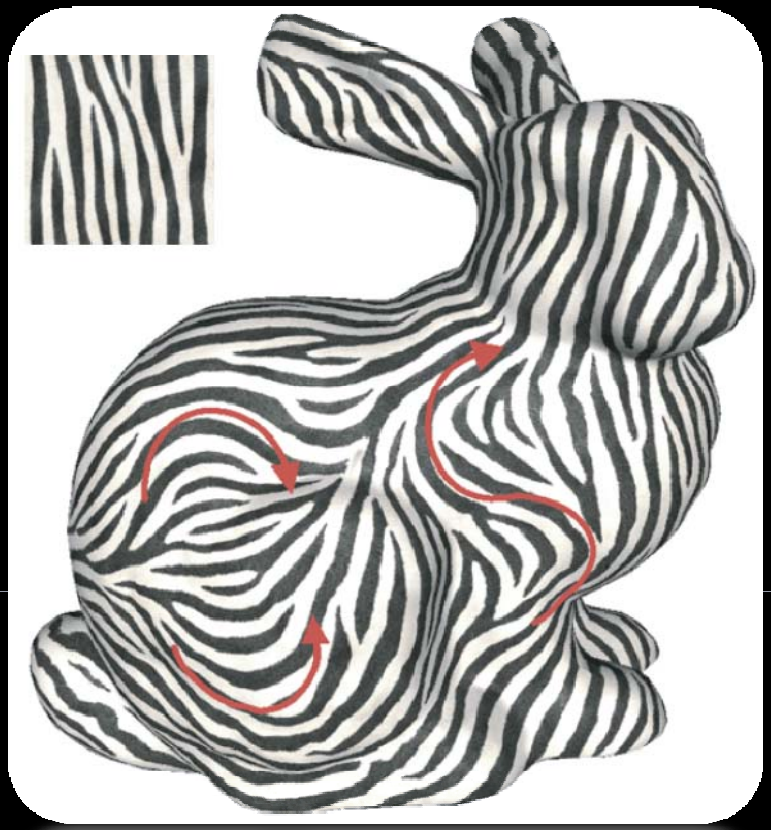


Fisher et al. "Design of Tangent Vector Fields" (SIGGRAPH 2007)

Discrete Exterior Calculus

Justin Solomon
Geometric Computing Group
Stanford University

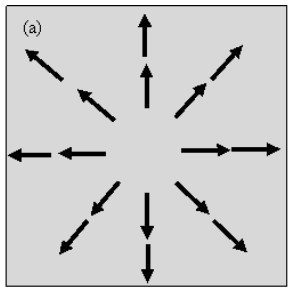
Gradient Operator

$$\operatorname{grad} f \equiv \nabla f \equiv \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

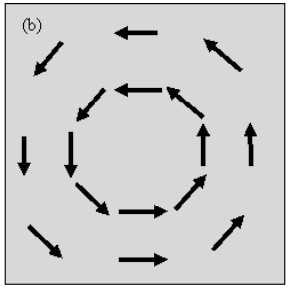
$$\nabla \equiv \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$$

Vector Calculus in R^3

$$\text{div } \vec{v} \equiv \nabla \cdot \vec{v} \equiv \sum_i \frac{\partial v_i}{\partial x_i}$$



$$\text{curl } \vec{v} \equiv \nabla \times \vec{v} \equiv \cdots$$



$$\Delta f \equiv \nabla \cdot \nabla f \equiv \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

Famous Theorems (in R^2)

$$\int_\Omega \operatorname{div}\ \vec{v}\ dA = \int_{\partial\Omega} \vec{v} \cdot \vec{n}\ dl$$

"Divergence Theorem"

$$\int_\Omega \operatorname{curl}\ \vec{v}\ dA = \int_{\partial\Omega} \vec{v} \cdot \vec{t}\ dl$$

"Green's Theorem"

Famous Theorems (in R^2)

$$\int_\Omega \operatorname{div} \vec{v}\, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{n}\, dl$$

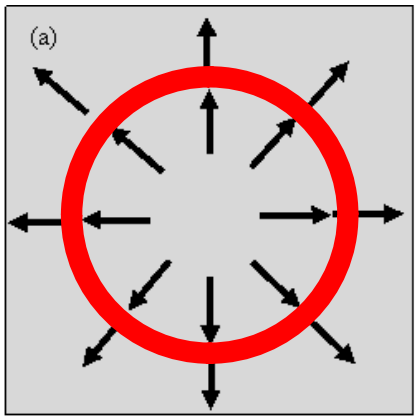
"Divergence Theorem"

$$\int_\Omega \operatorname{curl} \vec{v}\, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{t}\, dl$$

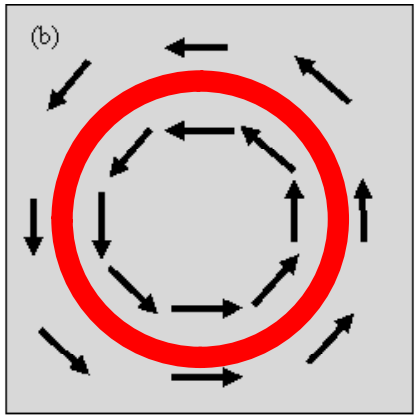
Scalars in R^2

"Green's Theorem"

Famous Theorems (in R^2)



"Divergence Theorem"



"Green's Theorem"

Some Questions

- Why are these theorems similar?
- How do we generalize?
- How do these work on surfaces?

Some Questions

- Why are these theorems similar?
- How do we generalize?
- How do these work on surfaces?

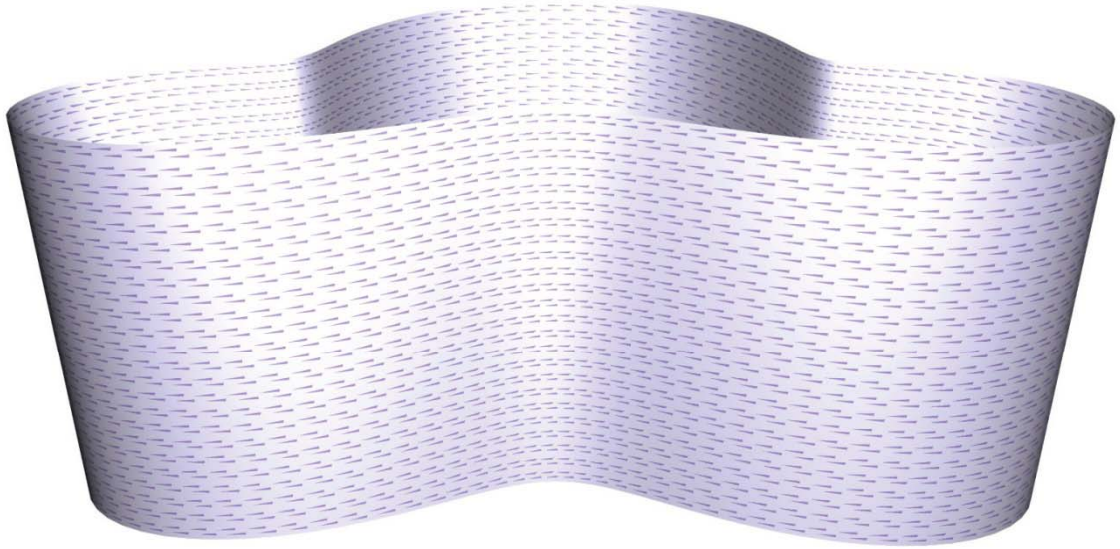
What's up with the cross product?

<math>

Introducing… Exterior Calculus

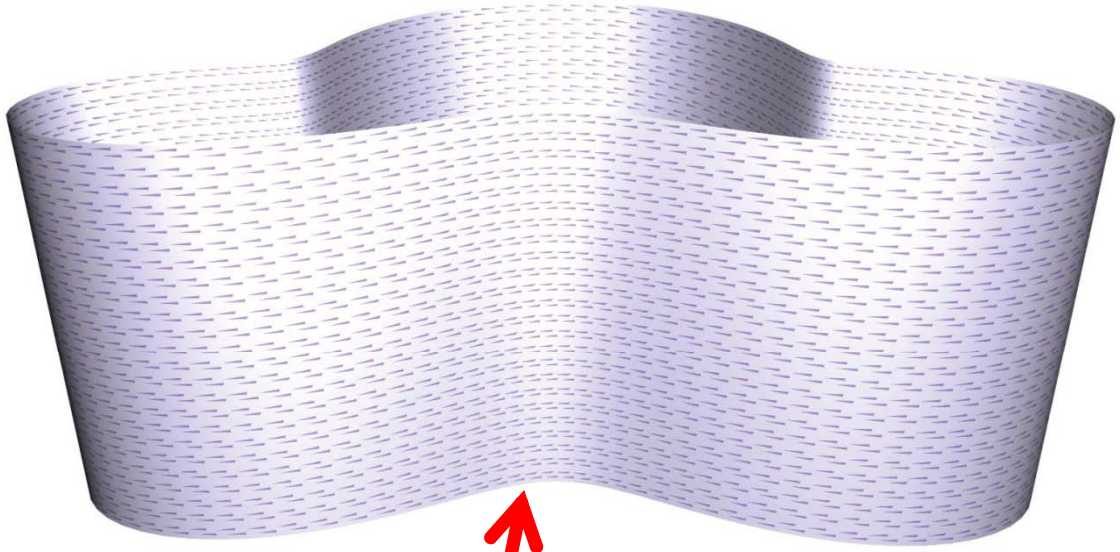
Extension of vector calculus to surfaces (and manifolds).

Basic Assumption



Everything must be **intrinsic**!

Basic Assumption



Vector fields are **tangent**!

Everything must be **intrinsic**!

