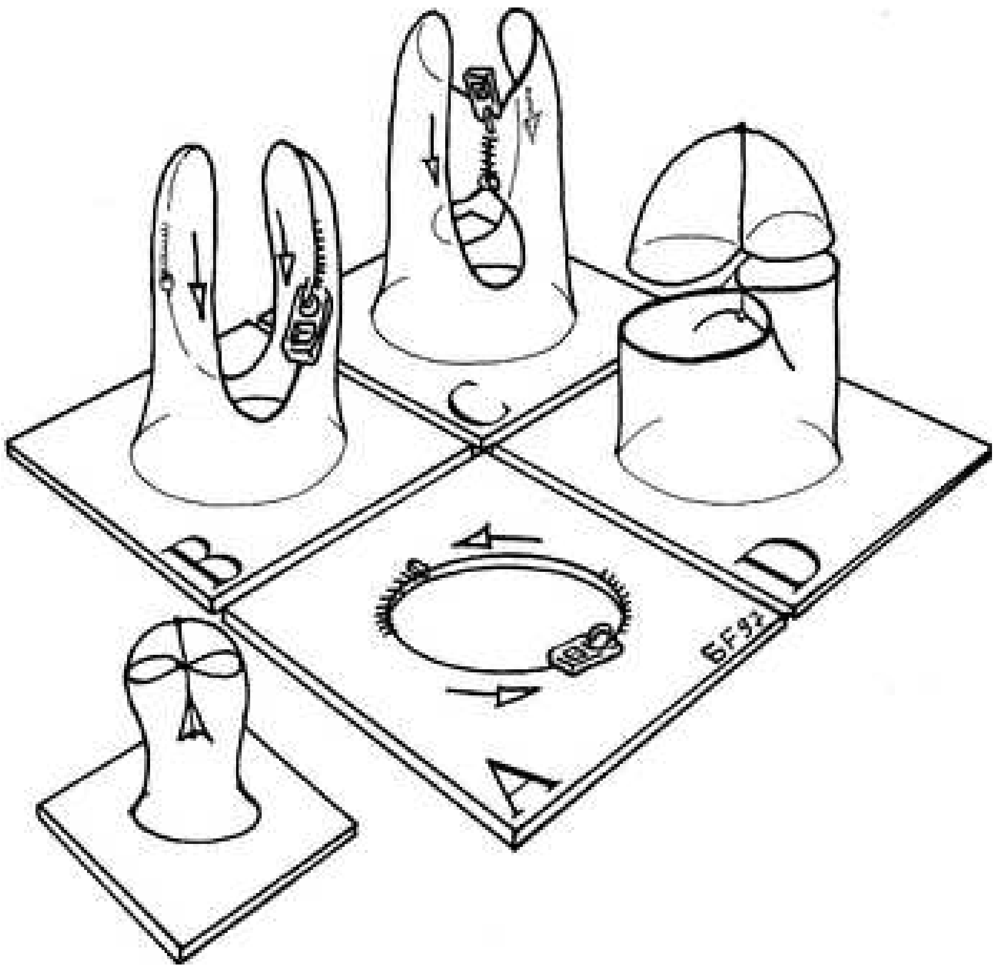
Classification Theorem

- **(Theorem)** Every closed compact surface is homeomorphic to a sphere, the connected sum of tori, or connected sum of projective planes.
- Known since 1860's
- Seifert and Threlfall proof
- Conway's Zero Irrelevancy Proof or ZIP (1992)
- Francis and Weeks (1999)

