Approximating the Diameter, Width, Smallest Enclosing Cylinder, and Minimum-Width Annulus

Timothy M. Chan

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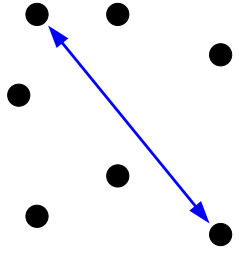
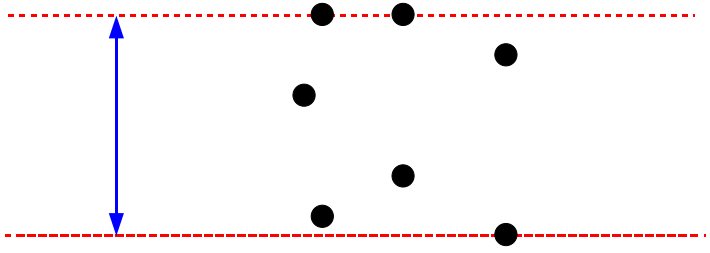
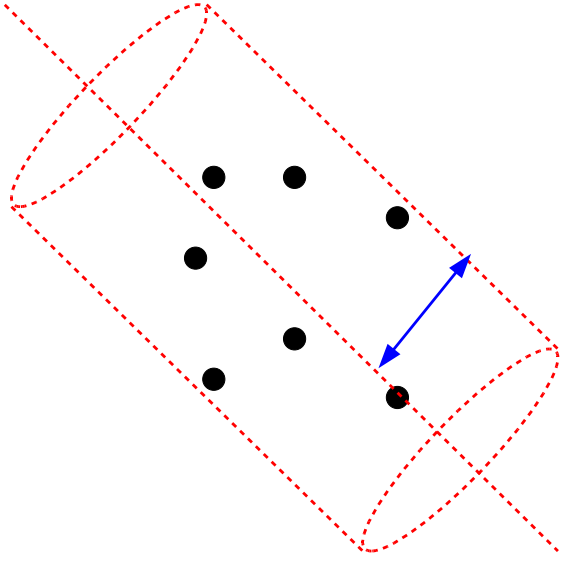
Problems



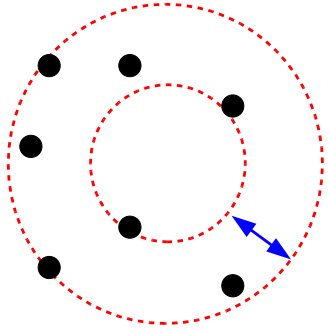
Diameter



Width



Smallest Enclosing Cylinder



Minimum-Width Annulus

Known Exact Algorithms

- **Diameter**
  - $\mathrm{O}(n \log n)$ for $d = 2,\ 3$
  - $\mathrm{O}(n^{2-2/(\lceil d/2 \rceil + 1)} \log^{\mathrm{O}(1)} n)$ for $d = 4$
  - Trivially $\mathrm{O}(n^2)$ for higher d
- **Width**
  - $\mathrm{O}(n \log n)$ for $d = 2$
  - $\mathrm{O}(n^{3/2+\delta})$ expected time for $d = 3$
  - $\mathrm{O}(n^{\lceil d/2 \rceil})$ by halfspace intersection algorithm for higher d

Known Exact Algorithms

- **Smallest Enclosing Cylinder**
  - Identical to width for $d = 2$
  - $\mathrm{O}(n^{3+\delta})$ for $d = 3$
  - $\mathrm{O}(n^{2d-1+\delta})$ for higher d
- **Minimum-Width Annulus**
  - $\mathrm{O}(n^{3/2+\delta})$ randomized for $d = 2$
  - $\mathrm{O}(n^{\lfloor d/2 \rfloor+1})$ for higher d