Differential Forms

For each point ***p*** on a surface:

Differential Forms

For each point ***p*** on a surface:

k-linear

Alternating

Differential Forms

For each point ***p*** on a surface:

k-linear

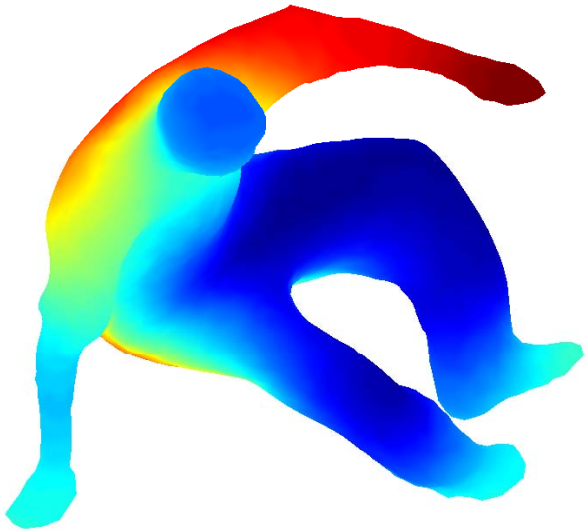
Alternating

[Challenge: In *n* dimensions, *p*-forms are zero for *p* > *n*.]

Differential Forms

For each point ***p*** on a surface:

k vectors in the tangent space at ***p*** → ***Differential k-form*** → $\mathbb{R}$



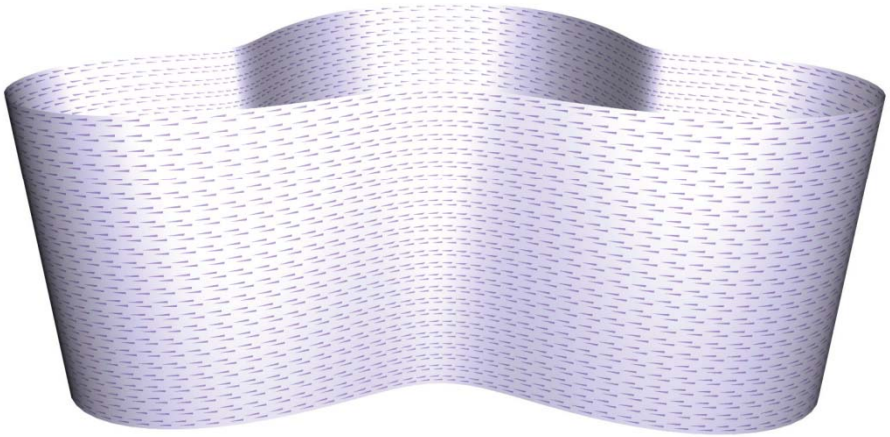
http://perception.inrialpes.fr/Publications/2011/ZBH11a/scale-9.png

$$f : \Sigma \to \mathbb{R}$$

↓

0-form

Expressing Vector Fields



Vector field

$$\vec{v} : \Sigma \to T\Sigma \subset \mathbb{R}^3$$

1-form ω:

$$\omega(\vec{x}) = \vec{v} \cdot \vec{x}$$

Differentiating *k*-Forms

$$d\omega$$

$$d\omega(\vec{v}, \vec{v}_1, \ldots, \vec{v}_k)$$

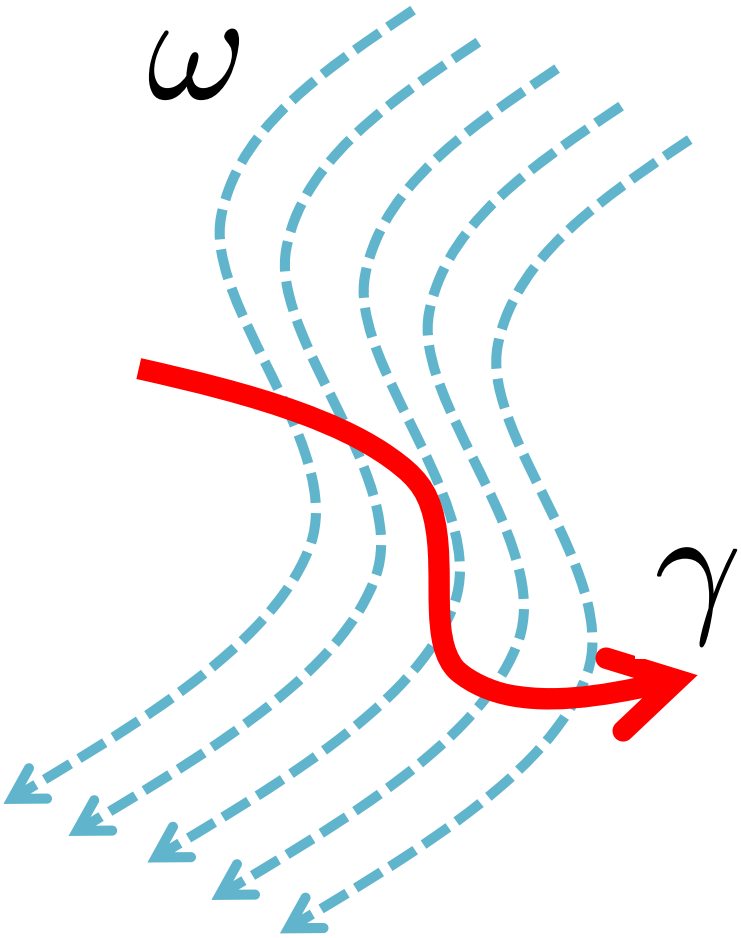
gives derivatives of

$$\omega(\vec{v}_1, \ldots, \vec{v}_k)$$

in the $\vec{v}$ direction

Yields (k+1)-forms!

Integrating k-Forms



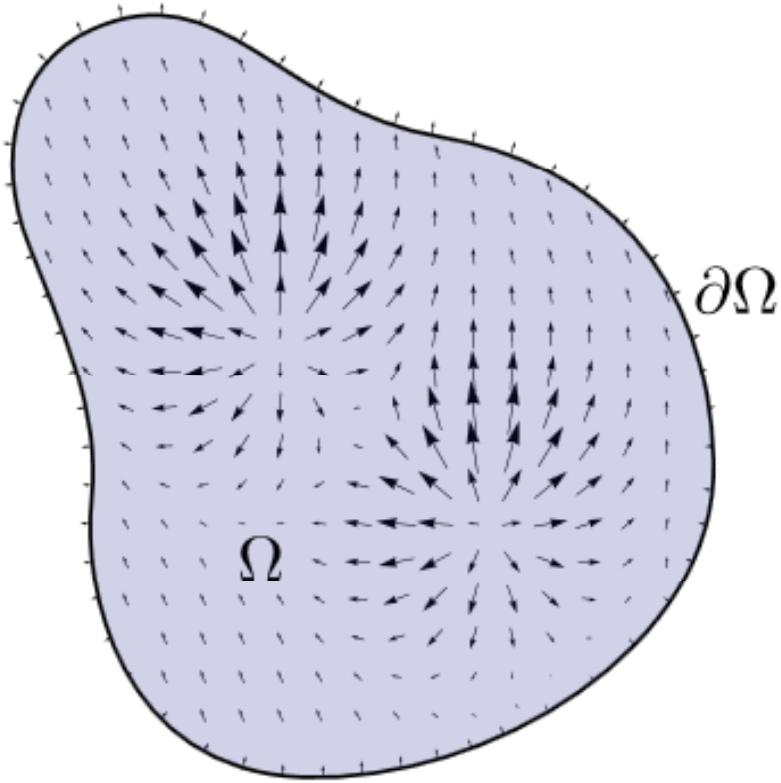
$$\int_\gamma \omega$$

measures amount of ω parallel to γ

Integrate on k-dimensional objects

Stokes' Theorem

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$



http://brickisland.net/cs177/wp-content/uploads/2011/11/ddg_divergence_theorem.svg

Zoo of Operators

$\omega^\sharp$ 1-form to vector

$\vec{v}^\flat$ Vector to 1-form

$\omega_1 \wedge \omega_2$ Product of forms

$\star\omega$ Dual

Zoo of Operators

$\omega^\sharp$ 1-form to vector

$\vec{v}^\flat$ Vector to 1-form

$\omega_1 \wedge \omega_2$ Product of forms

***k*,*p*-forms → (*k*+*p*)-form (cross product!)**

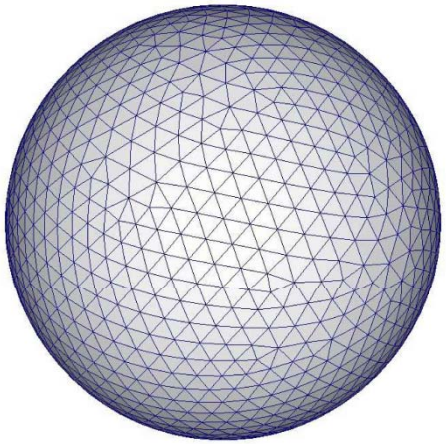
$\star\omega$ Dual

***k*-forms → (*n*-*k*)-form (plane to its normal)**

</math>

Discrete Differential Geometry

Discrete rather than **discretized** notion of differential geometry.