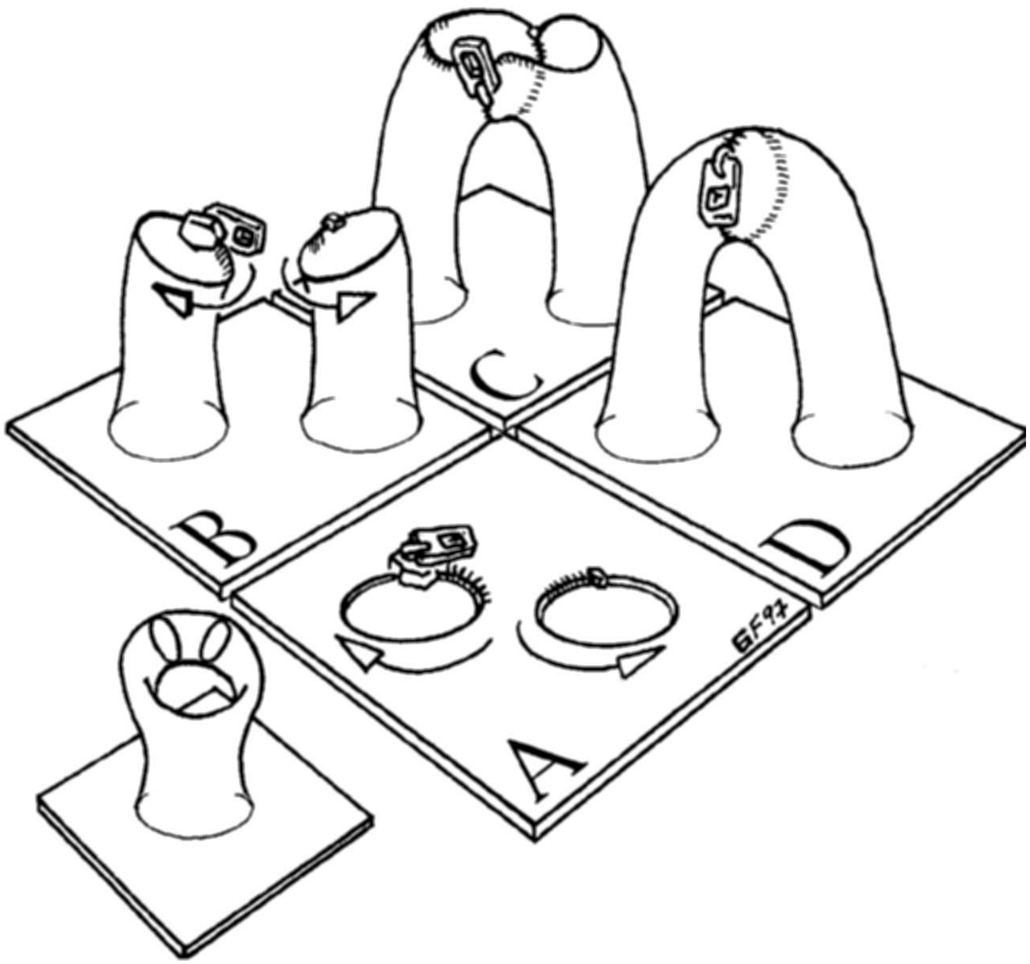
CONWAY'S ZIP:
CAP



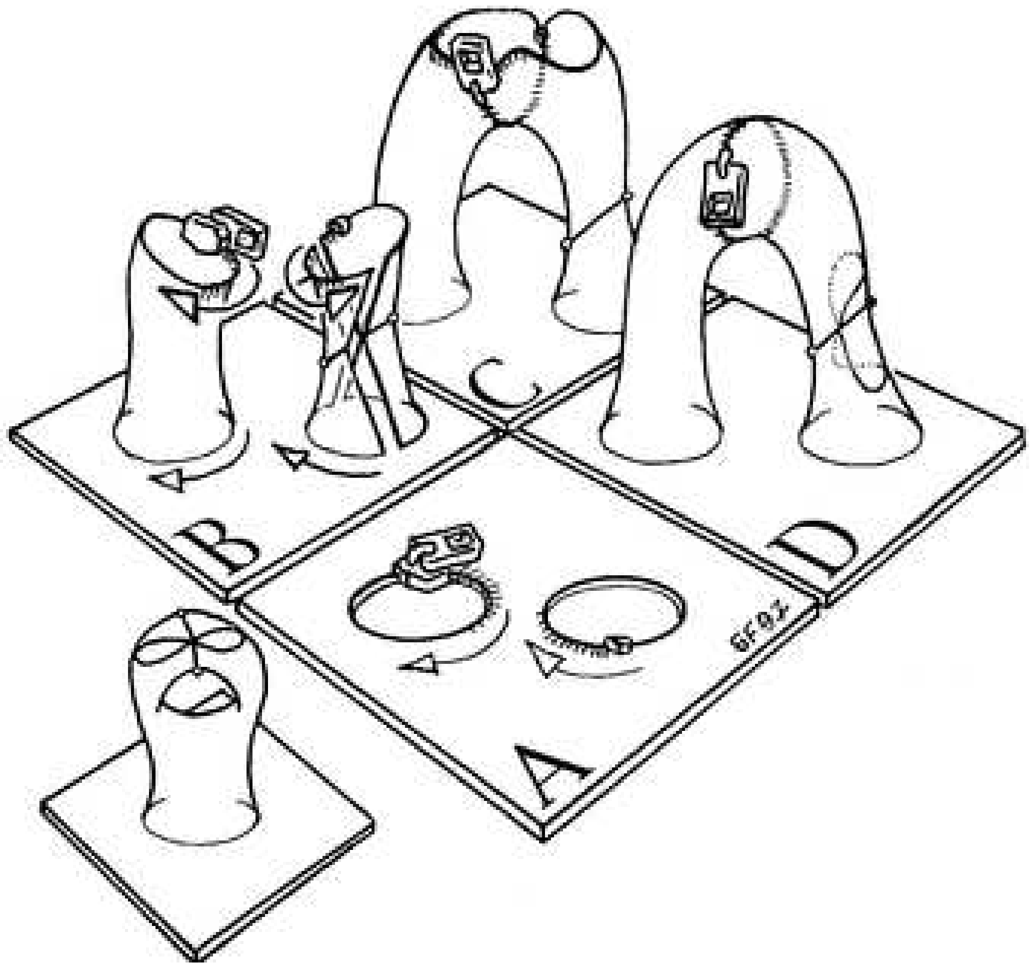
CONWAY'S ZIP:
