# Approximation Schemes for Euclidean k-Medians and Related Problems

S. Arora, P. Raghavan, S. Rao STOC '98

# A Nearly Linear-Time App. Scheme for the Euclidean k-median Problem

S. Kolliopoulos, S. Rao ESA '99

# k-medians Problem

Given

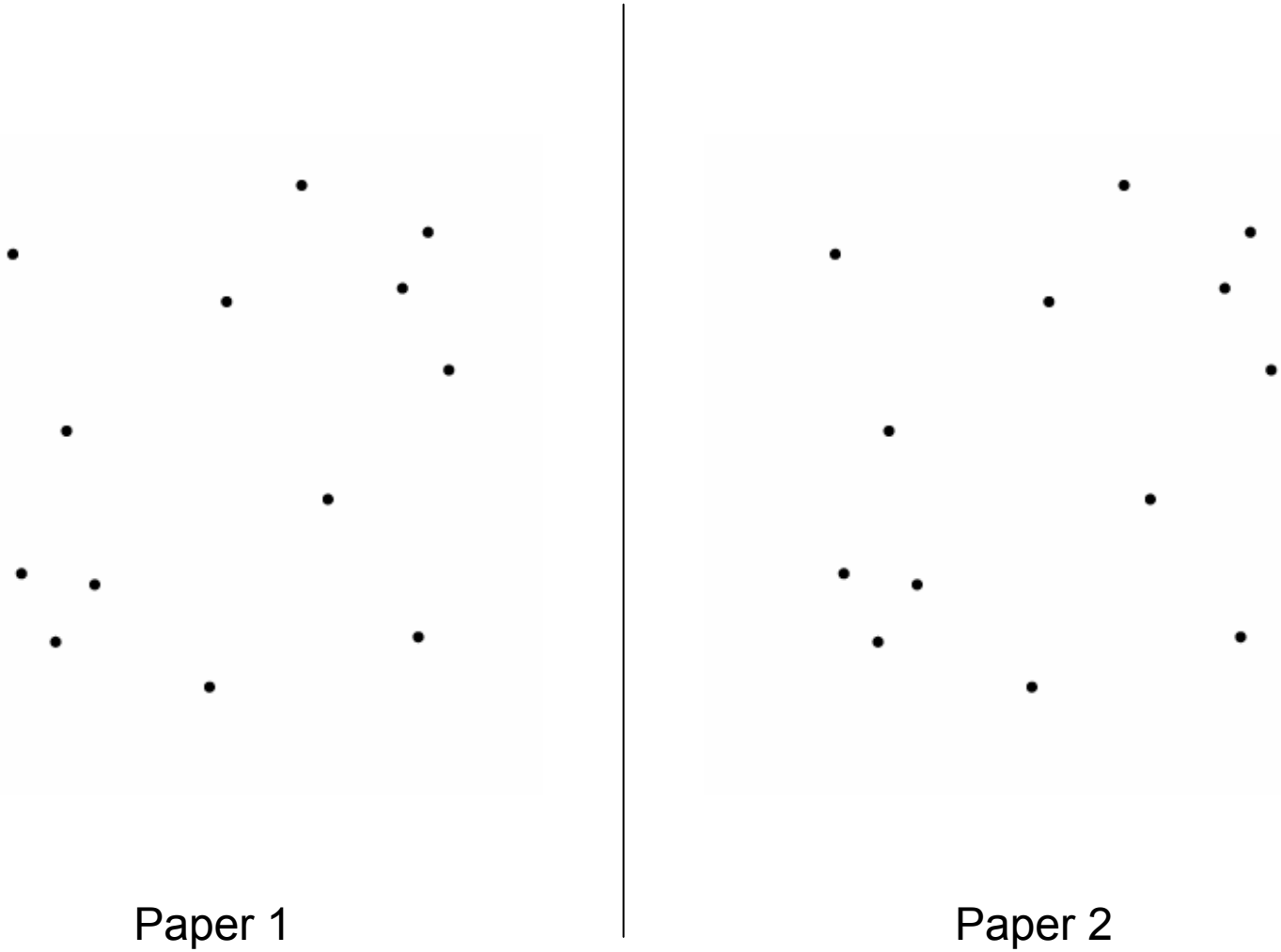
- S={$x_i$} be n points in metric space $R^d$
- Positive integer k