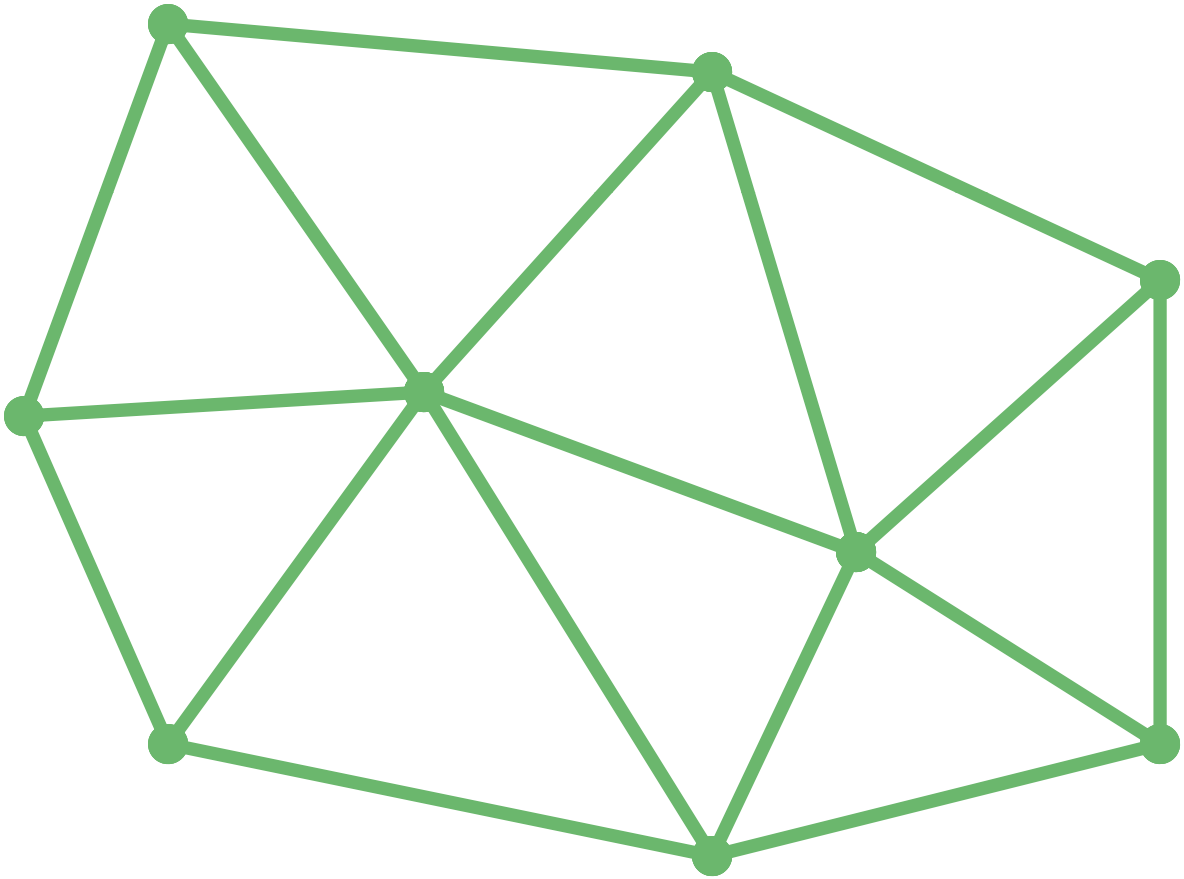
Discrete Exterior Calculus (DEC)

Discrete version of exterior calculus.

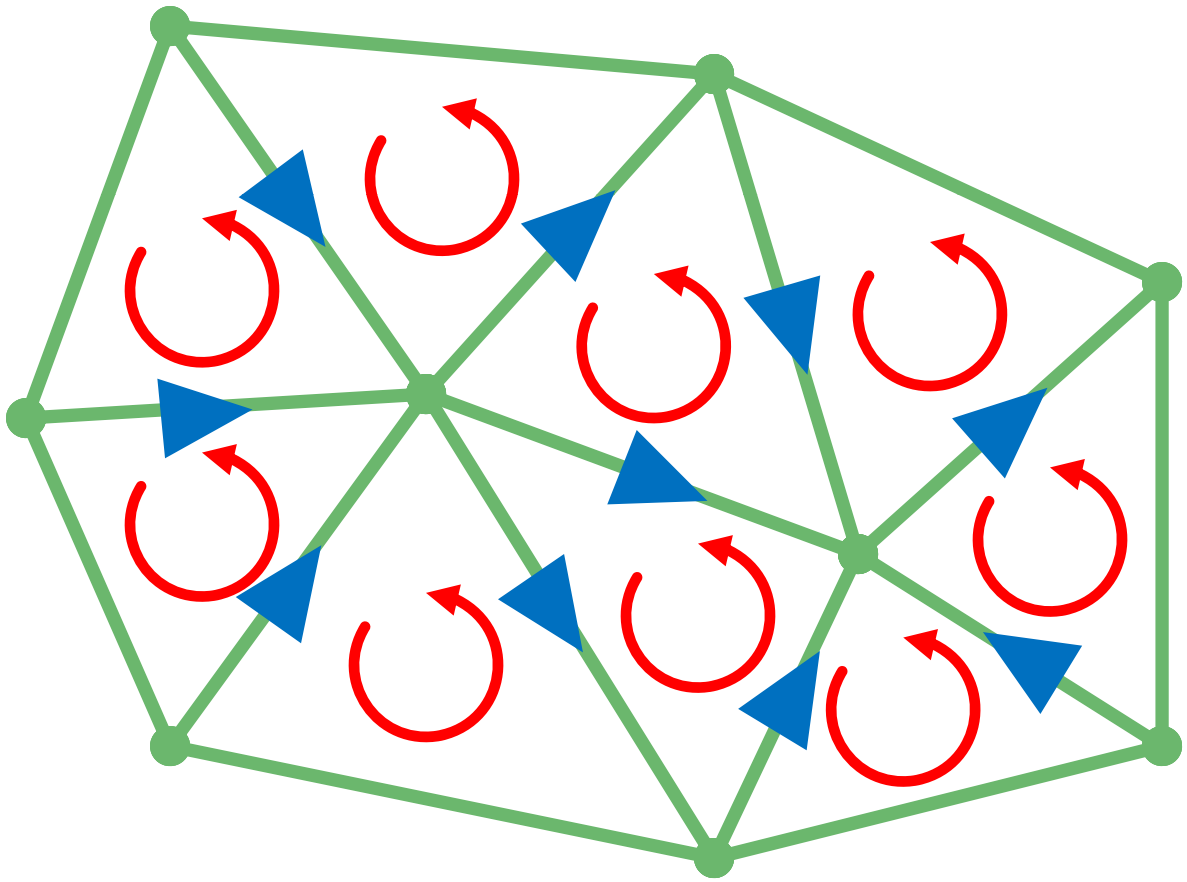
$$\omega^\sharp \quad \vec{v}^\flat \quad \omega_1 \wedge \omega_2 \quad \star\omega \quad d\omega$$

...

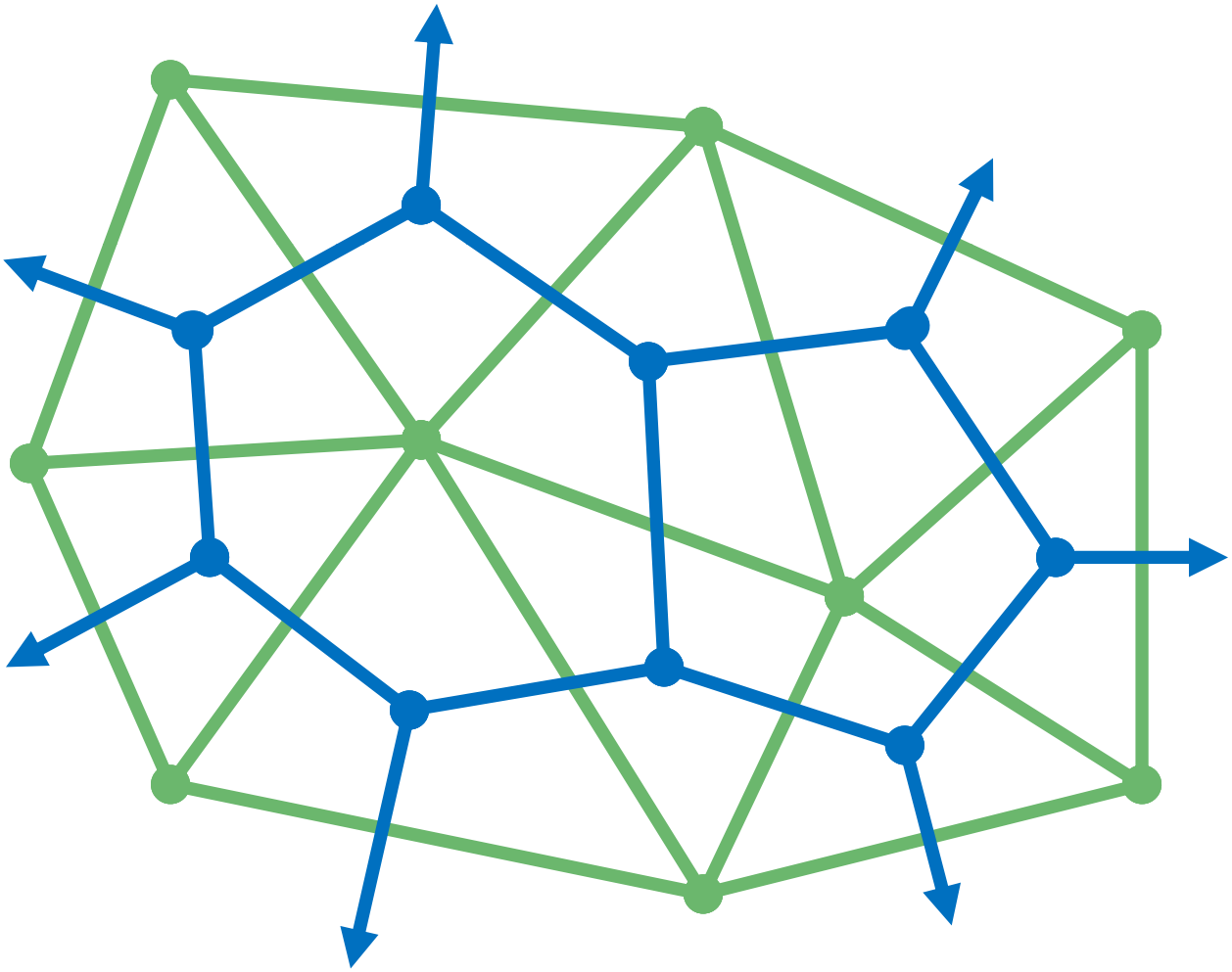
Simplicial Complex



Oriented Simplicial Complex



Dual Complex

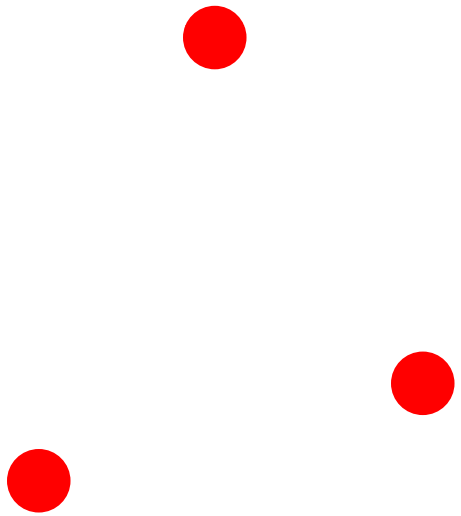


The Trick

Store *integrated* quantities!