CROSSCAP



CONWAY'S ZIP:
HANDLE



CONWAY'S ZIP:
CROSS HANDLE

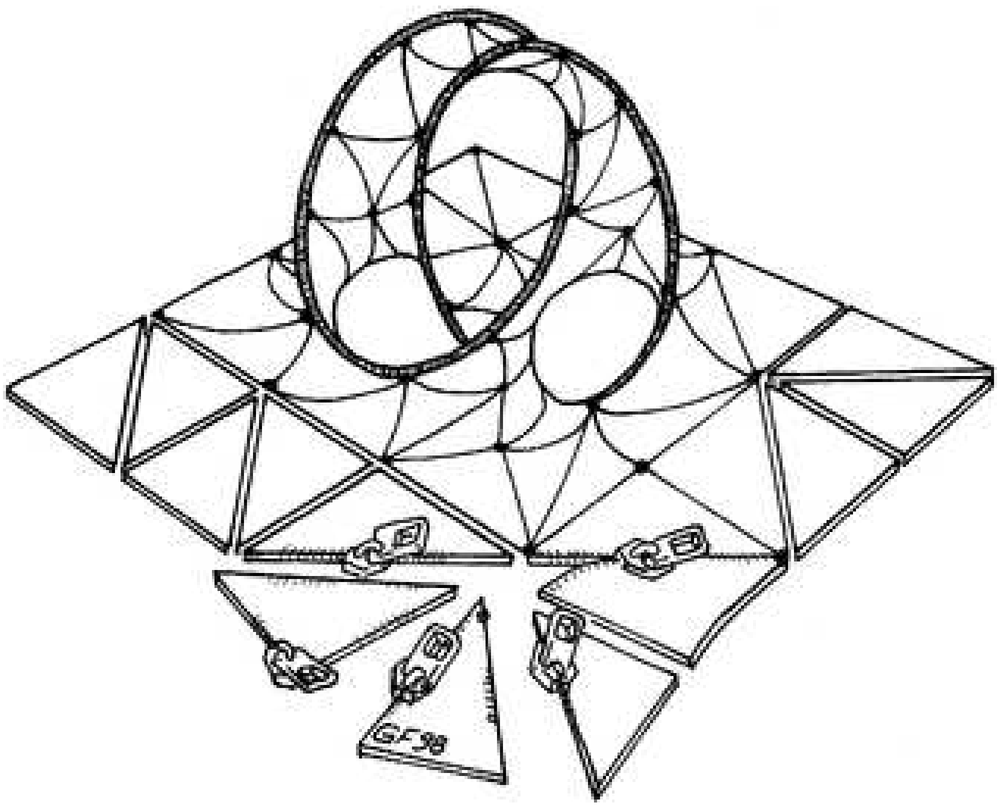


CONWAY'S ZIP:

ORDINARY SURFACES

- (Theorem) Every (compact) surface is homeomorphic to a finite collection of spheres, each with a finite number of
  - handles,
  - crosshandles,
  - crosscaps,
  - and perforations.
- That is, all surfaces are ordinary.

CONWAY'S ZIP:
ZIP-PAIRS

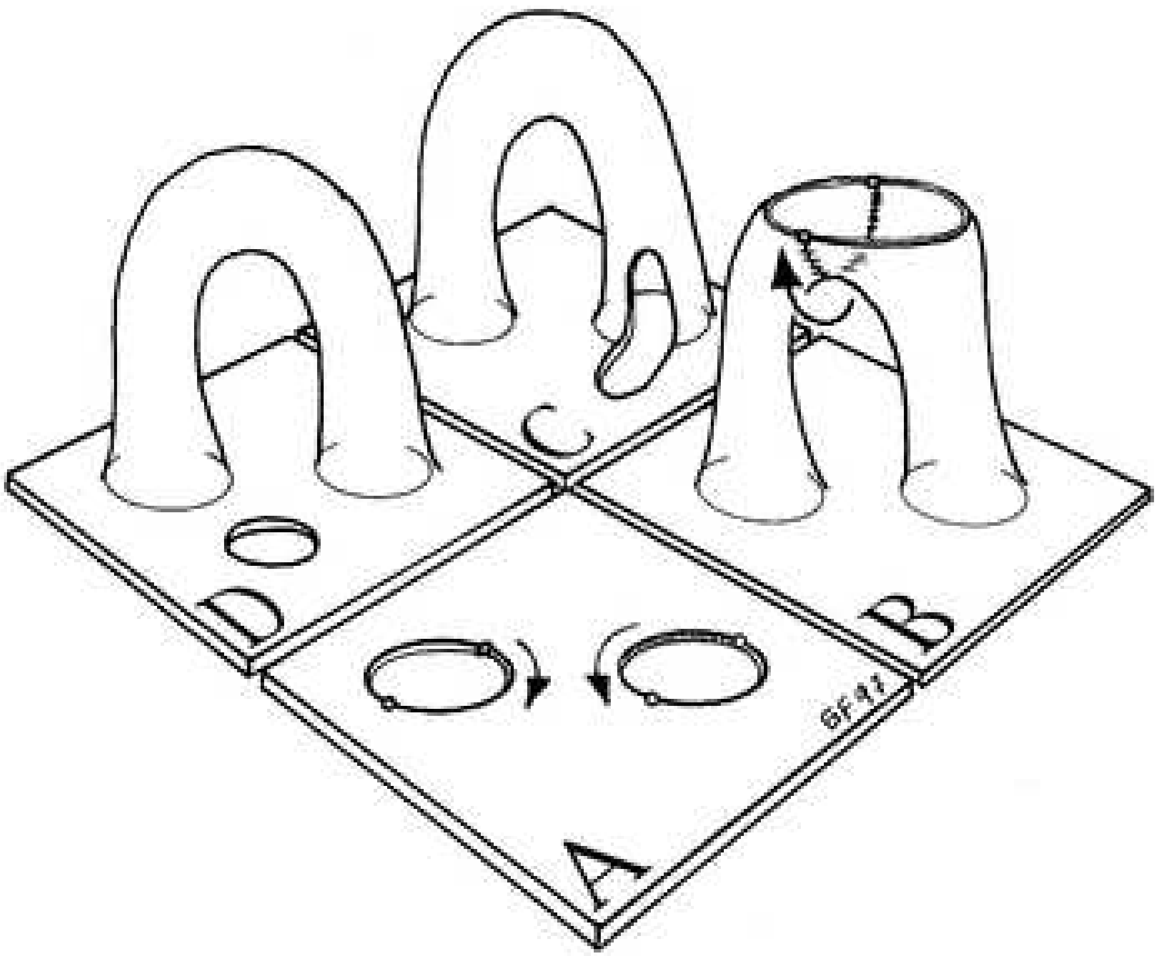


- Surface not necessarily connected
- We assume it's triangulated [Radó 1925]