Key Points in Approximation

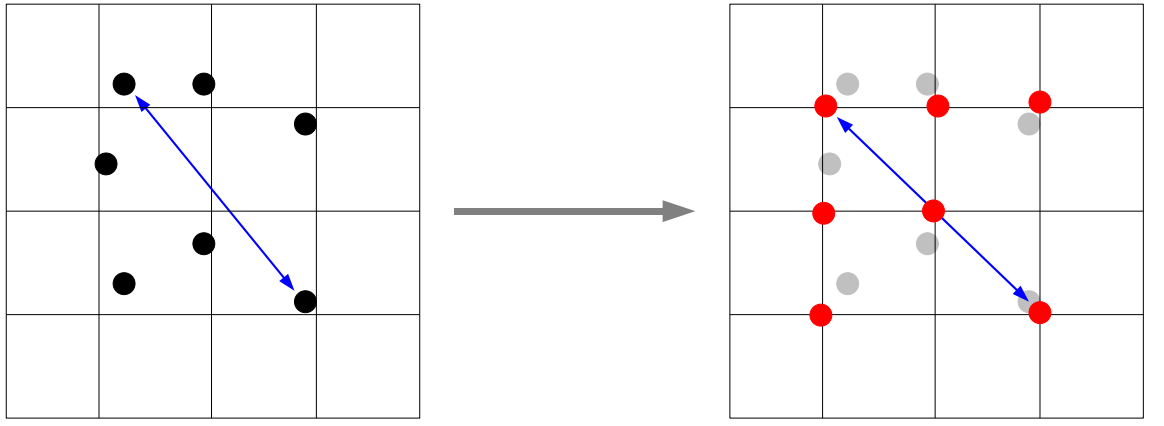
- Quantities must be approximated to within factor of $(1 + \mathrm{O}(\varepsilon))$.
- **Linear Time Approximation Scheme (LTAS)**: $\mathrm{O}^*(E^c n)$, where
  - $E = 1/\varepsilon$
  - O^* notation hides $\log^{\mathrm{O}(1)} E$ factors
- **Strong Linear Time Approximation Scheme**: $\mathrm{O}^*(n + E^c)$
  - Running time is linear in n as long as $E \leq n^{1/c}$
- The paper ignores constant factors depending on d (which are usually exponential), on the grounds that we are chiefly concerned with low-dimensional problems.

Key Points in Approximation

- Approximate solutions use combinations of the following simplifications:
  - Rounding to grid
  - Cones + Ray shooting
- The first problem (**Diameter**) is solved using just the above two techniques.
- The other three problems are approximated by linear/convex programs (LP's/CP's), then the input to these programs is simplified as above.

Rounding to Grid

- Ref. Barequet and Har-Peled paper presented by An last week.
- **Main idea**: Rounding points to their nearest neighbours on a regular lattice of spacing δ alters their linear properties by at most $\mathrm{O}(\delta)$.



- If $\delta = \mathrm{O}(\varepsilon\Delta)$ then Δ' is $(1 + \mathrm{O}(\varepsilon))$-approximation to Δ.