The Trick

Discrete 0-form

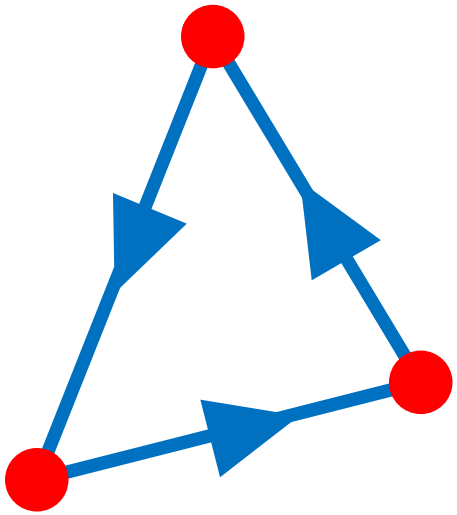


$$\int_v \omega = f(v) \to \mathbb{R}^{|V|}$$

Store *integrated* quantities!

The Trick

Discrete 1-form

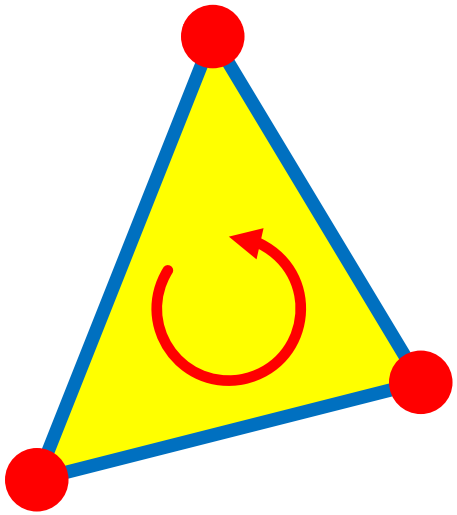


$$\int_e \omega \to \mathbb{R}^{|E|}$$

Store *integrated* quantities!

The Trick

Discrete 2-form



$$\int_t \omega \to \mathbb{R}^{|F|}$$

Store *integrated* quantities!

Exterior Derivative

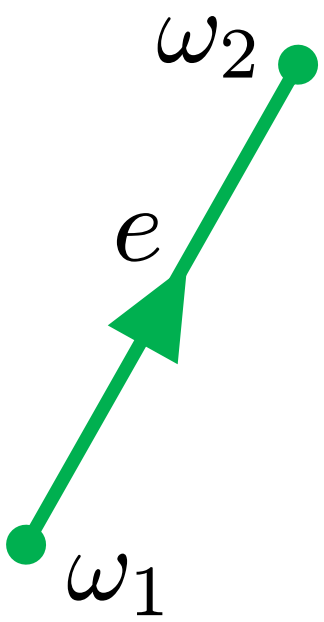
$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

Stokes' Theorem

Exterior Derivative

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

Stokes' Theorem

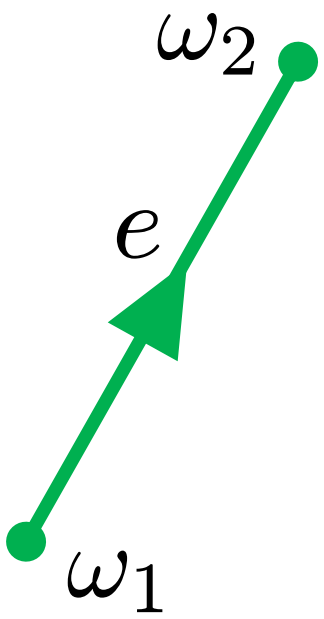


$$\int_{e} d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1$$

Exterior Derivative

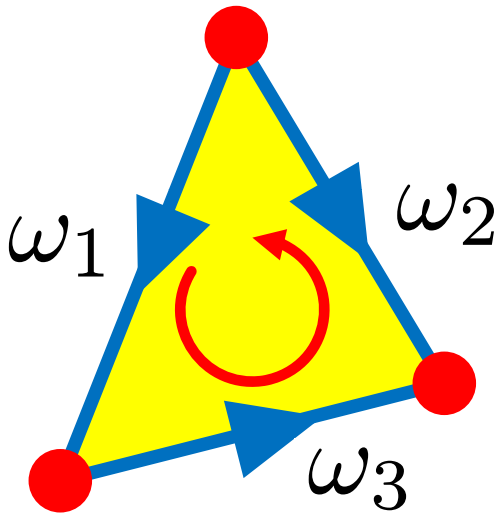
$$d \in \mathbb{R}^{|E| \times |V|}$$

consists of 1, 0, -1



$$\int_e d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1$$

Exterior Derivative

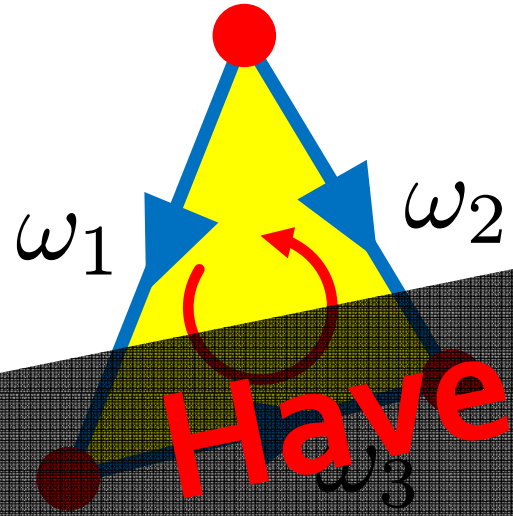


$$d \in \mathbb{R}^{|F| \times |E|}$$

consists of 1, 0, -1

$$\int_t d\omega = \int_{\partial t} \omega = \omega_1 - \omega_2 + \omega_3$$

Exterior Derivative



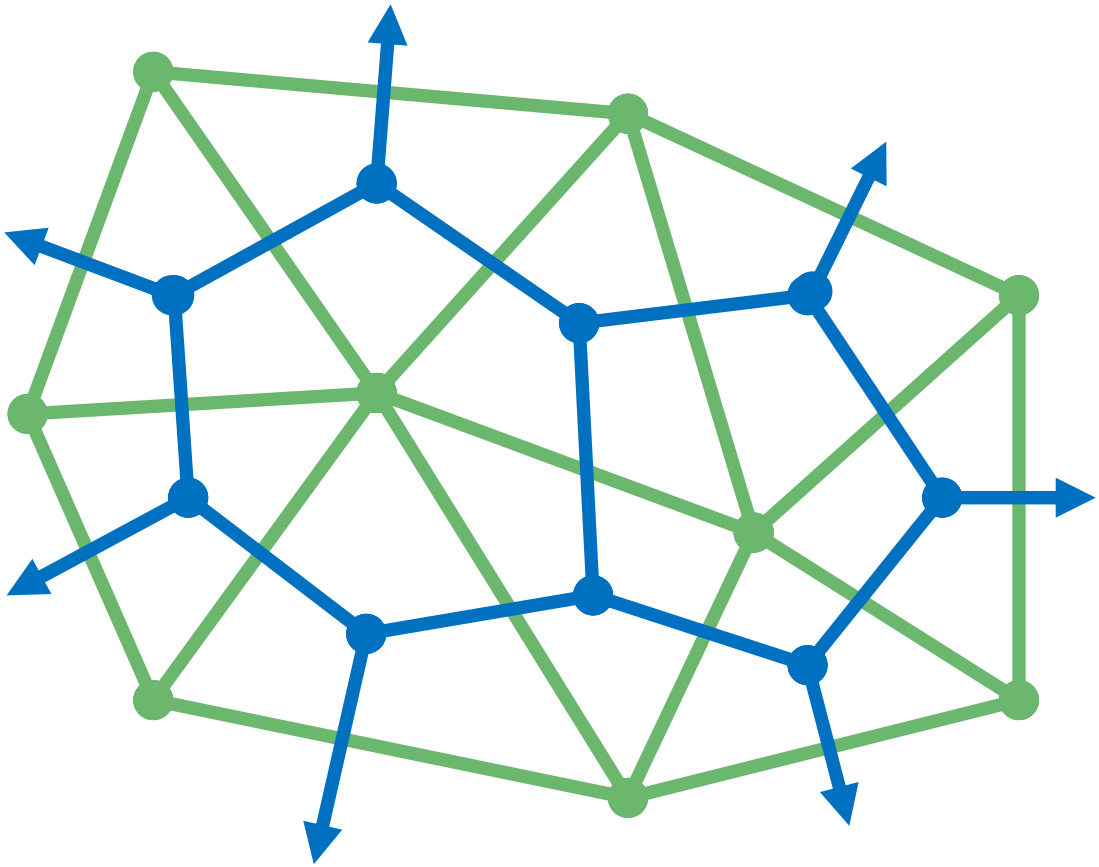
$$d \in \mathbb{R}^{|F| \times |E|}$$

consists of $-1, 0, 1$

Haven't made any approximations yet!