Goal

- Find M={$m_i$} $\in$ $R^d$ which minimizes

$$\sum_{i=1}^{n} \min_{1 \leq j \leq k} d(x_i, m_j)$$

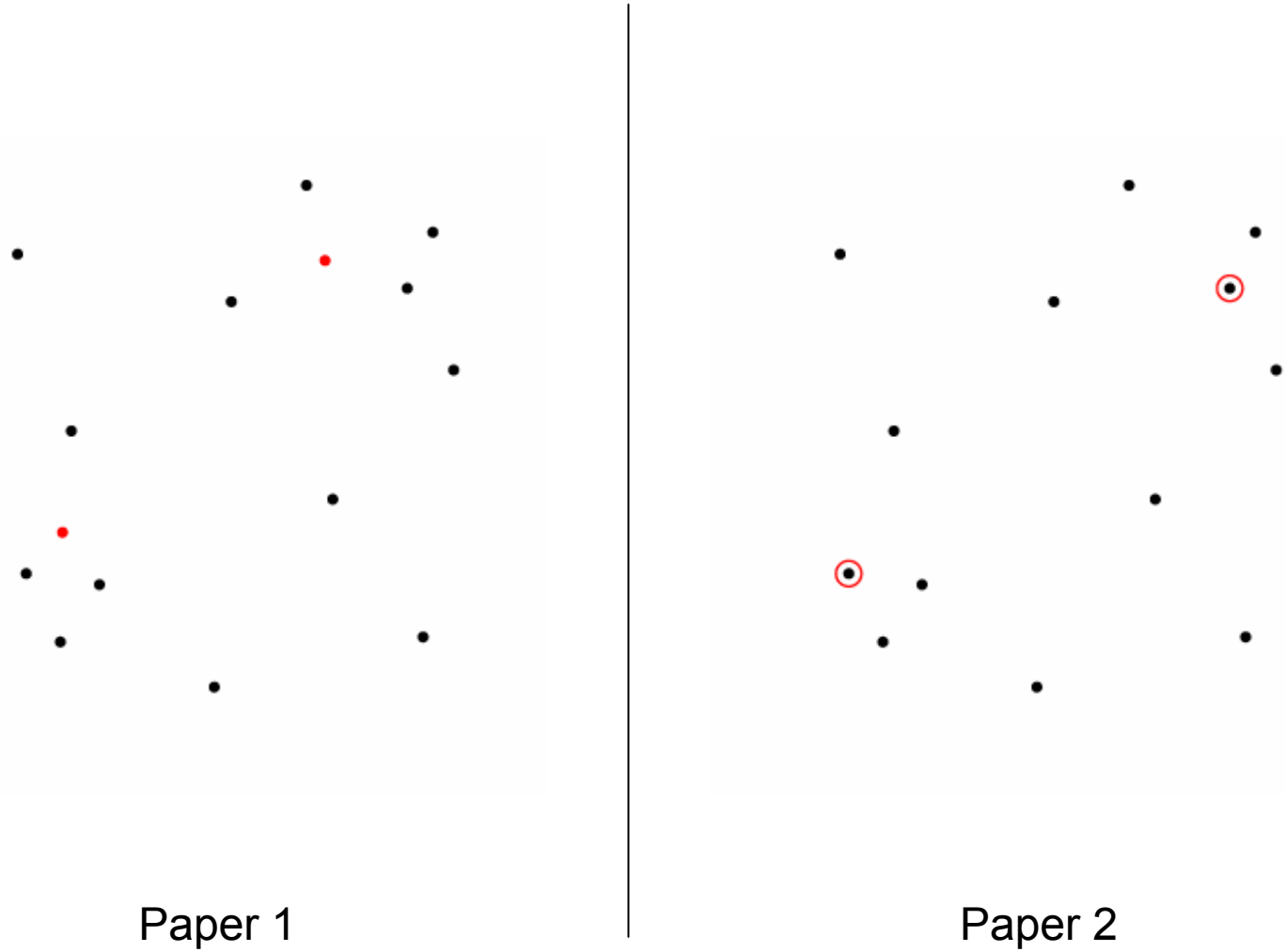
# k-medians Problem



Paper 1

Paper 2

# k-medians Problem



Paper 1

Paper 2

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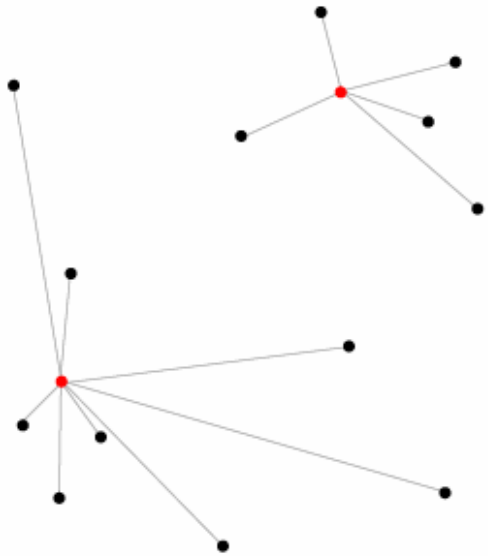
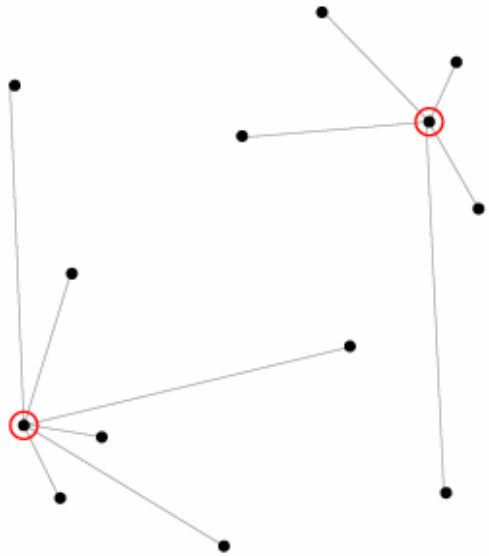


Paper 1



Paper 2