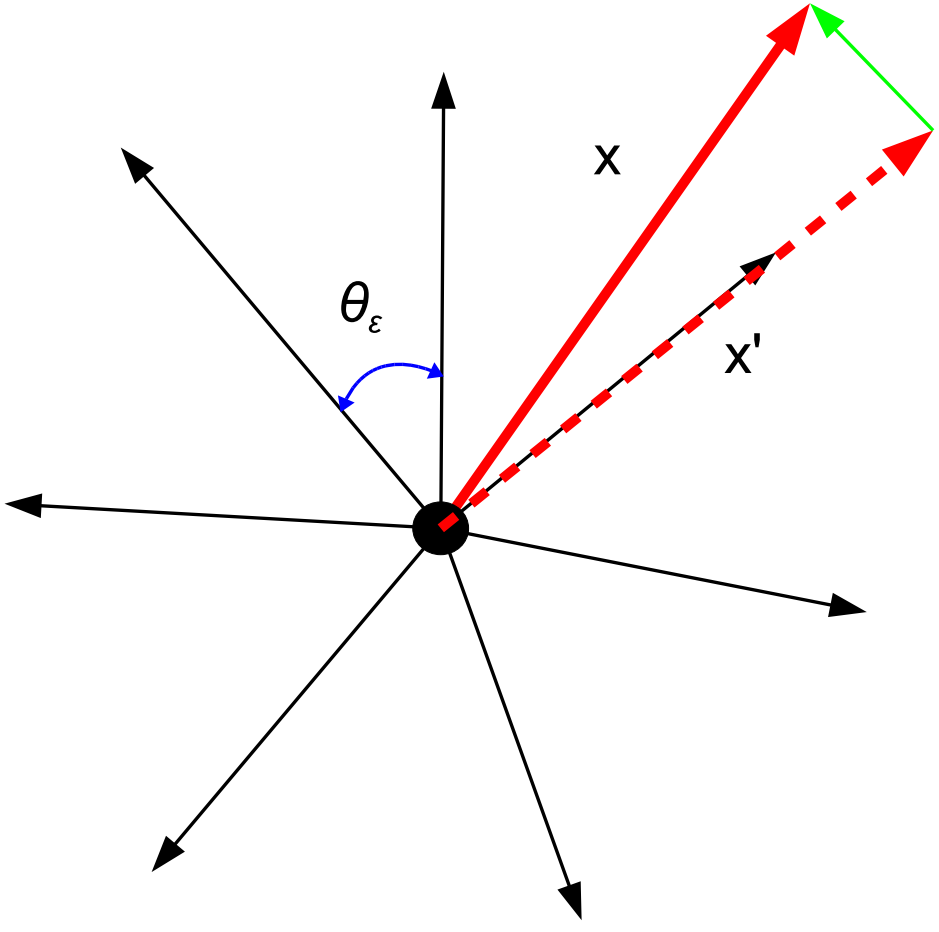
Cones + Ray Shooting

- **Main idea**: Let $\theta_\varepsilon = \arccos(1/(1+\varepsilon)) = \Theta(\sqrt{\varepsilon})$. It is possible to cover space of directions by $\mathrm{O}(1/\theta_\varepsilon^{d-1})$ cones of angle θ_ε.
- i.e. $\exists$ set V_d of $\mathrm{O}(1/\theta_\varepsilon^{d-1})$ unit vectors in $\mathbb{R}^d$ s.t. $\forall x \in \mathbb{R}^d$, $\exists a \in V_d$ s.t. $\angle(a, x) \leq \theta_\varepsilon$, or equivalently:

$$\|x\|/(1+\varepsilon) \;\leq\; \max_{a \in V_d} a \cdot x \;\leq\; \|x\|$$

- To maximize $\|p - q\|$ within an $(1+\varepsilon)$-factor approximation, we can maximize $a \cdot (p - q)$.
- Easily done by maximizing $a \cdot p$ and minimizing $a \cdot q$, in $\mathrm{O}(|V_d|n) = \mathrm{O}(E^{(d-1)/2}n)$ time each (brute force).
  - Known as "Ray Shooting" (RS) in a convex polytope (in the dual).

Cones + Ray Shooting



$$\|x\| \leq (1+\varepsilon)\|x'\|$$

Diameter

- **Previous best**: $\mathrm{O}(n + E^{2(d-1)d/(d+1)})$
- A constant-factor approximation $\Delta_0 = \Theta(\Delta)$ is easy to calculate in $\mathrm{O}(n)$ time (farthest-point distance of any point). Will be used as "seed" for grid separation.
- **Algorithm 1**
  - Round to grid of separation $\varepsilon\Delta_0$ ($\mathrm{O}(E^d)$ points).
  - Simplify further by retaining only top and bottom points of each vertical line in grid (diametrically opposite points must belong to this set).
  - Now compute diameter. Gives $(1 + \mathrm{O}(\varepsilon))$-approximation in brute force time $\mathrm{O}(n + E^{2(d-1)})$.
- **Algorithm 2**: Apply cones + RS in time $\mathrm{O}(E^{(d-1)/2}n)$.

Diameter

- **Algorithm 3** = Algorithm 1 + Algorithm 2
  - Instead of using brute force on rounded points, use Algorithm 2.
  - **Rationale**: a $(1+\varepsilon)$-approximation of a $(1+\varepsilon)$-approximation is a $(1+\mathrm{O}(\varepsilon))$-approximation.
  - More efficient strong LTAS: $\mathrm{O}(n + E^{3(d-1)/2})$.
- **Algorithm 4**: Out, damned spot!
- **Algorithm 5** (Grid + Cones + Dimension Reduction)
  - For a set of points $P \in \mathbb{R}^d$, $P' \subset P$ $(1+\varepsilon)$-simplifies P if for any $q \in \mathbb{R}^d$,

$$\max_{p' \in P'} \|p' - q\| \;\geq\; \max_{p \in P} \|p - q\| / (1+\varepsilon)$$

Diameter

- **Algorithm 5** (Grid + Cones + Dimension Reduction) (contd)
  - Round points to grid, obtaining P.
  - **Main idea**: Recursing on dimension, find a set P' of $\mathrm{O}(E^{(d-1)/2})$ points that $(1+\mathrm{O}(\varepsilon))$-simplify P.
    - Decompose P into $\mathrm{O}(E)$ subsets P_i, each lying on a grid hyperplane.
    - Recursively, find $P'_i \subset P_i$ of size $\mathrm{O}(E^{(d-2)/2})$ that $(1+\mathrm{O}(\varepsilon))$-simplifies P_i.
    - **Lemma**: $\cup_i P'_i$ $(1+\mathrm{O}(\varepsilon))$-simplifies P.