$$\int_t d\omega = \int_{\partial t} \omega = \omega_1 - \omega_2 + \omega_3$$

Hodge Star

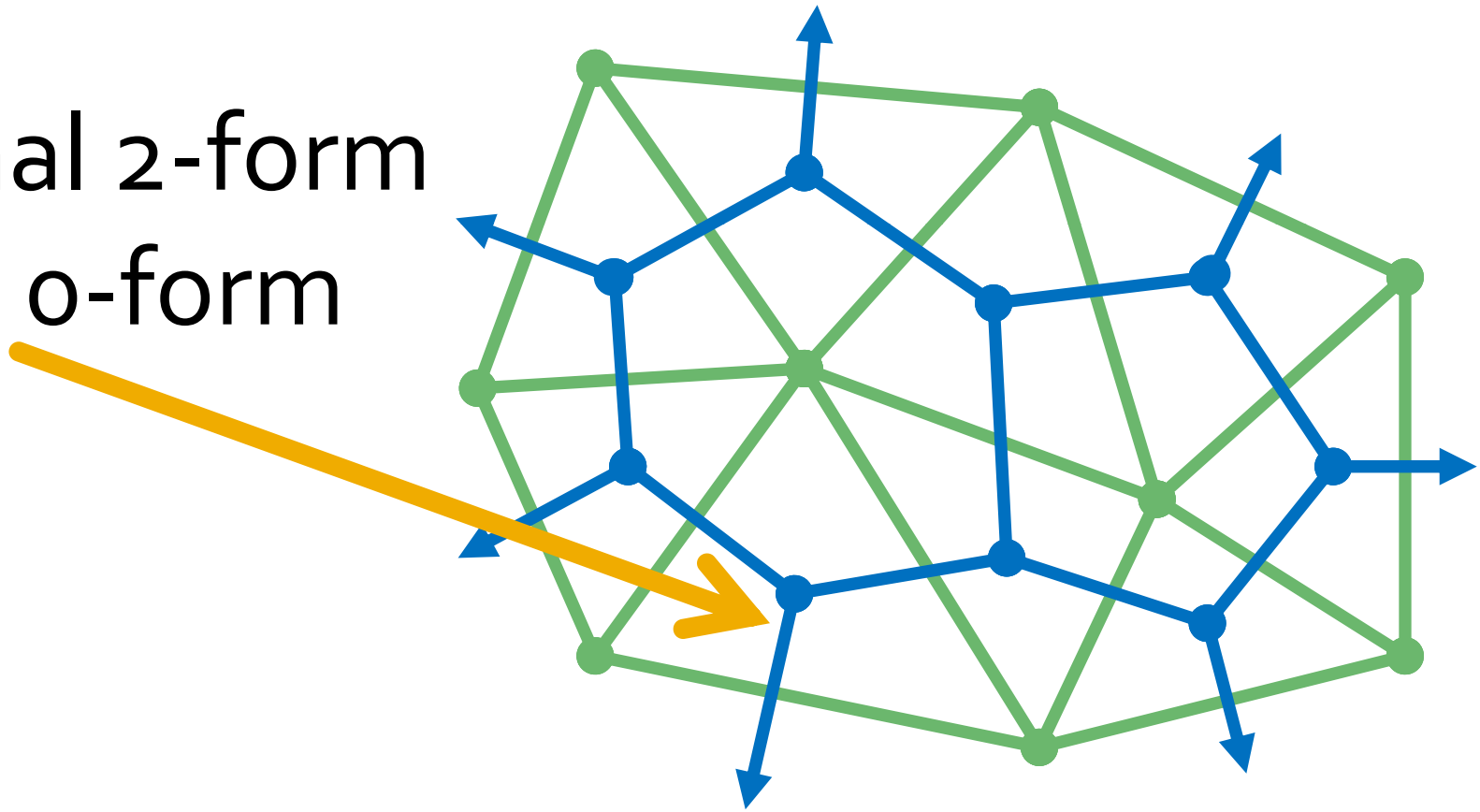


Moves to dual mesh

Hodge Star

Primal 2-form

Dual 0-form

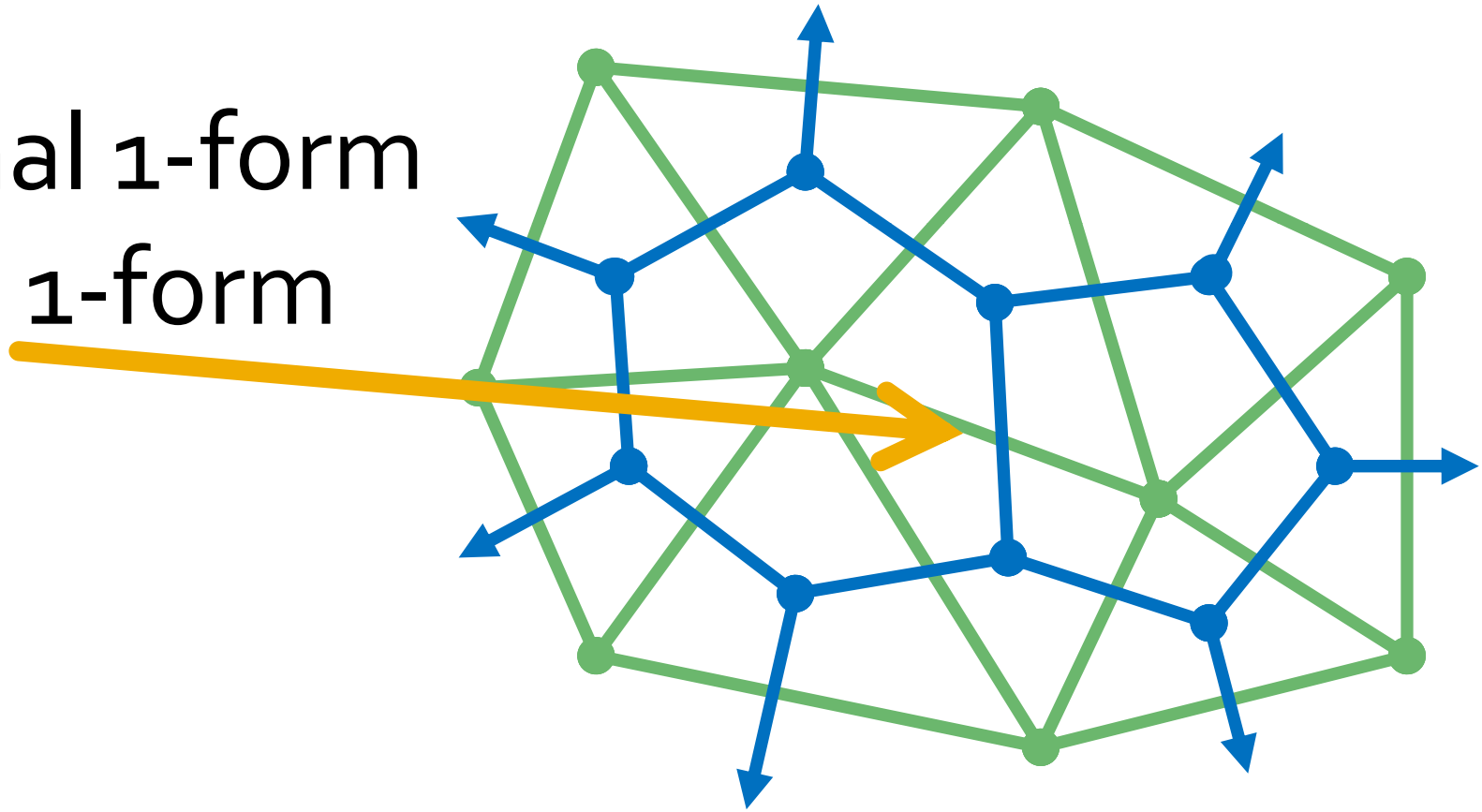


Moves to dual mesh