CONWAY'S ZIP:
ORDINARY PROOF

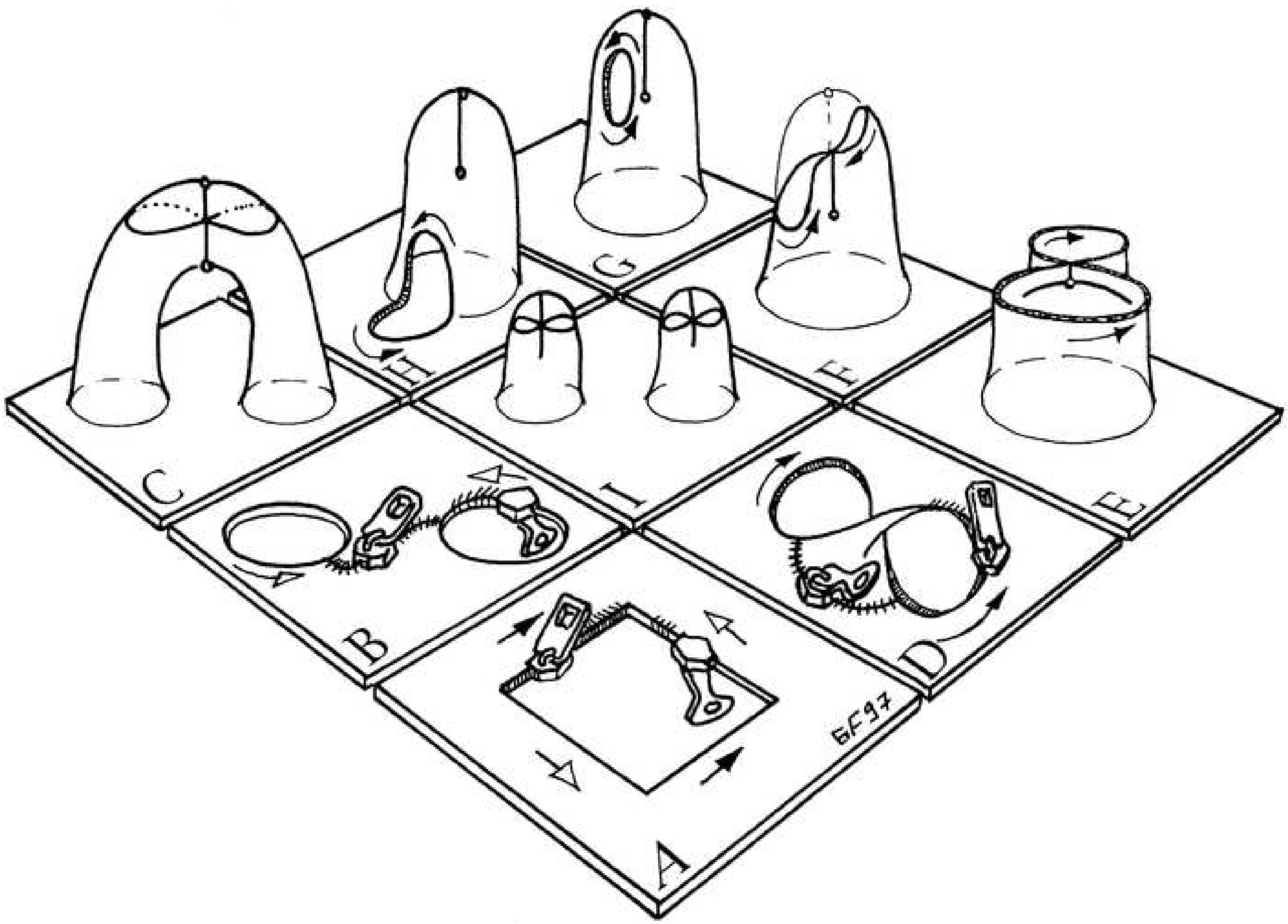
- Case 1: two zips completely occupy a boundary circle
  - same component: deform, get handle or crosshandle
  - different component: join components
- Case 2: share single boundary, occupy completely
  - cap
  - crosscap
- Case 3: do not occupy completely
  - zip, then slide out perforation
  - take off glasses, zip, put them on.

