Homework 3

Due: Tuesday, December 1, 2009.

Problem 1

Filtration to PL function. (20 points.) Given a simplicial complex K and a monotonic function $\hat{f} : K \to \mathbb{R}$ (i.e. a function that assigns to every simplex a value not exceeding that of any of its cofaces), assume that $\hat{f}$ assigns unique values to all the simplices. Give a piecewise linear continuous function $f : |K| \to \mathbb{R}$ such that the sublevel sets of f are homotopy equivalent to the sublevel sets of $\hat{f}$, $f^{-1}(-\infty, a] \simeq \hat{f}^{-1}(-\infty, a]$, for all a.

Problem 2

Examples of switches. (20 points.) Recall the Figure 1 from class where transposing different simplices resulted in switches in pairing. Give analogous examples for the three types of switches, but in one dimension up compared to the Figure.

Problem 3

Homology inference from the Vietoris-Rips complex. (30 points.) Suppose there is a compact set $\mathbb{X}$ with a sufficiently large homological feature size $\mathrm{hfs}(\mathbb{X})$, and we are given a point set P whose Hausdorff distance from $\mathbb{X}$ is $d_H(\mathbb{X}, P) = \varepsilon$. Consider the filtration of the Vietoris-Rips complex $R_r(P) \subseteq R_{r'}(P)$ for $r < r'$. Use the Stability Theorem for Filtrations to shade the regions of the persistence diagrams that can be occupied by points. Indicate which region contains the number of points equal to the Betti numbers of the space $\mathbb{X}$.

(*Hint: Recall that if we denote the Čech complex for radius r as $\check{C}_r(P)$, then we have the following inclusions $\check{C}_r(P) \subseteq R_{2r}(P) \subseteq \check{C}_{2r}(P)$. Furthermore, observe that* $\log$ *is a monotone function and consider filtrations $\check{C}_{\log r}(P)$ and $R_{\log r}(P)$.*)

Extra credit

ε-simplification. (30 points.) Call a Morse function $g : \mathbb{M} \to \mathbb{R}$ an ε-simplification of a Morse function $f : \mathbb{M} \to \mathbb{R}$ if $\|g - f\|_\infty \leq \varepsilon$ and the persistence diagrams of g, $\mathrm{Dgm}(g)$, consist only of those points in the diagrams of f, $\mathrm{Dgm}(f)$, whose persistence is greater than ε.

Prove that not every Morse function $f : \mathbb{M} \to \mathbb{R}$ has an ε-simplification for all ε.

(*Hint: Every Morse function on a 2-manifold has an ε-simplification for all ε.*)

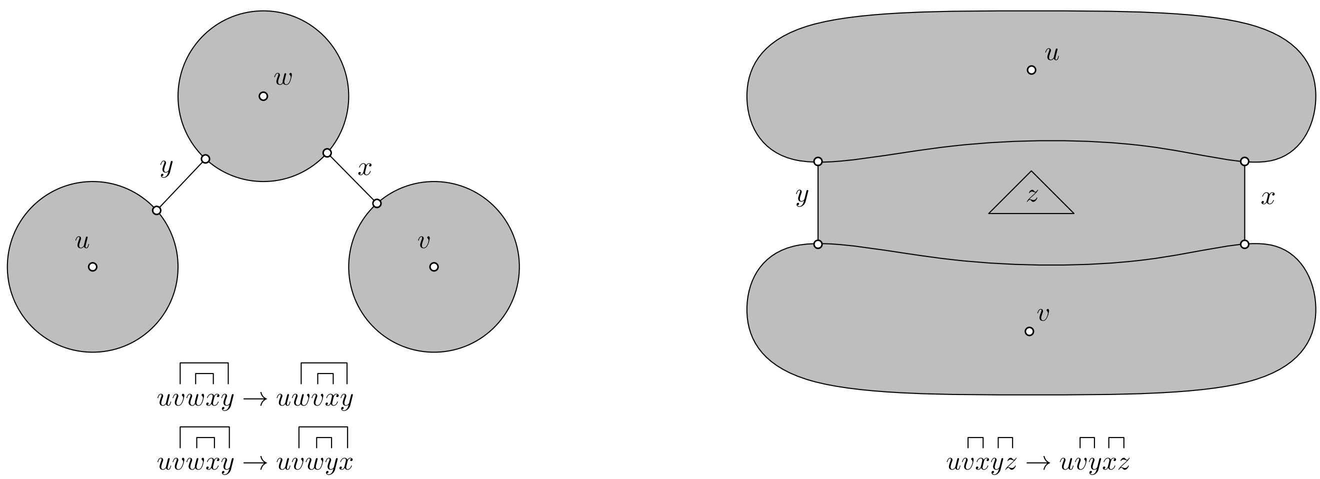

Figure 1: Examples of switches.