# Facility Location Problem

Given

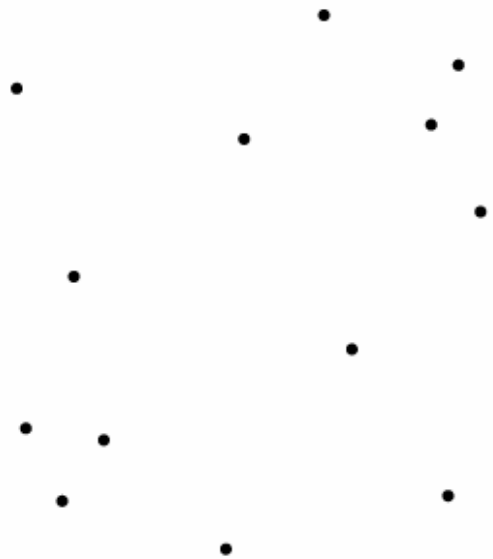
- S={$x_i$} be n points in metric space $R^d$
- Positive cost function c()

Goal

- Find M={$m_i$} $\in$ S which minimizes

$$\sum_{m_j \in M} c(m_j) + \sum_{i=1}^{n} \min_{1 \le j \le k} d(x_i, m_j)$$

# Facility Location Problem



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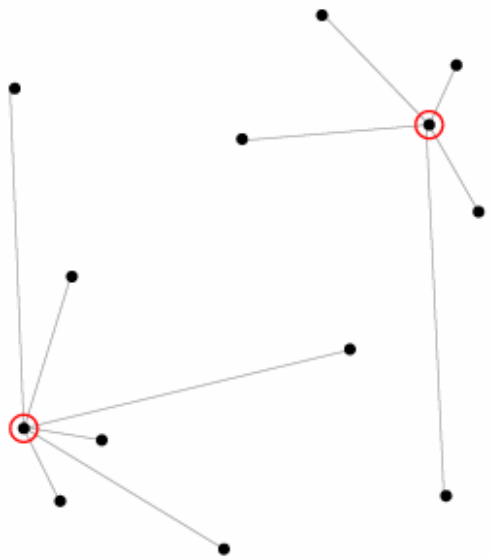