Diameter

- **Algorithm 5** (Grid + Cones + Dimension Reduction) (contd)
  - **Main idea** (contd)
    - $\cup_i P_i'$ has size $\mathrm{O}(E^{d/2})$ — still too large. Reduce by cones:
    - For each $a \in V_d$ find point $p_a \in \cup_i P_i'$ maximizing $a \cdot p$.
    - $P' = \{p_a \mid a \in V_d\}$ has size $\mathrm{O}(E^{(d-1)/2})$.
    - Computable by RS in time $\mathrm{O}(E^{(d-1)/2}|\cup_i P_i'|) = \mathrm{O}(E^{d-1/2})$.
    - **Lemma**: P' $(1+\varepsilon)$-simplifies $\cup_i P_i'$.
    - And again, $(1+\mathrm{O}(\varepsilon)) \times (1+\mathrm{O}(\varepsilon)) = (1+\mathrm{O}(\varepsilon))$ ...

Diameter

- **Algorithm 5** (Grid + Cones + Dimension Reduction) (contd)
  - Finally, apply Algorithm 2 (cones) to $(1+\mathrm{O}(\varepsilon))$-approximate the diameter of P', which in turn $(1+\mathrm{O}(\varepsilon))$-approximates the diameter of P ("easily seen").
    Time taken = $\mathrm{O}(E^{(d-1)/2}|P'|) = \mathrm{O}(E^{d-1})$.
  - **Total time**: solve

$$T_d(n) = \sum_{i=1}^{\mathrm{O}(E)} T_{d-1}(n_i) + \mathrm{O}(E^{d-1/2})$$

where $\sum_i n_i = n$. Answer: $\mathrm{O}(n + E^{d-1/2})$.

The LP/CP problems

- Variables $x, u \in \mathbb{R}^d, \quad y, z \in \mathbb{R}$.
- **Width**

$$\begin{aligned} \text{minimize} \quad & (z-y)/\|x\| \\ \text{subject to} \quad & y \le x \cdot p \le z \qquad (\forall p \in P) \end{aligned}$$

- **Smallest Enclosing Cylinder**

$$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & \left\| x + \left(\frac{u \cdot (p-x)}{u \cdot u} \right) u - p \right\| \le z \qquad (\forall p \in P) \end{aligned}$$

- **Minimum-Width Annulus**

$$\begin{aligned} \text{minimize} \quad & (z-y) \\ \text{subject to} \quad & y \le \|x-p\| \le z \qquad (\forall p \in P) \end{aligned}$$

The LP/CP problems

- **Basic approach**
  - Construct a grid of suitable separation (ε times constant-factor approximation) to reduce the number of points to $\mathrm{O}(E^{d-1})$.
  - Use cones + change of variables to approximate the problem with an LP/CP instance.
  - Solve LP/CP in linear time (known algorithms).

Width

- Simplest example of three steps of previous slide

1. **Rounding to grid**
   - Barequet and Har-Peled proved existence of box B such that
     - cB can be translated to fit inside **conv**(P), for constant c
     - B can be computed in linear time
   - Construct a grid where each cell is an instance of εcB and round P to P'.
   - If P is contained in slab S^* of width w^*, then P' is contained in slab of width $(1+2\varepsilon)w^*$ since P' fits in $P \oplus \varepsilon cB$, which fits in $P \oplus \varepsilon$**conv**$(P) \subset S^* \oplus \varepsilon S^*$.