Hodge Star

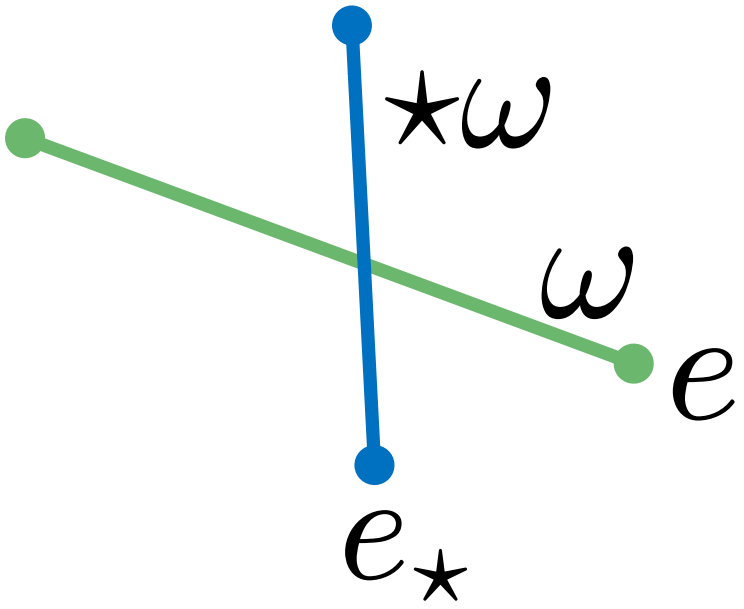
Primal 1-form

Dual 1-form



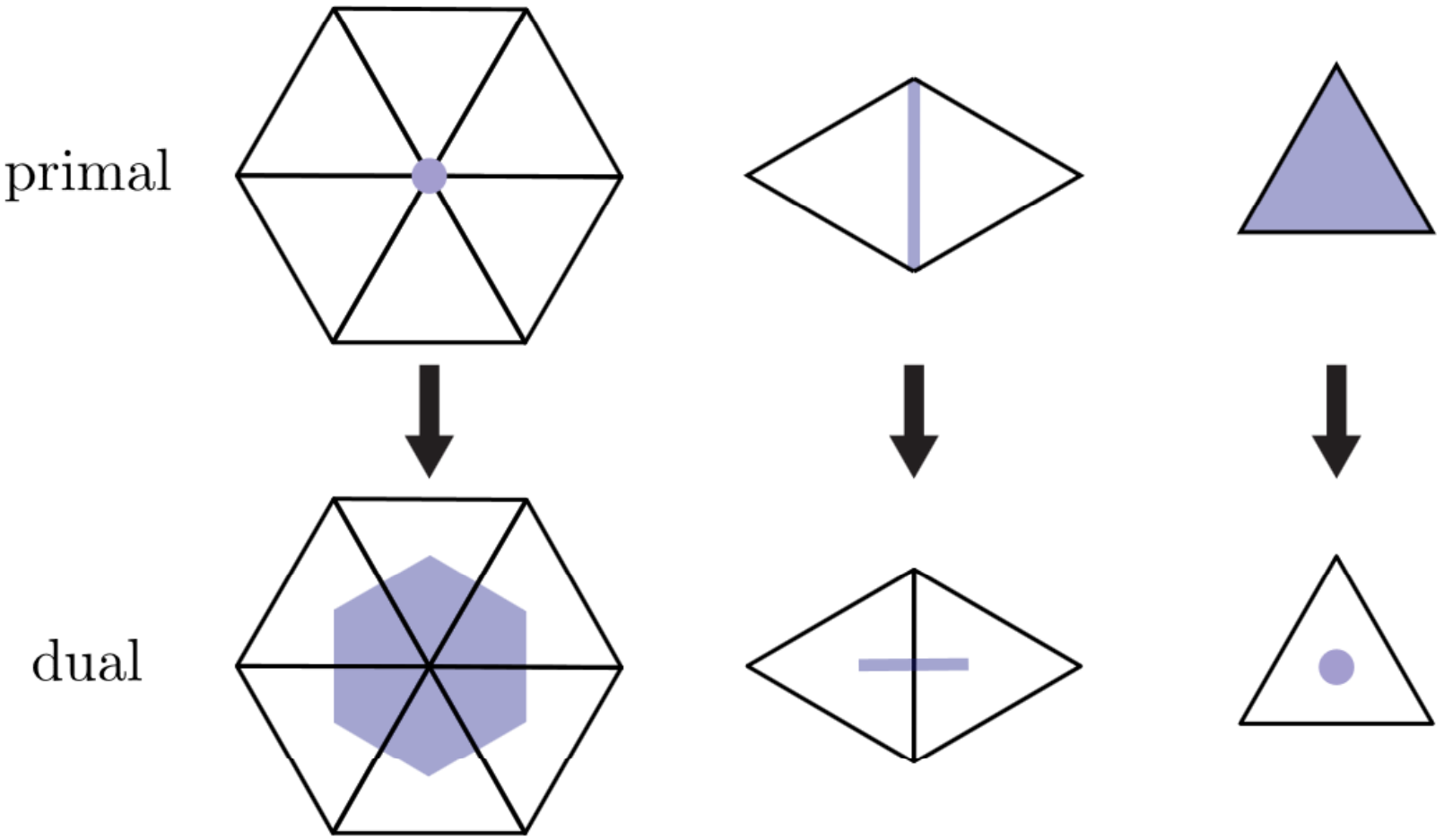
Moves to dual mesh

Hodge Star for 1-Forms



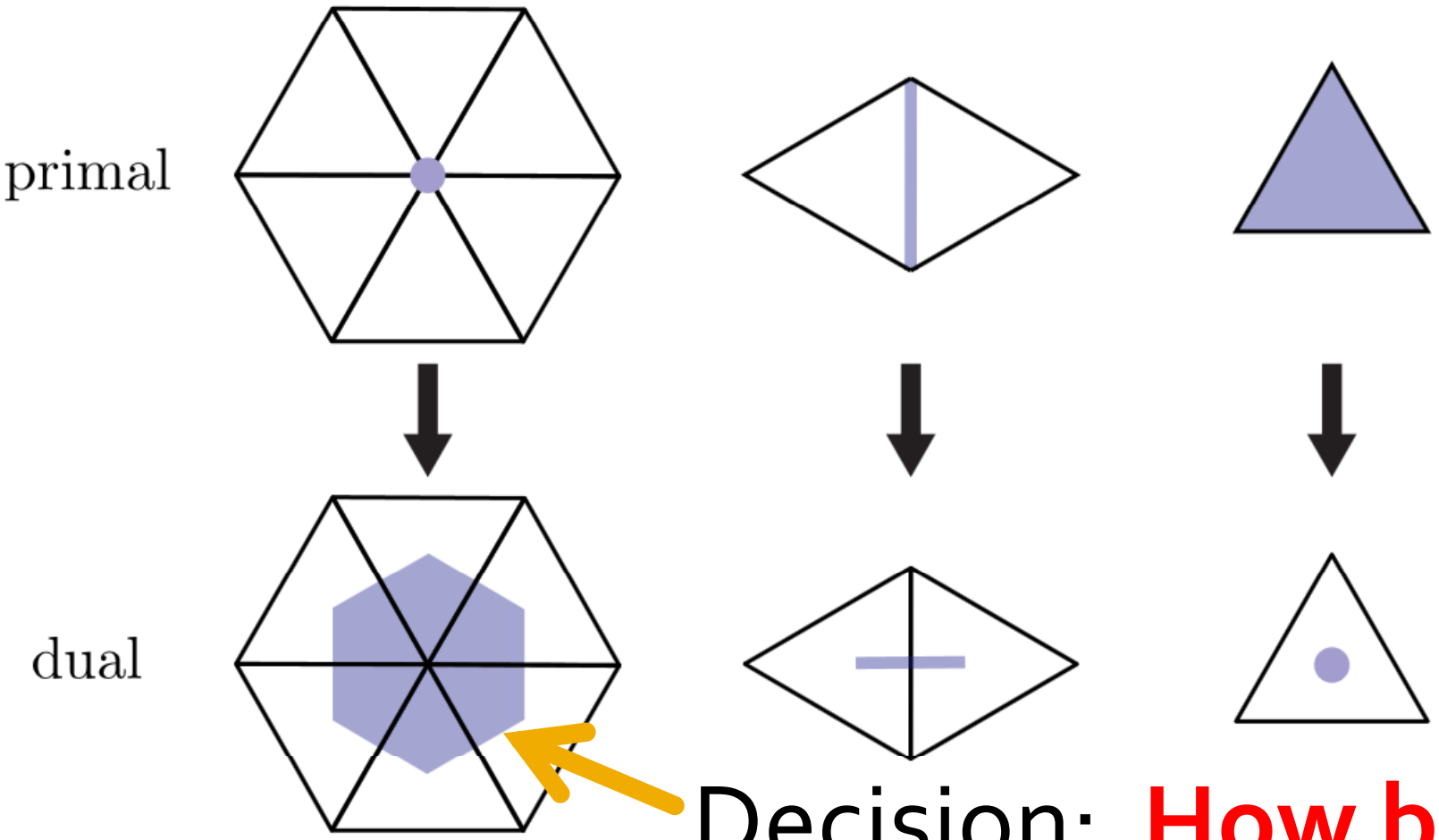
$$\star\omega = \frac{|e_\star|}{|e|}\omega$$