# Previous Results: Facility Location

| | |
|---|---|
| O(logn) approx | Hochbaum '82 |
| 3.16 approx | Shmoys et al. '97 |
| 2.41 approx | Guha and Khuller '98 |
| 1.74 approx | Chudak '98 |

but no $(1+\varepsilon)$ approx before this paper

# Previous Results: k-medians

$(1+\varepsilon)$ approx using $(1+1/\varepsilon)(1+\ln n)k$ medians

Lin and Vitter '92

$2(1+\varepsilon)$ approx using $(1+1/\varepsilon)k$ medians

Lin and Vitter '92

# Results: Paper 1

In 2D, for the *k-median problem*,
given any constant positive $\varepsilon$ with probability
1-o(1) achieve solution at most $(1+\varepsilon)$OPT
in time $O(n^{O(1/\varepsilon)}nk \log n)$

- Proof of existence
- Dynamic programming: bound table size

# Results: Paper 1

In 2D, for the *facility problem*,

given any constant positive $\varepsilon$ with probability 1-o(1) achieve solution at most $(1+\varepsilon)$OPT

in time $O(n^{1+O(1/\varepsilon)}\log n)$

# Results: Paper 2

In 2D, for the *k-median problem*,
given any constant positive $\varepsilon$ with probability
1-o(1) achieve solution at most $(1+\varepsilon)$OPT
in time $O(2^{O(1+\log(1/\varepsilon)/\varepsilon)}nk\log n)$

- Proof of existence
- Dynamic programming: bound table size

# Approximation Schemes for Euclidean k-Medians and Related Problems

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# Integer Point Coordinates

- Given n points, *minmax solution* D is 2OPT for *facility assignment*
- Optimal facility cost in [D/2,Dn]
- Arora-TSP '98: Mapping each point to $jD/n^2$ increases cost by maximum O(D/n) ($<\varepsilon$?)

- Length of BBox(S)
  - $L=O(n^4)$

# Terminology

- Dissection vs quadtree
- $L = O(n^4)$
- Number of nodes $O(L^2)$
- Minimum size of box = 1
- Depth of tree log(L)

- Dissection with (a,b) shift
- Quadtree derived from (a,b)-shifted dissection

# A Result from Arora-TSP-'98

S={$l_i$} is a collection of line segments

t(S,l) number of lines in S crossing l

- $|l_i| \geq 4$
- $\sum_{l_i \in S} |l_i| = cost(S)$

$$\sum_i t(S, l_i) = \theta(cost(S))$$

# More Notations

- BBox at level 0
  its 4 children at level 1, …
- Level of an edge
  level of the corresponding box
- Edge is in i level $\Rightarrow$ in (i+1), (i+2), .., logL
- maximal level

$$\Pr_a[l \text{ is at level } i] = 2^i/L$$

# Charging Scheme

R-charging

For a maximal edge which is crossed g times by S, charge

$$g/R \times \text{length of edge}$$

# Charging Lemma

- Expected total cost for top i-level edges is $O(i\text{cost}(S)/R)$
- Maximal level of grid line l be j
  - Length $L/2^j$
  - Charge $t(S,l)/R\ L/2^j$
  - Probability $2^j/L$

by linearity of expectation,

$$\sum_{j \le i} 2^j/L\ \ t(S,l)/R\ \ L/2^j \le i/R\ \ t(S,l)$$

# Charging Lemma

by linearity of expectation,

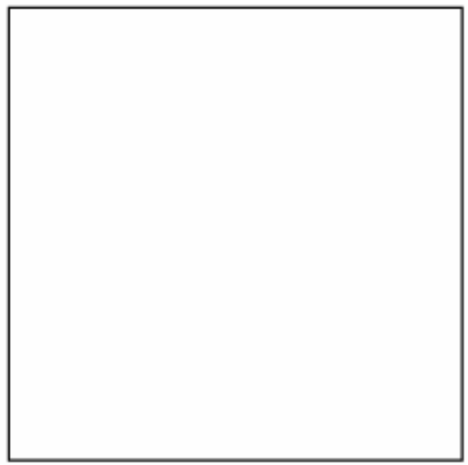
$$\sum_{j \le i} 2^j/L \;\; t(S,l)/R \;\; L/2^j \le i/R \;\; t(S,l)$$

- i ≤ logL
- Choose R = logL/$\varepsilon$

- Cost ≤ $\varepsilon$ t(S,l)
- Total ≤ O($\varepsilon$ cost(S))

# m-portal

- m-regular portals for a shifted dissection
- Total number of points $\Rightarrow$ 4m