CONWAY'S ZIP:
PERFORATIONS



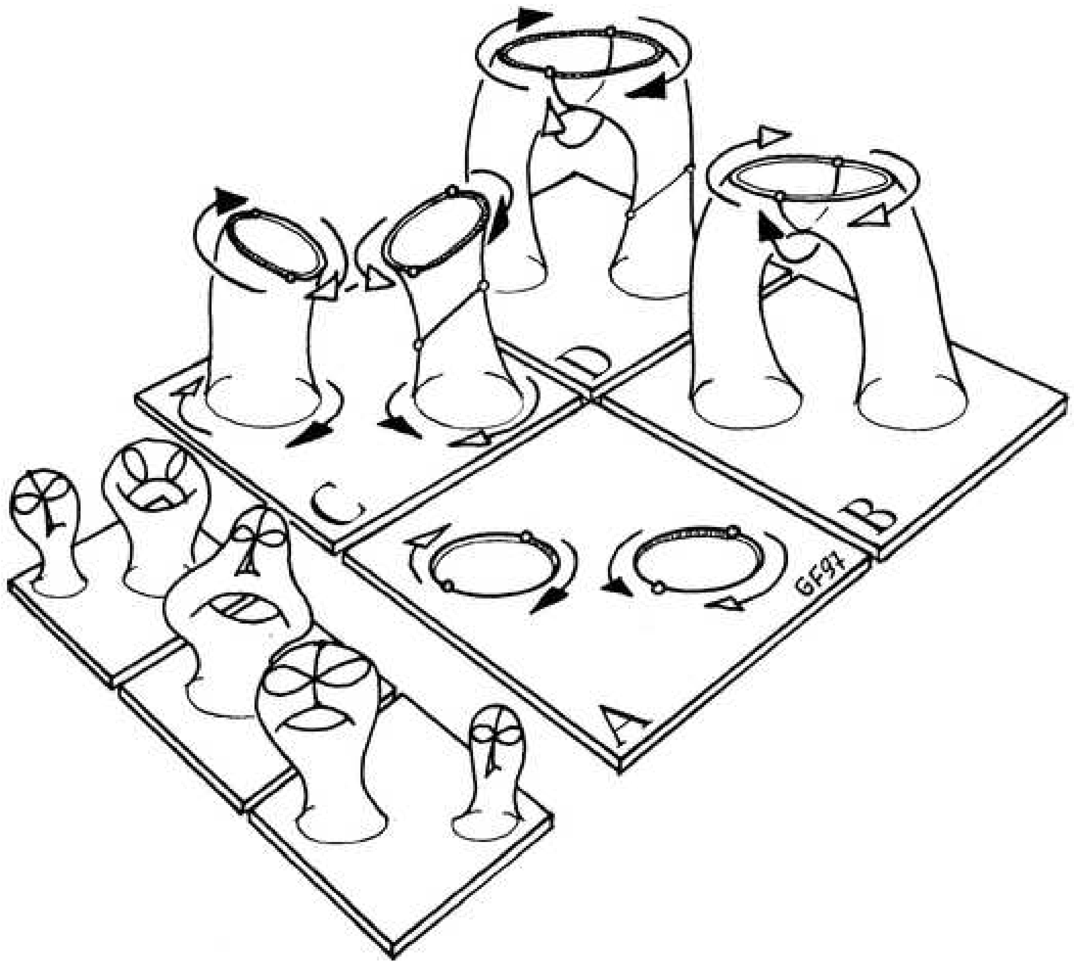
CONWAY'S ZIP:

LEMMA 1: XHANDLE = 2 XCAPS



CONWAY'S ZIP:

LEMMA 2: XHANDLE + XCAP = HANDLE + XCAP



[Dyck 1888]