General Hodge Star Formulation



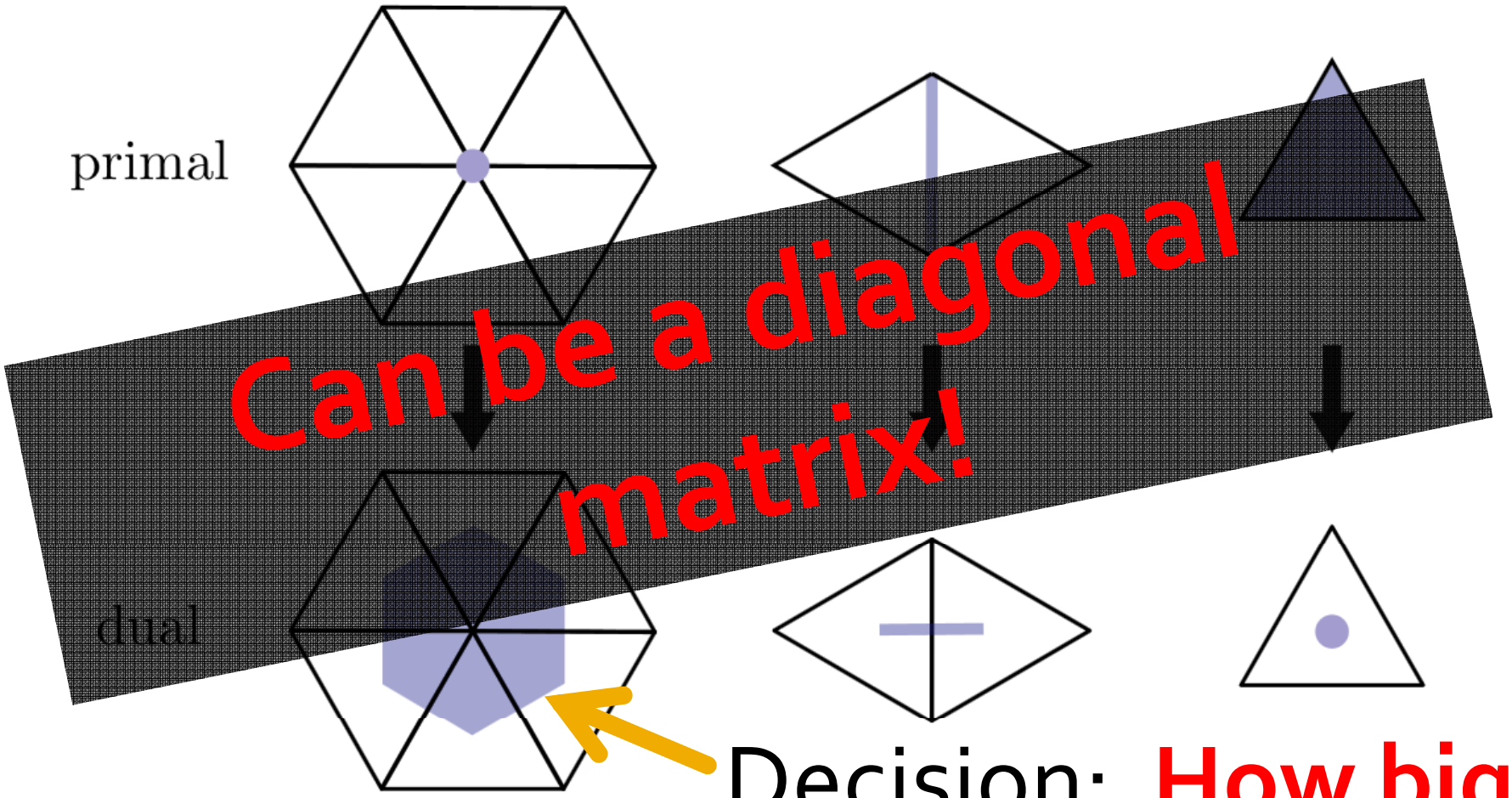
http://brickisland.net/cs177/

General Hodge Star Formulation



Decision: **How big?**

General Hodge Star Formulation



Decision: **How big?**

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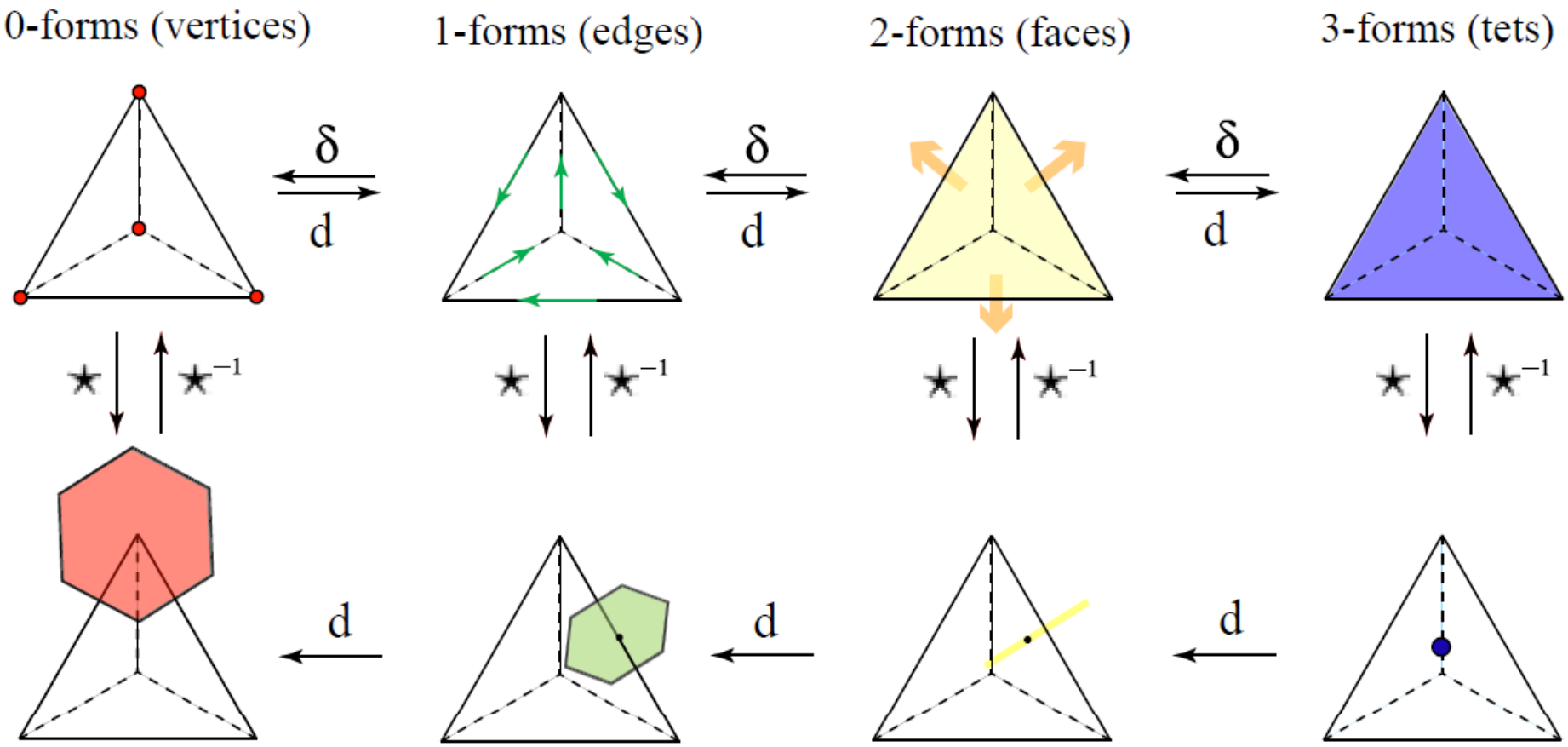
Discrete deRham Complex



http://ddg.cs.columbia.edu/SIGGRAPH06/DDGCourse2006.pdf

Laplacian

$$\Delta = d \star d \star + \star d \star d$$

0-Form Laplacian

$$\Delta = \cancel{d \star d \star} + \star d \star d$$

0-Form Laplacian

$$\Delta = \cancel{d \star d \star} + \star d \star d$$

Cotangent Laplacian

0-Form Laplacian

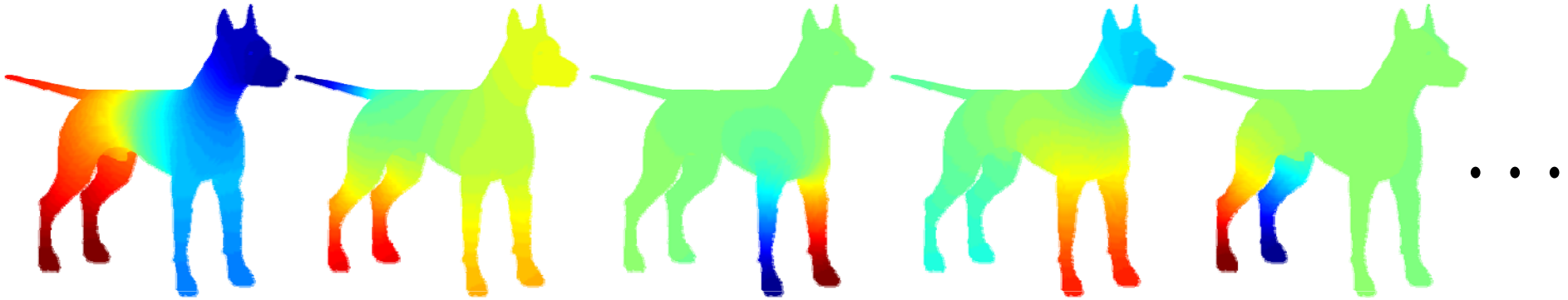
$$\Delta = \cancel{d \star d \star} + \star d \star d$$

Area weights

0-Form Laplacian

$$\Delta = \cancel{d \star d \star} + \star d \star d$$

Laplace-Beltrami Operator



Only Scratching the Surface…

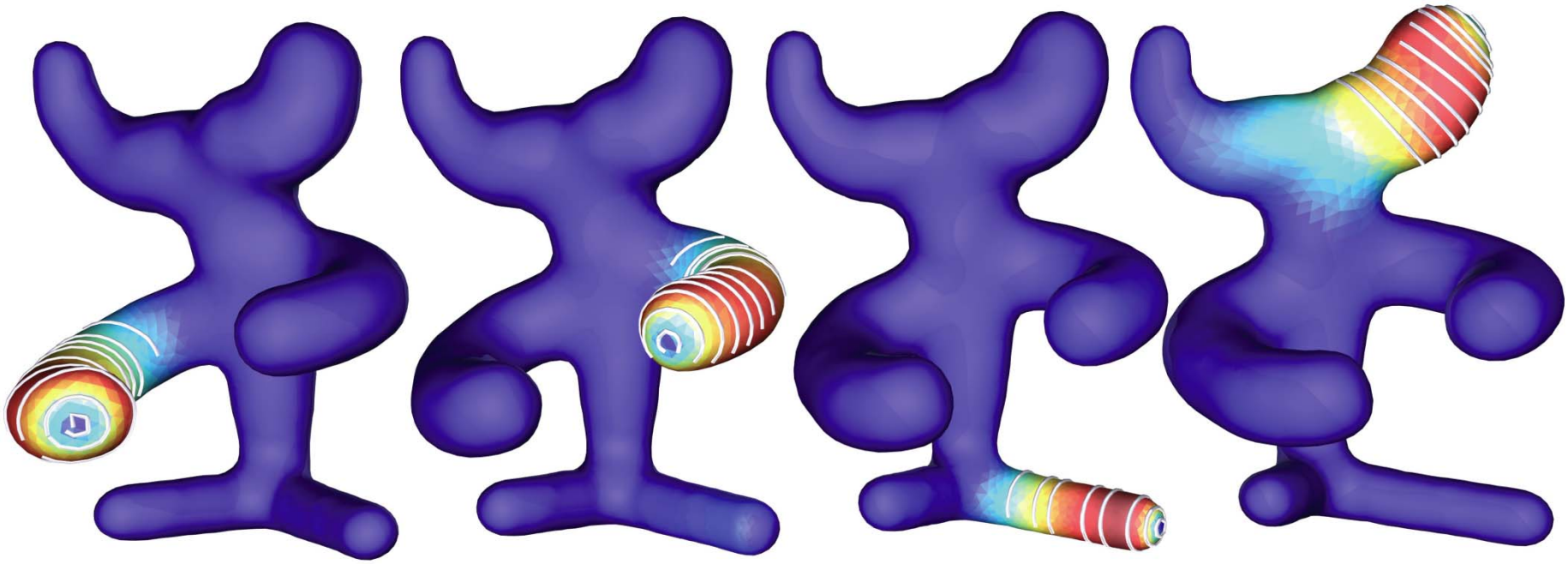
- Additional operators
- Discrete DEC theorems
- Limits of this theory

Why Bother?