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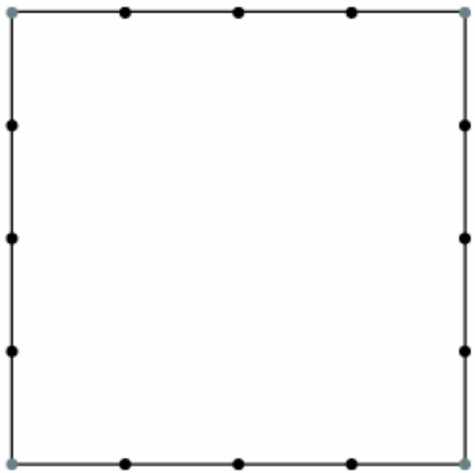
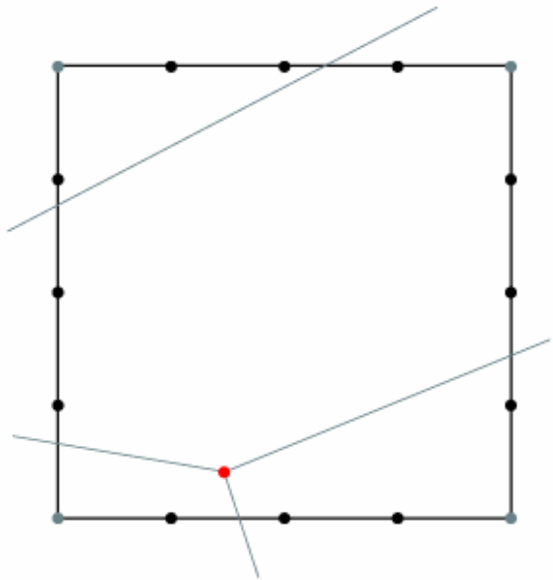
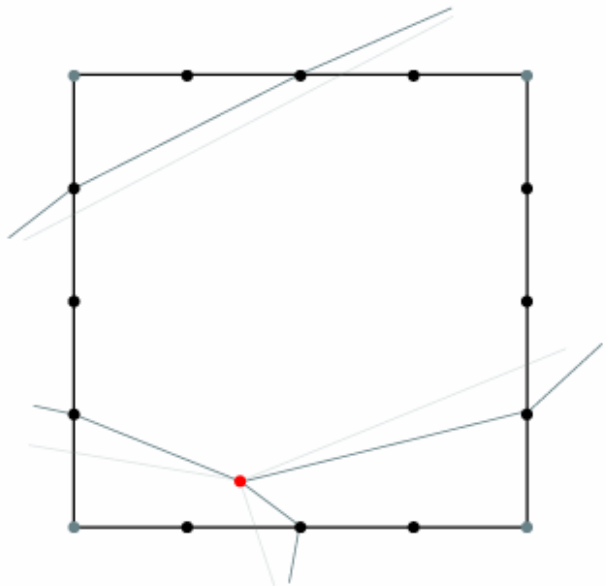


3-portal

# m-light Solution



# m-light Solution



# Existence

- m>1 and (a,b)$\in$U[0,L]
- OPT $\Rightarrow$ deflect to form m-portal
  - for l sized square, cost of deflection O(l/m)
  - m-charging scheme
- w.p. ½ or in expectation or w.h.p. m-light solution exists for (a,b) shifted dissection with cost at most

$$\sum_{l} O(i/R \;\; t(S,l))$$

$$\Rightarrow \;\; O(logL/m \;\; OPT)$$

# Existence

$$O(i/R \;\; t(S,l))$$

$$\Rightarrow \;\; O(logL/m \;\; OPT)$$

Solution is O( (1+logL/m)OPT )

- L=O($n^4$)
- *Choose m* to get O( (1+$\varepsilon$)OPT )

# Dynamic Programming: Sketch

- Finds solution within (1+1/4m) of m-light OPT
- $\Rightarrow$ $O((1+\varepsilon/4\log n)(1+\varepsilon)OPT)$
  $= O((1+\varepsilon)OPT)$

Dynamic Programming

1. Nearest facility within (1+1/4m)
2. Nearest facilities similar for neighbors

# Dynamic Programming: Sketch

Given f and sub-boxes (children boxes $S_i$-s) solve for current level

- Table build for all choices f($\leq$k) and S
- Table size $O(n^c)$
  $\Rightarrow$ worst case time for algorithm