Width

2. **Conversion to approximate LP**

- Given V_d (cone directions), transform to

$$\begin{aligned} &\text{minimize} && (z-y)/|a \cdot x| \\ &\text{subject to} && y \le x \cdot p' \le z \qquad (\forall p' \in P') \end{aligned}$$

- At the minimum, a must be θ_ε-close to x. Then we have $\|x\|/(1+\varepsilon) \le |a \cdot x|$.
- So a solution of the new system is a $(1+\varepsilon)$-approximation of the minimum width of P'.
- Set $X = x/(z-y)$ and $Y = y/(z-y)$ to get LP:

$$\begin{aligned} &\text{maximize} && |a \cdot X| \\ &\text{subject to} && Y \le X \cdot p' \le Y + 1 \qquad (\forall p' \in P') \end{aligned}$$

Width

3. **Solving LP**
   - Use a known linear-time LP algorithm to solve separately for each $a \in V_d$.
     - Total time = $\mathrm{O}(|V_d||P'|)$ = $\mathrm{O}(E^{3(d-1)/2})$
   - Take the maximum solution.

- Algorithm runs in $\mathrm{O}(n + E^{3(d-1)/2})$ time.

Smallest Enclosing Cylinder

1. **Rounding to grid**
   - Let z^* be optimal radius.
   - Compute constant-factor approximation $z_0 \leq cz^*$:
     - use Step 2 (below) with $\varepsilon = c - 1$, or
     - take enclosing cylinder with center line through approximate diametrical pair.
   - Build uniform grid of separation εz_0 and round points to P', retaining only top and bottom points on each vertical line as usual.
   - Smallest enclosing cylinder of P' (radius z') is $(1 + \mathrm{O}(\varepsilon))$-approximation to smallest enclosing cylinder of P.

Smallest Enclosing Cylinder

2. **Conversion to approximate CP**
   - Given V_d (cone directions), transform to

$$\begin{aligned} &\text{minimize} && z \\ &\text{subject to} && \left\| x + \left(\frac{a \cdot (p' - x)}{a \cdot u} \right) u - p' \right\| \leq z \qquad (\forall p' \in P') \end{aligned}$$

   - **Lemma**: If (x, u, z, a) is the optimal solution of this system, then (x, u, z) is feasible for the original program and $z \leq (1 + \varepsilon) z'$.
   - Set $X = x - \left(\frac{a \cdot x}{a \cdot u} \right) u$, $Y = \frac{u}{a \cdot u}$, $Z = z^2$ to get CP:

$$\begin{aligned} &\text{minimize} && Z \\ &\text{subject to} && \| X + (a \cdot p')Y - p \|^2 \leq Z \qquad (\forall p' \in P') \end{aligned}$$

Smallest Enclosing Cylinder

3. **Solving CP**
   - Use a known linear-time CP algorithm to solve separately for each $a \in V_d$.
     - Total time = $\mathrm{O}(|V_d||P'|)$ = $\mathrm{O}(E^{3(d-1)/2})$
   - Take the minimum solution.

- Algorithm runs in $\mathrm{O}(n + E^{3(d-1)/2})$ time.

Minimum-Width Annulus

- Start with a constant-factor approximation:
- Assume one p is the origin and consider the problem:

$$\begin{array}{ll} \text{minimize} & (z^2 - y^2)/|a \cdot x| \\ \text{subject to} & y \leq \|x - p\| \leq z \qquad (\forall p \in P) \end{array}$$

- **Lemma**: This system reduces to an LP by change of variables. The solution (x, y, z, a) of this LP gives a constant-factor approximation $w_0 = z - y$ to the true width $w^* = z^* - y^*$ of the thinnest annulus.
- **Narrow case**: $z^* \leq (1 + \varepsilon)y^*$. Add a new constraint (linearizes with above):

$$w_0 \leq \varepsilon(1 + \varepsilon)a \cdot x$$

Minimum-Width Annulus

- **Lemma**: If (x, y, z, a) is the optimal solution of the new system, then $z - y \leq (1 + \mathrm{O}(\varepsilon))w^*$.
- $\Rightarrow$ Narrow case solved in $\mathrm{O}(E^{(d-1)/2}n)$ time.
- **Wide case**: $z^* > (1 + \varepsilon)y^*$
  - Round to grid of separation εw_0. *Only in this case* can the number of rounded points be bounded, by $\mathrm{O}(E^{2d-1})$.
  - Further restrict possible centre points to grid ($\mathrm{O}(E^{2d})$ candidates) and calculate nearest/farthest point for each candidate by brute force.
- $\Rightarrow$ Wide case solved in $\mathrm{O}(n + E^{4d-1})$ time.
- Complete LTAS: $\mathrm{O}(E^{(d-1)/2}n + E^{4d-1})$

Thank You

Exit, pursued by a bear