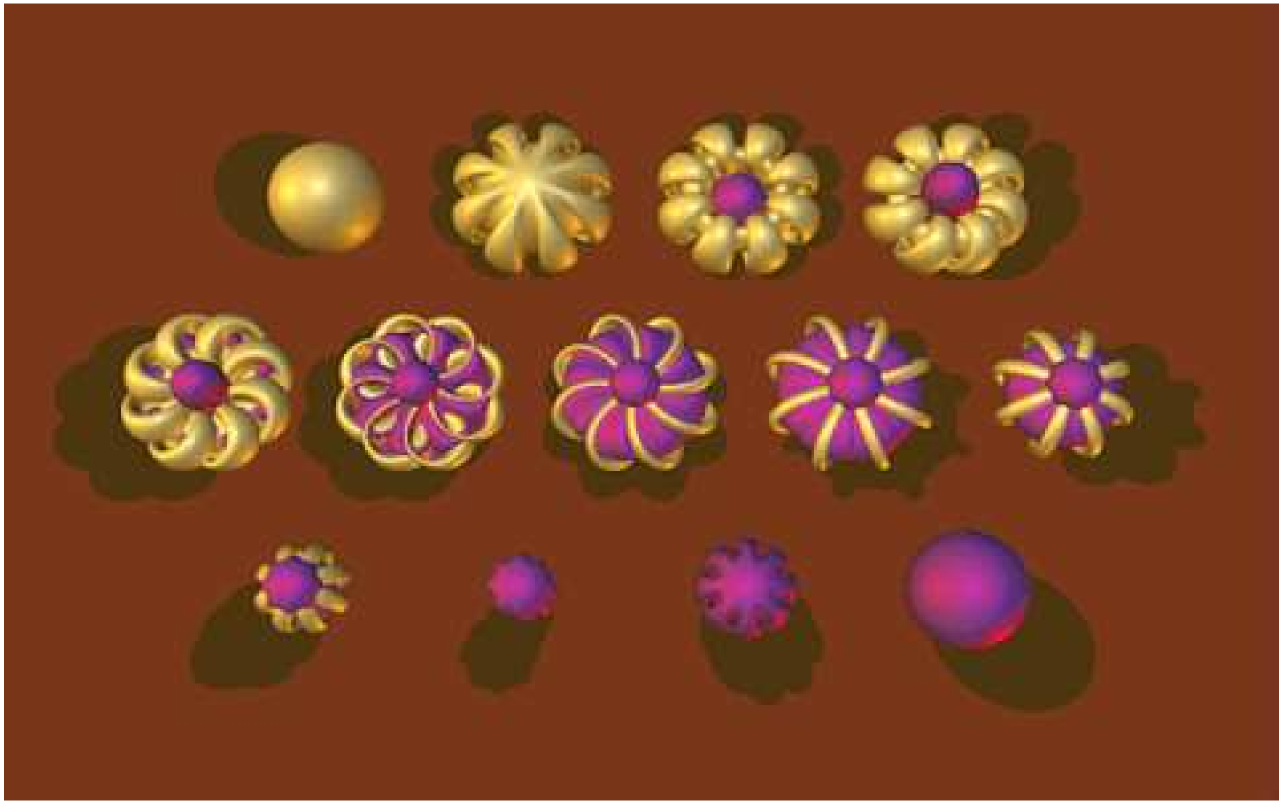
CONWAY'S ZIP:
PROOF

- Otherwise, we have handles, crosshandles, and crosscaps
- Lemma 1: crosshandle = 2 crosscaps
- Lemma 2: crosshandle + crosscap = handle + crosscap
- So, handle + crosscap = 3 crosscaps
- We get sphere, sphere with handles, or sphere with crosscaps. QED

SPHERE EVERSIONS

- Sphere is orientable (two-sided)
- So, turn it inside out!
- Smale 1957
- Morin 1979
- “Turning a Sphere Inside Out” [Max 1977]
- “Outside In” [Thurston 1994]
- “The Optiverse” [Sullivan 1998]

TWO EVERSIONS



Outside In [Thurston 94]

- The Optiverse [Sullivan 98]