# Summary

In 2D, for the *k-median problem*,
given any constant positive $\varepsilon$ with probability
1-o(1) achieve solution at most $(1+\varepsilon)$OPT
in time $O(n^{O(1/\varepsilon)}nk \log n)$

- Proof of existence
- Dynamic programming: bound table size

# A Nearly Linear-Time App. Scheme for the Euclidean k-median Problem

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# k-medians

– Given
  - S={xi} be n points in metric space Rd
  - Positive integer k

– Goal
  - Find M={mi} $\in$ S in metric space Rd which minimizes

$$\sum_{i=1}^{n} \min_{1 \le j \le k} d(x_i, m_j)$$

# Paper 2

$O(n^{O(1/\varepsilon)}nk \log n)$

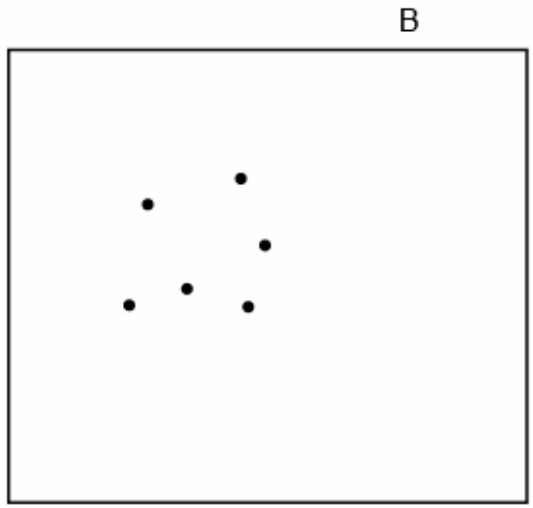
reduced to

$O(O(1/\varepsilon)nk \log n)$

- Reduce the number of portals from $O(\log n/\varepsilon) \Rightarrow O(1+\log(1/\varepsilon)/\varepsilon)$
- Different construction (no (a,b) shifting)

# Adaptive Dissection

Sub-rectangle



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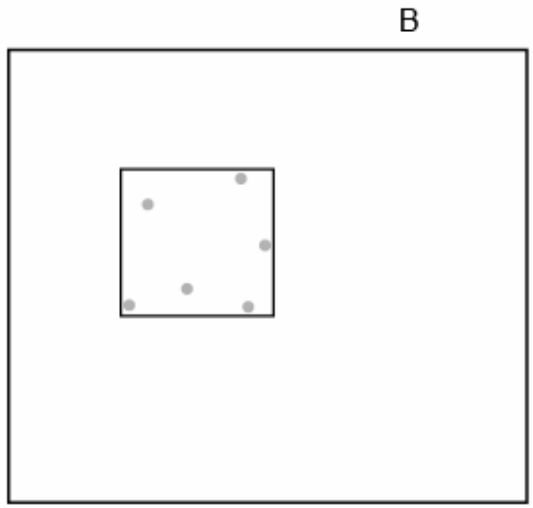

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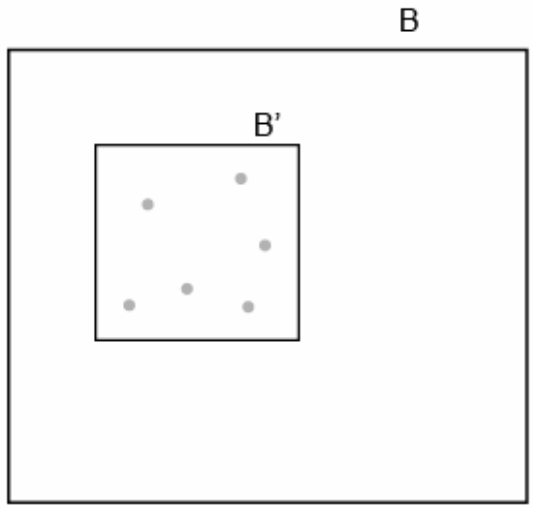
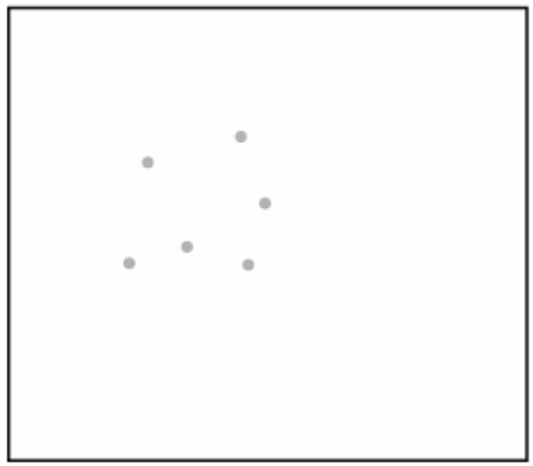



$$B_s = B \cap B'$$

# Adaptive Dissection

Cut-rectangle (randomization)