Fisher et al. "Design of Tangent Vector Fields" (SIGGRAPH 2007)

Treatment of tangent vector fields

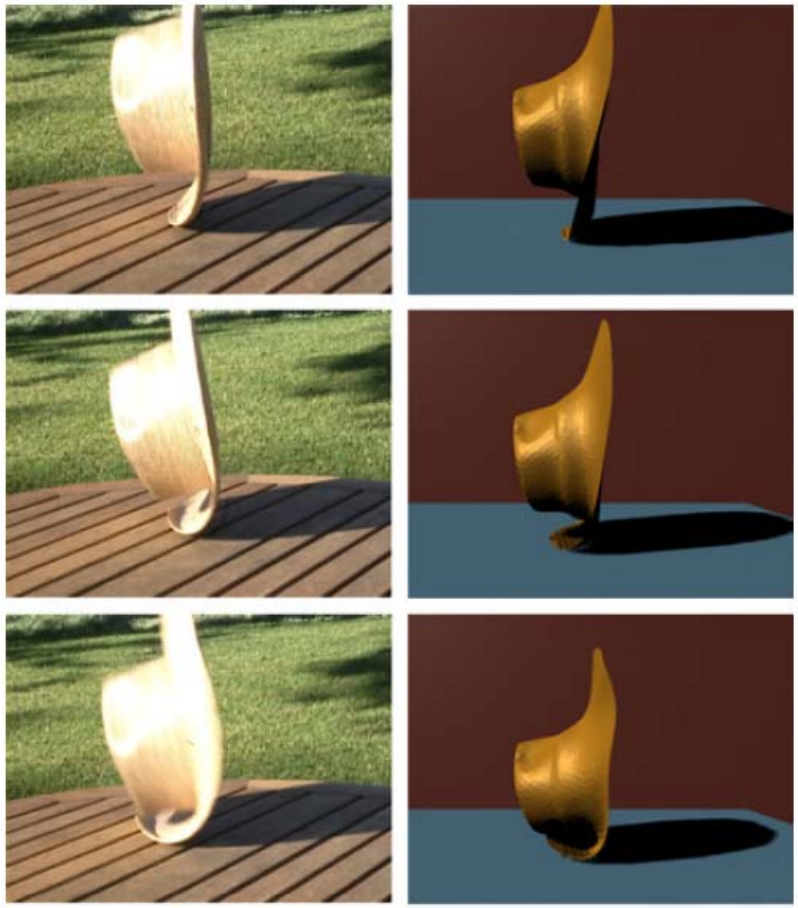
Why Bother?

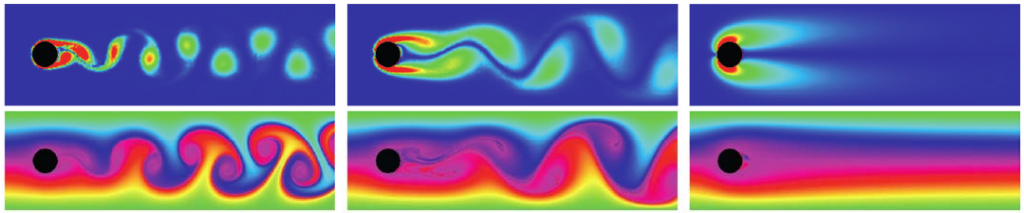


Ben Chen et al. "On Discrete Killing Vector Fields and Patterns on Surfaces" (SGP 2010)

Geometric PDEs

Why Bother?

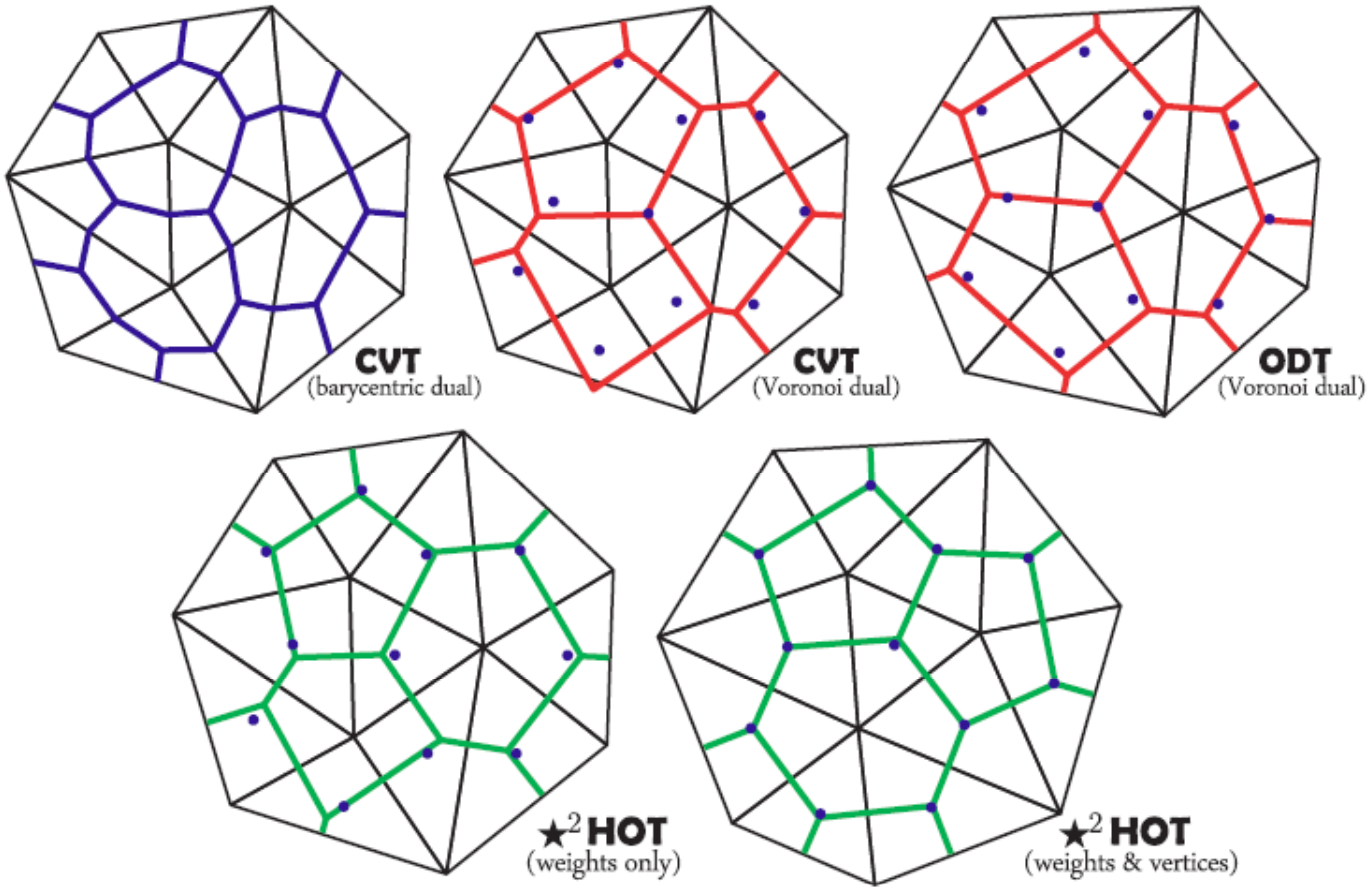




Grinspun. "A Discrete Model of Thin Shells."
Elcott et al. "Stable, Circulation-Preserving, Simplicial Fluids." (TOG)

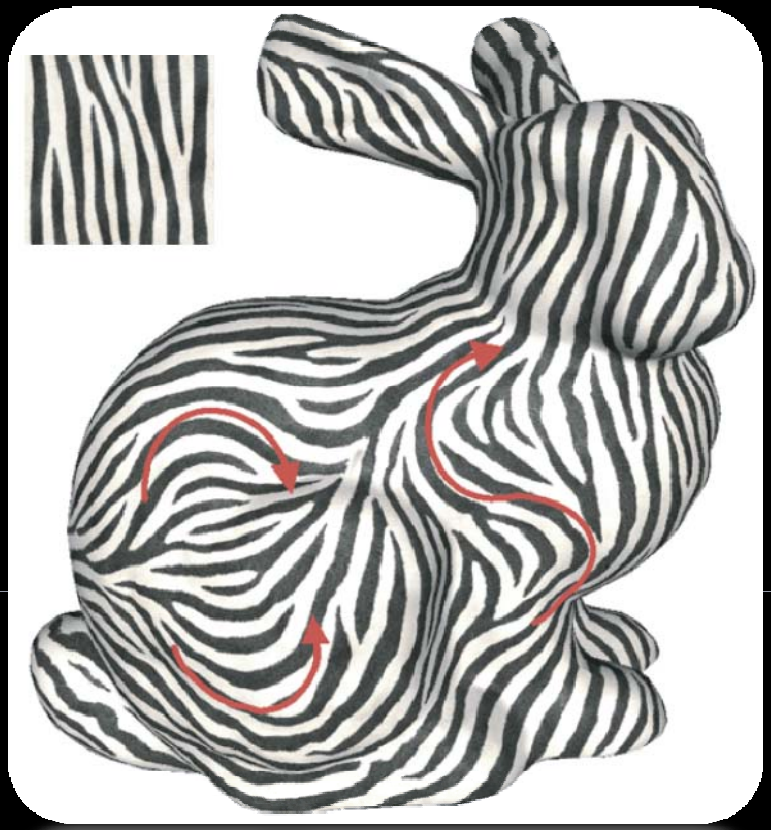
Simulation

Why Bother?



Mullen et al. "HOT: Hodge-Optimized Triangulations" (SIGGRAPH 2011)

Discrete Problems



Fisher et al. "Design of Tangent Vector Fields" (SIGGRAPH 2007)

Discrete Exterior Calculus

Questions?