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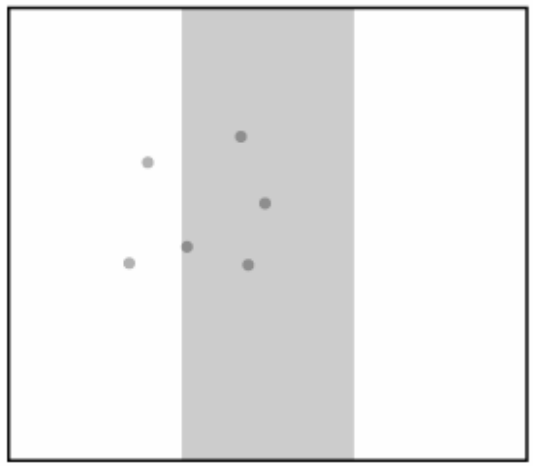
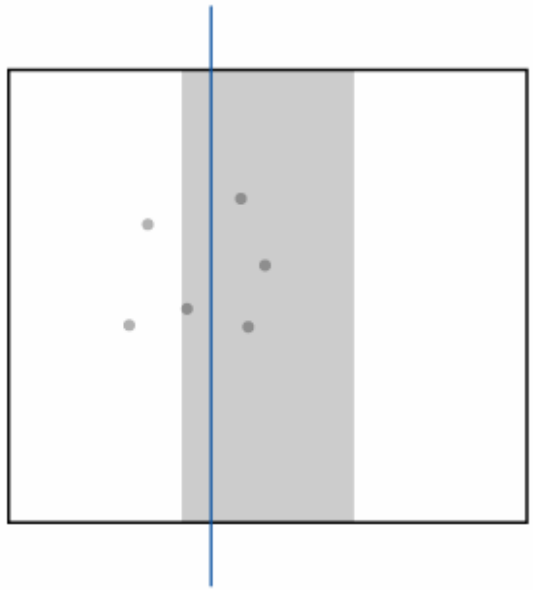


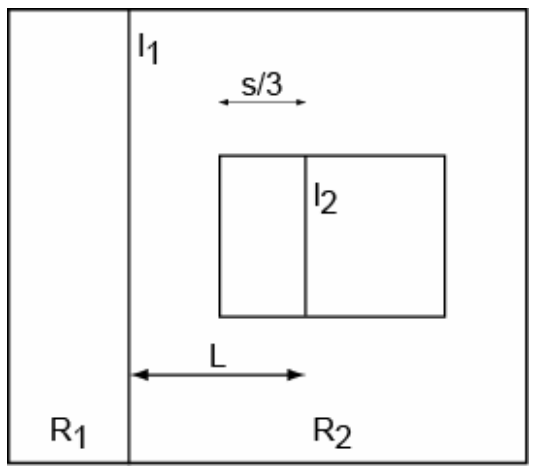
# Adaptive Dissection

Cut-rectangle (randomization)



# Lemma (just one of many)

- Given two parallel cut-lines (due to cut-rectangle) are L apart, the line segments has side length less than 3L.



# Structure Theorem

Error due to assignment using m-portal respecting paths is bounded by

$$O(\max\{1, \log(m)\}/m \, OPT)$$

Choose

$$\log(m)/m \leq c\epsilon$$

# Extensions

- d-dimension $\Rightarrow$ $m^{d-1}$-portal
- Facility location
  - Same structure
- Capacitated k-median
  - Tweak dynamic programming
- Few medians (small k)
  - Guess position of facilities
  - Number of choices $\binom{m^2}{k}$

# k-medians Problem

$$\sum_{i=1}^{n} \min_{1 \le j \le k} d(x_i, m_j)$$

# k-centers Problem

$$\sum_{i=1}^{n} \min_{1 \le j \le k} d^2(x_i, m_j)$$